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On the ultimate boundedness of solutions of systems of differential equations

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Abstract

In this paper we give conditions under which all solutions of a system of differential equations are equi bounded and ultimate bounded. We apply our result to a system which contain to well known Liénard's equation.

1. Introduction

We consider a system of ordinary differential equations:

$$x' = f(t, x). \tag{1}$$

Where $f : I \times \mathbb{R}^m \to \mathbb{R}^m$, $I = [0, +\infty)$, is continuous. Also consider that the solutions of the system are uniquely determined by initial conditions.

In Liapunov's Second Method the study of various qualitative properties associated with solutions of (1) was originated in the fundamental memoir of the russian mathematician Liapunov. Since that time this area has been extensively (perhaps even exhaustively) investigated. Many of the results require intermediate, or direct, use energy (Liapunov) functions.

In [8] Salvadori used scalar functions $V(t, x)$ and $W(t, x)$, and $C(r) > 0$ which satisfy:

$$V'_{(1)}(t, x(t)) \leq -c(W(t, x(t))), \tag{2}$$

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and studied the asymptoptic stability of (1) with $f(t,0) \equiv 0$. In [1] two Liapunov functions have been used to investigate the ultimate boundedness and the equi–ultimate boundedness of the solutions of (1) and

$$V'_{(1)}(t, x(t)) \leq -c(W(t,x)) + \lambda(t)\phi(V(t,x)), \tag{3}$$

where $c(r) > 0$, $\int_0^\infty \lambda(t)dt < \infty$ and $\int_0^\infty \frac{du}{\phi(u)} = \infty$.
The purpose of this paper is to investigate the ultimate boundedness and the equi–ultimate boundedness of the solutions of (1) by using two scalar functions too. In particular, our results generalize those in [1]. In section 3 as an application, we consider the system $x' = \alpha(y) - \beta(y)\ f(x)$, $y' = -a(t)g(x)$ where α, β, f, g and a are continuous.
Let $\mathbb{R}^m$, denote Euclidean m space and $\|\cdot\|$ denote any norm in $\mathbb{R}^m$, $x(t; t_0, x_0)$ denote a solution of (1) with $x(t_0; t_0, x_0) = x_0$.
We will use the following notations:

$$S_r = \{x \in \mathbb{R}^m : ||x|| < r\},$$

$C(X)$ and $CI(X)$ denote the families of continuous functions and continuous increasing functions, respectively, on a real intervall X $(0 \in \operatorname{int} X)$ and $I = [0, +\infty)$.
We consider the following functional class:

$$\begin{aligned}
\mathcal{L} &= \left\{\lambda(t) \in C(I) : \lambda(t) \geq 0, \int_0^\infty \lambda(t)dt < \infty\right\}, \\
P &= \left\{p(t) \in C(I) : p(t) \geq 0, \int_0^\infty p(t)dt = \infty\right\}, \\
\mathcal{F} &= \left\{\phi(u) \in C(\mathbb{R}) : \phi(u) > 0 \text{ no decreasing and } \int_0^\infty \frac{du}{\phi(u)} = \infty \right\}
\end{aligned}$$

For definitions of boundedness of the solutions of (1), we refer the reader to [9].
We shall use some auxiliary functions continuous in (t, x) and locally Lipschitzian in x. Define:

$$V'_{(1)}(t,x) := \lim_{h \to 0^+} \sup h^{-1} \left\{V(t+h, x+hf(t,x)) - V(t,x)\right\}.$$

Now we give two lemmas which will play an important role in the proof of our main result.

Lemma 1. *Suppose that there exists a Liapunov function $V(t,x)$ defined on $I \times \mathbb{R}^m$, which satisfies the following conditions:*

(i) $a(\|x\|) \leq V(t,x)$*, where* $a(r) \in CI(\mathbb{R}^+)$ *and* $a(r) \to \infty$ *as* $r \to \infty$,

(ii) $V'_{(1)}(t,x) \leq \lambda(t)\left[r(t)\phi(V) - q(t)\right]$*, where* $\lambda \in \mathcal{L}$; $q, r \in \mathcal{P}$ *and* $\phi \in \mathcal{F}$.

Then the solutions of (1) are equi–bounded.

Lemma 2. *Let $\lambda(t)$ be continuous on $I, \phi \in \mathcal{F}$, $\Phi(v)$ be a primitive of $1/\phi(v)$ and suppose that $v' \leq \lambda'(t)\phi(v)$. Then we have:*

$$v(t) \leq \Phi^{-1}\left(\Phi(v(t_0)) + \int_{t_0}^{t} \lambda(s)ds\right)$$

for all $t \geq t_0$.

Remark 1. *The proof of lemma 1, is a variant of the proofs of Theorems 10.1 and 10.2 of [9] and the Lemma 2 is a trivial application of the theory of differential inequalities.*

2. Our general theorem

We now state our main result.

Theorem 1. *Suppose that there exist two Liapunov functions $V(t,x)$ and $W(t,x)$, defined on $I \times \mathbb{R}^m$, satisfying the conditions:*

1. $k \leq V(t,x)$, *where* $k \in \mathbb{R}$,
2. $b(\|x\|) \leq W(t,x)$, $b(r) \in CI$ *and* $b(r) \to \infty$, *as* $r \to \infty$,
3. $V'_{(1)} \leq -p(t)\, c(W(t,x)) + \lambda_1(t)\,\phi_1(V(t,x))$, *where* $c(r) \in C(\mathbb{R}^+)$ *and* $\lim\limits_{r \to \infty} \inf c(r) > 0$, $p \in \mathcal{P}$, $\lambda_1 \in \mathcal{L}$, *y* $\phi_1 \in \mathcal{F}$,
4. $W'_{(1)}(t,x) \leq \lambda_2(t)\left[r(t)\,\phi_2(W) - q(t)\right]$, *where* $\lambda_2 \in \mathcal{L}$; $q, r \in \mathcal{P}$, $\phi_2 \in \mathcal{F}$ *with* $r(t)\,\phi_2(W) - q(t) > 0$, *for all* $t \in I$ *and* x *fixed.*

Then the solutions of (1) are equi–bounded and equi–ultimately bounded.

Proof. The conditions 2 and 4 satisfy the lemma 1 and the solutions of (1) are equi–bounded. ∎

There exist positive constants γ and k such that:

$$\inf_{r\geq k} c(r)=\gamma, \quad b^{-1}(s)>0 \quad \text{for any } s\geq k,$$

since $\lim_{r\to\infty}\inf c(r)>0$ and $b(r)\to\infty$ as $r\to\infty$. For the proof of the equi–ultimate boundedness of solutions of (1), we first show the following facts.

We suppose the there exist $t_0\in I$ and $\alpha>0$ such that for all $T>0$,there exist $x_0\in S_\alpha$ and $x(t;t_0,x_0)$ such that:

$$W\left(t,x\left(t;t_0,x_0\right)\right)\geq b\left(B\right) \quad for \quad t_0\leq t\leq t_0+T,$$

which means the no ultimate boundedness of the solutions of (1).

By the condition 3 and Lemma 2, there exist $N>0$ such that:

$$\phi_1\left(V\left(t,x\left(t;t_{0,}x_0\right)\right)\right)\leq N \quad for \quad t\geq t_{0.}$$

Then it would follow the condition 3 that:

$$V'_{(1)}\left(t,x\right)\leq -\gamma p\left(t\right)+N\lambda_1\left(t\right) \quad \text{for} \quad t_0\leq t\leq t_0+T$$

and therefore that:

$$V\left(t_0+T,x\left(t_0+T\right)\right)\leq V\left(t_0,x_0\right)-\gamma\int_{t_0}^{t_0+T}p(t)dt+N\int_{t_0}^{t_0+T}\lambda_1(t)dt.$$

Since $p(t)\in\mathcal{P}$,the last member becomes $-\infty$ as $T\to+\infty$,and this contradicts the condition 1.

Let $\bar{B}>b^{-1}\left(F^{-1}\left(F\left(b\left(B\right)\right)\right)+L\right)$,where $F\left(r\right)$ is a primitive of $\frac{1}{r(t)\phi_2(u)-q(t)}$ and $L=\int_0^\infty\lambda_2(t)dt$

From the condition 4 we have:

$$\frac{W'_{(1)}(t,x)}{r(t)\phi_2(W(t,x))-q(t)}\leq\lambda_2(t).$$

Integrating this inequality from $t_1$and t $(t\geq t_1)$ we have:

$$F\left(W\left(t,x\right)\right)\leq F\left(b\left(B\right)\right)+L$$

and therefore:

$$W\left(t,x\right)\leq F^{-1}\left(F\left(b\left(B\right)\right)+L\right),t\geq t_1,$$

then $W\left(t,x\right)<b(\bar{B})$, $t\geq t_0+T$. Thus the condition 2 shows that:

$$\|x(t)\|<B \quad \text{for} \quad t>t_0+T$$

and the proof of Theorem 1 is completed. ∎

Remark 2. *In [1, Th 3.3] the authors obtained the same conclusion if the condition 4 is replaced by the more restrictive condition:*

$$W'_{(1)}(t,x) \leq \lambda_2(t)\phi_2(W),$$

where $\lambda_2 \in \mathcal{L}$ and $\phi_2 \in \mathcal{F}$.

3. An application

We consider the system:

$$\begin{aligned} x' &= \alpha(y) - \beta(y)f(x), \\ y' &= -a(t)g(x), \end{aligned} \tag{4}$$

where the dots indicate differentiation with respect to t. In what follows the functions α, β, f and g are taken to be continuous real–valued and a positive continuously differentiable on $[0,+\infty)$.
We also asumme that the following conditions are fulfilled:

a) $\alpha \in CC\,(\mathbb{R})$,

b) $\beta \in CP_b\,(\mathbb{R})$,

c) $f, g \in CS\,(\mathbb{R})$,

d) $a \in \left\{CP\,([0+\infty)) \cap C^1\,([0,+\infty))\right\}$,

where:

$$\begin{aligned} CS\,(\mathbb{R}) &= \{h \in C\,(\mathbb{R}) : xh(x) > 0 \text{ for all } x\}, \\ CC\,(\mathbb{R}) &= \{h \in (CI\,(\mathbb{R}) \cap CS\,(\mathbb{R}))\}, \\ CP_K\,(\mathbb{R}) &= \{h \in C\,(\mathbb{R}) : h(x) \geq k > 0 \text{ for all } x\}, \\ CP\,(\mathbb{R}) &:= CP_0\,(\mathbb{R}), \end{aligned}$$

with $C^1\,(\mathbb{R})$ are the families of functions with first derivate continuous on $\mathbb{R}$.
We define $G\,(x) = \int_0^x g\,(s)\,ds$ and $A\,(y) = \int_0^y \alpha(r)dr$.
In [5] we proved that under these conditions all solutions of (4) are continuable to the future.
In many papers of author (for example [2–7]), we discussed the asymptotic behavior of solutions of system (4) under several conditions on functions involved in this.
In this section we shall give sufficient conditions for the equi and ultimate boundedness of solutions of (4).

Theorem 2. *Under conditions a)–d suppose that:*

1. *There exist some constants* ϵ *and* $\underline{a}$ *such that* $a(t) > \underline{a} > 0$ *for* $t \geq 0$ *and* $\lim_{t\to\infty} \frac{2a(t)}{a'(t)} < +\infty$,
2. $A(\pm\infty) = \pm\infty$ *or* $G(\pm\infty) = \pm\infty$

Then the solutions of (4) are equi–bounded and ultimately bounded.

Proof. Let:

$$V(t,x,y) = \frac{2a(t)}{a'(t)} \exp\left(A(y) + a(t)G(x)\right),$$

then: $V(t,x,y) \geq \epsilon$, thus $V(t,x,y)$ satisfies the condition 1 of the Theorem 1. We next obtain:

$$V'_{(4)}(t,x,y) \leq \left\{-abf(x)g(x) + \left[\frac{2a(t)}{a'(t)}\right]'\right\} \exp\left(A(y) + a(t)G(x)\right),$$

from this we have:

$$\begin{aligned} V'_{(4)}(t,x,y) \leq\ & \{-abf(x)g(x)+ \\ & +\ln\left[\frac{2a(t)}{a'(t)}\right]' \left\{\frac{2a(t)}{a'(t)}\right\} \exp\left(A(y) + a(t)G(x)\right)\right\} \end{aligned}$$

taking

$$W(t,x,y) = \frac{1}{\alpha}\frac{2a(t)}{a'(t)} \left[\exp\left(A(y) + a(t)G(x)\right) - 1\right]$$

and

$$c(W) = \underline{a}bf(x)g(x)\left[W + 1\right]$$

is clear that $b(x,y) = A(y) + \underline{a}G(x)$ satisfies the condition 2 of the Theorem 1. Also:

$$W'_{(4)}(t,x,y) \leq \left[\frac{2a(t)}{a'(t)}\right]' \left\{\frac{1}{\alpha}W(t,x,y) + \left[\frac{1}{\alpha} - 1\right]\right\},$$

putting $p(t) \equiv 1$, $\lambda_1(t) = \left[\ln \frac{2a(t)}{a'(t)}\right]'$, $\phi_1(r) = r$, $\lambda_2(t) = \left[\frac{2a(t)}{a'(t)}\right]'$, $r(t) \equiv \frac{1}{\alpha}$ y $q(t) = \frac{1}{\alpha} - 1$, the conditions 3 and 4 in Theorem 1 are fulfilled. Hence, all solutions of (4) are equi–bounded and ultimate bounded. ∎

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