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Function Spaces with Generalized Distances*

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In [1] we presented a generalization of the idea of a metric, a *generalized distance*. We now consider how these distances can be used to determine convergence structures for function spaces.

Given a collection $\mathfrak{F}$ of continuous functions into a distance space $\langle Y, \delta \rangle$, we produce a definition for a distance on $\mathfrak{F}$ generalizing that of the uniform norm for functions into metric spaces. If $\mathfrak{F}$ is given the topology induced by this distance, then a sufficient condition that the evaluation function be continuous is that δ be a *summable* distance (equivalently, that the topological space Y be regular.)

We use this distance concept to describe the compact-open topology on a function space by means of a base for the topology in place of the more common subbase.

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Recall from [1] the following definitions and results:

Definition 1: By a distance space we will mean a set Y together with a function δ from $Y \times Y$ to a partially ordered set P such that:

D_1. for any $x, y \in Y$, if $\delta(x, y) < p \in P$, then $\delta(x, x) < p$ and $\delta(y, y) < p$.

D_2. $\delta(x, y) = \delta(y, x)$ for all $x, y \in Y$

D_3. if $\delta(x, y) < \sigma$, then there exists some $\mu \in P$ such that $\delta(y, y) < \mu$ and such that $\delta(y, z) < \mu$ implies that $\delta(x, z) < \sigma$.

D_4. If $\delta(x, y) < \mu$ and $\delta(x, y) < \nu$, then there exists some $\sigma \in P$ such that $\delta(x, y) < \sigma$, $\sigma \leq \mu$ and $\sigma \leq \nu$.

D_5. For any $x, y \in Y$, there exists some $p \in P$ such that $\delta(x, y) < p$.

The partially ordered set P is called a distance set for Y and the function δ is called a distance function. We denote by $N_\varepsilon(x)$ the collection $\{ y \in Y : \delta(x, y) < \varepsilon \}$. A set $N_\varepsilon(x)$ is said to be a distance neighborhood (or a δ neighborhood) of x. Please note that distance neighborhoods may be empty.

A distance space is a triple $\langle X, \delta, P\rangle$ where P is a distance set for X and $\delta : X \times X \longrightarrow P$ is a distance function. If there exists an element $0_\delta \in P$ such that $\delta(x, x) = 0_\delta$ for all $x \in X$, then $\langle X, \delta, P\rangle$ is called a zeroed distance space.

If $\langle X, \delta, P\rangle$ and $\langle Y, \gamma, Q\rangle$ are distance spaces and if f is a function from X to Y, then f is said to be (distance) continuous provided that for any $\varepsilon \in Q$ and any $x \in X$, if $\gamma(f(x), f(x)) < \varepsilon$, then there exists some $\sigma \in P$ such that $\delta(x, x) < \sigma$ and such that $\delta(x, z) < \sigma$ implies that $\gamma(f(x), f(z)) < \varepsilon$. The collection of all distance spaces and all distance continuous functions forms a category *DST*. The full subcategory whose objects are zeroed distance spaces will be designated *ZDST*.

For any distance space $\langle X, \delta, P\rangle$, the collection

$$\{ N_\varepsilon(x) : x \in X, \varepsilon \in P \}$$

is a base for a topology $\mathcal{T}_\delta$ on X and the association which maps $\langle X, \delta, P\rangle$ to $\langle X, \mathcal{T}_\delta\rangle$ induces a functor F_{DT} from *DST* onto the category *TOP* of all topological spaces and all continuous functions. The image under F_{DT} of *ZDST* is the category of R_0 spaces (see [2].) Any two distance spaces with the same image (or homeomorphic images) under F_{DT} are isomorphic. Thus the isomorphism equivalence classes of *DST* form a category equivalent to *TOP*.

Given any topological space $\langle X, \mathcal{T}\rangle$, we can define a partial order on $\mathcal{P}(X)$, the collection of subsets of X, by saying that $A \leq B$ provided that $A \subseteq B$ and $B \in \mathcal{T}$. We designate this partially ordered set as $\mathcal{P}_\mathcal{T}$. Define $\delta_\mathcal{T}(x, y)$ to be $\{x, y\}$, an element of $\mathcal{P}_\mathcal{T}$. Then $\langle X, \delta_\mathcal{T}, \mathcal{P}_\mathcal{T}\rangle$ is a distance space. The association $\langle X,\mathcal{T}\rangle \longrightarrow \langle X, \delta_\mathcal{T}, \mathcal{P}_\mathcal{T}\rangle$ induces a functor F_{TD} from *TOP* into *DST* and the composition $F_{DT} \circ F_{TD}$ is the identity on *TOP*.

Given any topological space $\langle X, \mathcal{T}\rangle$, we denote by $\mathcal{P}(X \times X)$ the collection of all subsets of the product $X \times X$. The collection $\mathcal{P}(X \times X)$ can be partially ordered by requiring that $A \leq B$ only if $A \subseteq B$, that B be symmetric and open in $X \times X$ and that the diagonal $\Delta_X = \{ (x, x) : x \in X \}$ be contained in B. We denote this partially ordered set as $\mathcal{Z}_\mathcal{T}$. We define a function $\zeta_\mathcal{T}$ from $X \times X$ to $\mathcal{Z}_\mathcal{T}$ by $\zeta_\mathcal{T}(x, y) = \Delta_X \cup \{ (x, y), (y, x) \}$. Then $\langle X, \zeta_\mathcal{T}, \mathcal{Z}_\mathcal{T}\rangle$ is a zeroed distance space, the association relating $\langle X, \mathcal{T}\rangle \longrightarrow \langle X, \zeta_\mathcal{T}, \mathcal{Z}_\mathcal{T}\rangle$ induces a functor Z_{TD} from *TOP* into *ZDST* and the composition $F_{DT} \circ Z_{TD}$ is the identity on the category of R_0 spaces.

A distance space $\langle X, \delta, P\rangle$ is said to be summable if for any $x \in X$ and any $\varepsilon \in P$, if $\delta(x, x) < \varepsilon$ then there exists some $\gamma \in P$ such that

a) $\delta(x, x) < \gamma$

b) $\delta(x, y) < \gamma$ and $\delta(y, z) < \gamma$ imply that $\delta(x, z) < \varepsilon$.

If <X, δ, P> is a summable zeroed distance space then F_{DT}(<X, δ, P>) is a regular space (i.e. an R_1 space in the terminology of [2]) and if <X, 𝒯> is a regular space, then Z_{TD}(<X, 𝒯>) is a summable zeroed distance space.

The following two definitions are NOT included in [1]. Their inclusion will simplify both the statements and the proofs of the results in this paper.

Definition 2: A zeroed distance space <X, δ, P> will be said to be a T_1 distance space provided that for any pair of distinct points x, y ∈ X there exists some ε ∈ P such that x ∈ $N_\varepsilon(x)$ and y ∉ $N_\varepsilon(x)$.

Definition 3: A distance space <X, δ, P> will be said to be a *lower bound* distance space provided that for any p, q ∈ P, if for some x, y ∈ X we have δ(x, y) < p and δ(x, y) < q then there exists some r ∈ P such that s < r implies s < p and s < q and such that s < p and s < q implies s ≤ r. We call this element r the *minimum* of p and q.

The relation between T_1 distance spaces and T_1 topological spaces is the obvious one.

Theorem 1: If <X, δ, P> is a zeroed distance space then the image F_{DT}(<X, δ, P>) is a T_1 topological space if and only if the distance space <X, δ, P> is a T_1 distance space.

proof: Suppose that <X, δ, P> is a T_1 distance space. Then for any two points x, y ∈ X, there exists ε ∈ P such that x ∈ $N_\varepsilon(x)$ and y ∉ $N_\varepsilon(x)$. Since x ∈ $N_\varepsilon(x)$, the zero element 0_δ is less than ε and so, as δ(y, y) = 0_δ < ε, we have that y ∈ $N_\varepsilon(y)$. Since δ(x, y) is NOT less than ε, (i.e. y ∉ $N_\varepsilon(x)$) we know that x ∉ $N_\varepsilon(y)$. Hence, x and y each has a neighborhood which does not contain the other, and so, <X, $\mathcal{T}_\delta$> is a T_1 space. Suppose now that <X, $\mathcal{T}_\delta$> is a T_1 space. For any two points x, y ∈ X there exist open sets U, V ∈ $\mathcal{T}_\delta$ such that x ∈ U, y ∈ V, x ∉ V and y ∉ U. From the construction of $\mathcal{T}_\delta$, there must exist some ε ∈ P such that x ∈ $N_\varepsilon(x)$ ⊆ U. This, then implies that y ∉ $N_\varepsilon(x)$, and so <X, δ, P> is a T_1 distance space.

We note that for any topological space $\langle X, \mathcal{T}\rangle$, the image $F_{TD}(\langle X, \mathcal{T}\rangle)$ is a lower bound distance space and that for any R_0 space $\langle X, \mathcal{T}\rangle$, the image $Z_{TD}(\langle X, \mathcal{T}\rangle)$ is also a lower bound distance space. Thus, we lose little in limiting ourselves to lower bound distance spaces.

The following is an elementary exercise:

Proposition 1: If $\langle X, \delta, P\rangle$ is a lower bound distance space, if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is any finite subset of P and if for some pair of points $x, y \in X$ we have $\delta(x, y) < \varepsilon_i$ for each $i = 1, 2, \ldots, n$, then there exists some $\varepsilon_0 \in P$ such that:

1. $p \le \varepsilon_0$ implies $p \le \varepsilon_i$ for each $i = 1, 2, \ldots, n$
2. $p \le \varepsilon_i$ for each $i = 1, 2, \ldots, n$ implies $p \le \varepsilon_0$

The following description of a distance on a collection of functions is motivated by the *uniform metric* (see [3]) on a collection of functions into a metric space.

Definition 4: Suppose that $\mathfrak{F}$ is a collection of functions from a set X into a lower bound T_1 distance space $\langle Y, \delta, P\rangle$. Let P^* denote P if P contains an element m such that $\delta(y, z) < m$ for all $y, z \in Y$. If P contains no such element, let P^* denote the set $P \cup \{m\}$ with partial order of P together with the rule that $m > p$ for all $p \in P$. Let $\mathfrak{D}_{\mathfrak{F}}$ denote the collection of functions from X into P^*. We give $\mathfrak{D}_{\mathfrak{F}}$ the partial order defined by defining $\sigma < \tau$ provided that:

1. τ is a constant function $\tau(x) = p$ for all $x \in X$, with $p > 0_\delta$ the zero element of P.
2. there exists some $\varepsilon \in P$ such that:
   a) $\delta(y, y) < \varepsilon$ for any $y \in Y$
   b) for any $x \in X$, $\delta(y, z) \le \sigma(x)$ and $\delta(z, w) < \varepsilon$ implies that $\delta(y, w) < p$.

For any two elements $f, g \in \mathfrak{F}$ we define $\delta_{\mathfrak{F}}(f, g)$ to be the function which carries each x in X to the point $\delta(f(x), g(x))$ of P.

With these definitions, we will have a distance space. It will be easier to follow, if we develop this result as a sequence of lemmas. In the following lemmas, assume that $\mathfrak{F}$ is a collection of functions from a set X to a lower bound T_1 distance space $\langle Y, \delta, P\rangle$.

Lemma 1: If $\delta_{\mathfrak{F}}(f, g) < h \in \mathfrak{D}_{\mathfrak{F}}$ then $\delta_{\mathfrak{F}}(f, f) < h$.

proof: By the definition of the partial order on $\mathfrak{D}_{\mathfrak{F}}$, h must be a constant function $h(x) = p$ for all $x \in X$. Clearly for any three points $y, z, w \in Y$, if:

$$\delta(y, z) \leq \delta_{\mathfrak{F}}(f, f)(x) = \delta(\ f(x),\ f(x)\ ) = 0_\delta$$

then, since $\langle Y, \delta, P\rangle$ is a T_1 distance space, $y = z$. Thus, if $\delta(z, w) < p$ and $\delta(y, z) < \delta_{\mathfrak{F}}(f, f)(x)$, then $\delta(y, z) < p = h(x)$. Therefore $\delta_{\mathfrak{F}}(f, f) < h$.

Lemma 2: For any $f, g \in \mathfrak{D}_{\mathfrak{F}}$, $\delta_{\mathfrak{F}}(f, g) = \delta_{\mathfrak{F}}(g, f)$.

proof: For any $x \in X$, $\delta_{\mathfrak{F}}(f, g)(x) = \delta(\ f(x),\ g(x)\ )$ is, by D_2, equal to $\delta(\ g(x),\ f(x)\ ) = \delta_{\mathfrak{F}}(g, f)(x)$.

Lemma 3: If $\delta_{\mathfrak{F}}(f, g) < h \in \mathfrak{D}_{\mathfrak{F}}$ then there exists $r \in \mathfrak{D}_{\mathfrak{F}}$ such that $\delta_{\mathfrak{F}}(g, g) < r$ and such that $\delta_{\mathfrak{F}}(g, s) < r$ implies $\delta_{\mathfrak{F}}(f, s) < h$.

proof: If $\delta_{\mathfrak{F}}(f, g) < h$, then h is a constant function $h(x) = p$ and there exists some $q > 0_\delta$ in P such that for any $x \in X$ and any $y \in Y$, if $\delta(\ g(x),\ y) < q$ then $\delta(\ f(x),\ y) < p$. Define $r(x)$ to be the constant function $r(x) = q$. As was shown in lemma 1, $\delta_{\mathfrak{F}}(g, g) < r$ and it is immediate that $\delta_{\mathfrak{F}}(g, s) < r$ implies that $\delta_{\mathfrak{F}}(f, s) < h$.

Lemma 4: Suppose that $\delta_{\mathfrak{F}}(f, g) < \rho$ and $\delta_{\mathfrak{F}}(f, g) < \sigma$. Then there exists $\tau \in \mathfrak{D}_{\mathfrak{F}}$ such that $\delta_{\mathfrak{F}}(f, g) < \tau$, $\tau \leq \rho$ and $\tau \leq \sigma$.

proof: From the partial order on $\mathfrak{D}_{\mathfrak{F}}$, ρ is a constant function $\rho(x) = r$ and σ is a constant function $\sigma(x) = s$. There also exist r_0 and s_0 in P such that for any $x \in X$ and any $y \in Y$, if $\delta(\ g(x),\ y\ ) < r_0$ then $\delta(\ f(x),\ y\ ) < r$ and such that if $\delta(\ g(x),\ y\ ) < s_0$ then $\delta(\ f(x),\ y\ ) < s$. Since $\langle Y, \delta, P\rangle$ is a lower bound distance space, there exist z and z_0 in P such that $z \leq r$, $z \leq s$, $z_0 \leq r_0$ and $z_0 \leq s_0$, such that $t \leq r$ and $t \leq s$ implies that $t \leq z$ and such that $t \leq r_0$ and $t \leq s_0$ implies that $t \leq z_0$. Define $\tau(x)$ to be the element z of P. It is clear that $\tau \leq r$ and $\tau \leq s$.

Lemma 5: There exists some $\sigma \in \mathfrak{D}_{\mathfrak{F}}$ such that for any elements f and g of $\mathfrak{F}$, $\delta_{\mathfrak{F}}(f, g) < \sigma$.

proof: The set P^* contains an element m such that for any $y, z \in Y$ the image $\delta(y, z) < m$. Then, defining σ to be the constant function $\sigma(x) = m$, it is clear that for any $f, g \in \mathfrak{F}$, $\delta_{\mathfrak{F}}(f, g) < \sigma$.

Lemma 6: Given any two distinct elements $f, g \in \mathfrak{F}$, there exists some $\sigma \in \mathfrak{D}_{\mathfrak{F}}$ such that $f \in N_\sigma(f)$ and such that $g \notin N_\sigma(f)$.

proof: Since f and g are distinct functions, there exists some $x \in X$ such that $f(x) \neq g(x)$. Since <Y, δ, P> is a T_1 distance space, there is some $p \in P$ such that $\delta(f(x), f(x)) < p$ and such that $g(x) \notin N_p(f(x))$. Define σ to be the constant function $\sigma(x) = p$. From the proof of lemma 1, we can conclude that $\delta_{\mathfrak{F}}(f, f) < \sigma$ and it is immediate that $g \notin N_\sigma(f)$.

Lemma 7: For any three elements $\sigma, \tau, \lambda \in \mathfrak{D}_{\mathfrak{F}}$, if both $\lambda < \sigma$ and $\lambda < \tau$, then there exists $\gamma \in \mathfrak{D}_{\mathfrak{F}}$ which has the property that if $\xi \le \sigma$ and $\xi \le \tau$ then $\xi \le \gamma$ and which has the property that if $\xi \le \gamma$ then $\xi \le \sigma$ and $\xi \le \tau$.

proof: Since $\lambda < \sigma$, then σ must be a constant function $\sigma(x) = \sigma_0$. Since $\lambda < \tau$, then τ must be a constant function $\tau(x) = \tau_0$. For some (any) $x \in X$, we have both $\lambda(x) < \sigma_0$ and $\lambda(x) < \tau_0$. Since <Y, δ, P> is a lower bound distance space, there exists some γ_0 in P such that $\rho_0 \le \sigma_0$ and $\rho_0 \le \tau_0$ imply that $\rho_0 \le \gamma_0$ and such that $\rho_0 \le \gamma_0$ implies $\rho \le \sigma_0$ and $\rho_0 \le \tau_0$. Define γ to be the constant function $\gamma(x) = \gamma_0$. Then if $\alpha < \gamma$, there exists some $\varepsilon \in P$ such that $0_\delta < \varepsilon$ and such that $\delta(y, z) \le \alpha(x)$ and $\delta(z, w) < \varepsilon$ imply $\delta(y, w) < \gamma_0$ and, therefore that $\delta(y, w) < \sigma_0$ and $\delta(y, w) < \tau_0$. Thus, $\alpha < \sigma$ and $\alpha < \tau$. If $\alpha < \sigma$ and $\alpha < \tau$, then there exist ε_1 and ε_2 in P such that $\delta(y, z) \le \alpha(x)$ and $\delta(z, w) < \varepsilon_1$ imply that $\delta(y, w) < \sigma_0$ and such that $\delta(y, z) \le \alpha(x)$ and $\delta(z, w) < \varepsilon_2$ imply that $\delta(y, w) < \tau_0$. Since <Y, δ, P> is a lower bound distance space, there exists $\varepsilon_3 \in P$ such that $p < \varepsilon_3$ implies $p < \varepsilon_1$ and $p < \varepsilon_2$. Thus, $\delta(y, z) \le \alpha(x)$ and $\delta(z, w) < \varepsilon_3$ implies $\delta(y, w) < \sigma_0$ and $\delta(y, w) < \tau_0$ which, in turn, imply that $\delta(y, w) < \gamma_0$.

Having established these lemmas, we are now in a position to prove our theorem.

Theorem 2: If $\mathfrak{F}$ is a collection of functions from a set X to a lower bound T_1 distance space $\langle Y, \delta, P\rangle$, then $\langle \mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{D}_{\mathfrak{F}}\rangle$ is a lower bound T_1 distance space.

proof: From lemma 1, the system satisfies condition D_1 of definition 1. Lemma 2 implies that the system satisfies condition D_2. Lemma 3 implies that the system satisfies condition D_3. Lemma 4 implies that the system satisfies condition D_4. Lemma 5 implies that the system satisfies condition D_5. Hence, $\langle \mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{D}_{\mathfrak{F}}\rangle$ is a distance space. The constant function $0_{\mathfrak{F}}(x) = 0_{\delta}$ is obviously a zero element and so the system is a zeroed distance space. Lemma 6 gives us that $\langle \mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{D}_{\mathfrak{F}}\rangle$ is a T_1 distance space and lemma 7 implies that it is a lower bound distance space.

Given a distance structure for a collection of functions, it is only natural to inquire how this distance structure relates to the continuity of the functions.

Theorem 3: Suppose that $\langle X, \rho, Q\rangle$ is a distance space and that $\langle Y, \delta, P\rangle$ is a summable lower bound T_1 distance space. If $\mathfrak{F}$ is a collection of continuous functions from $\langle X, \rho, Q\rangle$ to $\langle Y, \delta, P\rangle$, then the evaluation function $e : \mathfrak{F} \times X \longrightarrow Y$ defined by $e(f, x) = f(x)$ is continuous (as a function from the product of the distance spaces $\langle \mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{D}_{\mathfrak{F}}\rangle$ and $\langle X, \rho, Q\rangle$ to the distance space $\langle Y, \delta, P\rangle$.)

proof: For a given $x \in X$ and $\varepsilon \in P$, if $\delta(f(x), f(x)) < \varepsilon$, then, since $\langle Y, \delta, P\rangle$ is summable, there exists some $\tau \in P$ such that $f(x) \in \cup \{ N_{\tau}(z) : z \in N_{\tau}(f(x)) \} \subseteq N_{\varepsilon}(f(x))$. Since f is a continuous function, there exists some $\upsilon \in Q$ such that $x \in N_{\upsilon}(x)$ and such that $z \in N_{\upsilon}(x)$ implies $f(z) \in N_{\tau}(f(x))$. Define σ to be the constant function $\sigma(x) = \tau$. If $\delta_{\mathfrak{F}}(f, g) < \sigma$ and $\rho(x, z) < \upsilon$, then $\delta(f(z), g(z)) < \tau$ and $\delta(f(x), f(z)) < \tau$, and so $g(z) \in \cup \{ N_{\tau}(y) : y \in N_{\tau}(f(x)) \}$.

Of course the above construction can be used to construct a topology on $\mathfrak{F}$ such that the evaluation function is a continuous function from the topological space $\mathfrak{F} \times X$ to the topological space Y. This construction does not, in general, provide the smallest topology for which the evaluation function is continuous. It is well known that, in the case that the space Y is locally compact, then the smallest topology on $\mathfrak{F}$ for which the evaluation function is continuous is the compact open topology. A modification of the above construction does, in fact, yield the compact open topology.

Definition 5: Suppose that $\mathfrak{F}$ is a collection of functions from a distance space $<X, \rho, Q>$ into a lower bound T_1 distance space $<Y, \delta, P>$. Let P^* denote P if P contains an element m such that $\delta(y, z) < m$ for all $y, z \in Y$. If P contains no such element, let P^* denote the set $P \cup \{m\}$ with partial order of P together with the rule that $m > p$ for all $p \in P$. Let Σ be a collection of subsets of X which is closed under finite unions and finite intersections, which contains the empty set and which has the property that every element of X has a neighborhood contained in some element of Σ. Let $\mathfrak{D}_\Sigma$ denote the collection of functions from X into P^*. By a "Σ distance construct" we will mean a pair $<\mathfrak{S}, \ell>$ where $\mathfrak{S}$ is a finite subset of Σ which is closed under the formation of unions and intersections, and where ℓ is a function from $\mathfrak{S}$ to P having the property that $\mathfrak{M} \subseteq \mathfrak{N}$ implies $\ell(\mathfrak{M}) \leq \ell(\mathfrak{N})$. Each Σ distance construct $<\mathfrak{S}_\upsilon, \ell_\upsilon>$ determines a "Σ distance function" τ_υ defined by:

$$\tau_\upsilon(x) = \begin{cases} m \text{ if } x \notin \cup \{ S : S \in \mathfrak{S}_\upsilon \} \\ \ell(\cap \{ S : x \in S \in \mathfrak{S}_\upsilon \}) \text{ if } x \in \cup \{ S : S \in \mathfrak{S}_\upsilon \} \end{cases}$$

We give $\mathfrak{D}_\Sigma$ the partial order obtained by defining $\sigma < \tau$ provided that:

1. τ is a Σ distance function
2. there exists some $\varepsilon \in P$ such that:
   a) $\delta(y, y) < \varepsilon$ for any $y \in Y$
   b) for any $x \in X$, if $\delta(y, z) \leq \sigma(x)$ and $\delta(z, w) < \varepsilon$, then $\delta(y, w) < \tau(x)$.

For any two elements $f, g \in \mathfrak{F}$ we define $\delta_{\mathfrak{F}}(f, g)$, as before, to be the function which carries each element x of X to the point $\delta(f(x), g(x))$ of P.

We will need a basic result concerning the order on $\mathfrak{D}_\Sigma$.

Proposition 2: Let <X, ρ, Q>, <Y, δ, P>, $\mathfrak{D}_\Sigma$, $\mathfrak{F}$ and $\delta_{\mathfrak{F}}$ be as described above and let ψ be an element of $\mathfrak{D}_\Sigma$. Suppose that $S, T \in \Sigma$ and $S \subseteq T$. Suppose also that $p, q \in P$ and that $p < q$. If $\psi < \tau_{qT}$ then $\psi < \tau_{pS}$.

proof: For any $x \in X$, it is clear that $\tau_{qT}(x) \le \tau_{pS}(x)$. Since $\psi < \tau_{qT}$, there exists some $\varepsilon \in P$ such that $\delta(y, y) < \varepsilon$ for each $y \in Y$, and such that for any $x \in X$, if $\delta(y, z) \le \psi(x)$ and if $\delta(z, w) < \varepsilon$, then $\delta(y, w) < \tau_{qT}(x) \le \tau_{pS}(x)$.

We must, of course, prove that the structure of definition 5 produces a distance space. Once again we do this as a sequence of lemmas, all very similar to the lemmas proved earlier. In each of the following lemmas, assume that <X, ρ, Q> is a distance space. Assume also that Σ is a collection of subsets of X which is closed under finite unions and which has the property that each element of X has a neighborhood which is contained in some element of Σ. Finally assume that <Y, δ, P> is a lower bound T_1 distance space.

Lemma 1A: If $\delta_{\mathfrak{F}}(f, g) < h \in \mathfrak{D}_\Sigma$ then $\delta_{\mathfrak{F}}(f, f) < h$.

proof: By the definition of the partial order on $\mathfrak{D}_\Sigma$, h must be a function of the form τ_{pS}. Clearly for any $x \in S$ and any three points $y, z, w \in Y$, if $\delta(y, z) \le \delta_{\mathfrak{F}}(f, f)(x) = \delta(\ f(x),\ f(x)\ )$ which is equal to 0_δ. Then, since <Y, δ, P> is a T_1 distance space, $y = z$. Thus, if $\delta(z, w) < h(x)$ and $\delta(y, z) \le \delta_{\mathfrak{F}}(f, f)(x)$, then $\delta(y, w) < h(x)$. Therefore $\delta_{\mathfrak{F}}(f, f) < h$.

Lemma 2A: For any $f, g \in \mathfrak{D}_\Sigma$, $\delta_{\mathfrak{F}}(f, g) = \delta_{\mathfrak{F}}(g, f)$.

proof: For any $x \in X$, $\delta_{\mathfrak{F}}(f, g)(x) = \delta(\ f(x),\ g(x)\ )$ is, by D_2, equal to $\delta(\ g(x),\ f(x)\ ) = \delta_{\mathfrak{F}}(g, f)(x)$.

Lemma 3A: If $\delta_{\mathfrak{F}}(f, g) < h \in \mathfrak{D}_\Sigma$ then there exists $r \in \mathfrak{D}_\Sigma$ such that $\delta_{\mathfrak{F}}(g, g) < r$ and such that $\delta_{\mathfrak{F}}(g, s) < r$ implies $\delta_{\mathfrak{F}}(f, s) < h$.

proof: If $\delta_{\mathfrak{F}}(f, g) < h$, then h is a function of the form τ_{pS} for some $S \in \Sigma$ and some $p \in P$. In addition there exists some $\varepsilon \in P$ such that $0_\delta < \varepsilon$ and such that $\delta(y, z) \le \delta(\ f(x), g(x)\ )$ and $\delta(z, w) < \varepsilon$ imply that $\delta(y, w) < h(x)$. Define $r(x)$ to be the function $\tau_{\varepsilon S}(x)$. With this definition, it is immediate that $\delta_{\mathfrak{F}}(g, g) < r$. Suppose that $\delta_{\mathfrak{F}}(g, s) < r$. For any $x \in X$, if $\delta(y, z) \le \delta(\ f(x), s(x)\ )$ and $\delta(z, w) < r(x)$ then either $x \notin S$ which implies that $\delta(y, w) < h(x) = m$, or $x \in S$ and $\delta(z, w) < \varepsilon$, and so $\delta(y, w) < h(x) = p$. In either case we have that $\delta(y, w) < h(x)$.

Lemma 4A: Suppose that $\delta_{\mathfrak{F}}(f, g) < \eta$ and $\delta_{\mathfrak{F}}(f, g) < \sigma$. Then there exists $\lambda \in \mathfrak{D}_\Sigma$ such that $\delta_{\mathfrak{F}}(f, g) < \lambda$, $\lambda \le \eta$ and $\lambda \le \sigma$.

proof: From the partial order on $\mathfrak{D}_\Sigma$, η is a function of the form $\eta(x) = \tau_{pS}(x)$ for some $p \in P$ and some $S \in \Sigma$, and σ is a function of the form $\sigma(x) = \tau_{qT}(x)$ for some $q \in P$ and some $T \in \Sigma$. Let U be the union $S \cup T$. By hypothesis, Σ is closed under finite unions, and so U is an element of Σ. Since $\langle Y, \delta, P\rangle$ is a lower bound distance space,there exists $w \in P$ such that $0_\delta < w$, such that $w \le p$, such that $w \le q$ and such that if $z \le p$ and $z \le q$ then $z \le w$. Define λ by:

$$\lambda(x) = \begin{cases} m \text{ if } x \in X\backslash(T \cup S) \\ p \text{ if } x \in S\backslash T \\ q \text{ if } x \in T\backslash S \\ w \text{ if } x \in S \cap T \end{cases}$$

Clearly λ is a Σ distance function. Since $\delta_{\mathfrak{F}}(f, g) < \eta$, there exists $\varepsilon_1 \in P$ such that $0_\delta < \varepsilon_1$ and such that for any $x \in X$ and any $r, s, t \in Y$, if $\delta(r, s) \le \delta(\ f(x), g(x)\ )$ and $\delta(s, t) < \varepsilon_1$ then $\delta(r, t) < \eta(x)$. Since $\delta_{\mathfrak{F}}(f, g) < \sigma$, there exists some $\varepsilon_2 \in P$ such $0_\delta < \varepsilon_2$ and such that for any $x \in X$ and any $r, s, t \in Y$, if $\delta(r, s) \le \delta(\ f(x), g(x)\ )$ and $\delta(s, t) < \varepsilon_2$, then $\delta(r, t) < \sigma(x)$. As $\langle Y, \delta, P\rangle$ is a lower bound space, there exists $\gamma \in P$ such that $\gamma \le \varepsilon_1$, $\gamma \le \varepsilon_2$ and such that $\delta(a, b) < \varepsilon_1$ and $\delta(a, b) < \varepsilon_2$ implies that $\delta(a, b) < \gamma$. Suppose now that $x \in X$, that $r, s, t \in Y$, that $\delta(r, s) \le \delta(\ f(x), g(x)\ )$ and that $\delta(s, t) < \gamma$. If $x \in X\backslash(S \cup T)$, then $\delta(r, t) < m = \lambda(x)$. If $x \in S\backslash T$ then

$\delta(s, t) < \gamma \le \varepsilon_1$ implies that $\delta(r, t) < p = \lambda(x)$. If $x \in T\setminus S$ then $\delta(s, t) < \gamma \le \varepsilon_2$ implies that $\delta(r, t) < q = \lambda(x)$. Finally, if $x \in S \cap T$, then $\delta(r, t) < p$ and $\delta(r, t) < q$ implies that $\delta(r, t) < w = \lambda(x)$. Hence $\delta_{\mathfrak{F}}(f, g) < \lambda$.

Lemma 5A: There exists some $\sigma \in \mathfrak{D}_\Sigma$ such that for any elements f and g of $\mathfrak{F}$, $\delta_{\mathfrak{F}}(f, g) < \sigma$.

proof: The set P^* contains an element m such that for any $y, z \in Y$ the image $\delta(y, z) < m$. Then, defining σ to be the constant function $\sigma(x) = m$, it is clear that for any $f, g \in \mathfrak{F}$, $\delta_{\mathfrak{F}}(f, g) < \sigma$.

Lemma 6A: Given any two distinct elements $f, g \in \mathfrak{F}$, there exists some $\sigma \in \mathfrak{D}_\Sigma$ such that $f \in N_\sigma(f)$ and such that $g \notin N_\sigma(f)$.

proof: Since f and g are distinct functions, there exists some $x \in X$ such that $f(x) \neq g(x)$. Since $\langle Y, \delta, P\rangle$ is a T_1 distance space, there is some $p \in P$ such that $\delta(f(x), f(x)) < p$ and such that $g(x) \notin N_p(f(x))$. There exists some $S \in \Sigma$ such that $x \in S$. Define σ to be the function

$$\sigma(z) = \begin{cases} p & \text{if } z \in S \\ m & \text{if } z \notin S \end{cases}$$

The function σ is clearly a Σ distance function, $\delta_{\mathfrak{F}}(f, f) < \sigma$ and it is immediate that $g \notin N_\sigma(f)$.

Lemma 7A: For any three elements $\sigma, \vartheta, \lambda \in \mathfrak{D}_\Sigma$, if both $\lambda < \sigma$ and $\lambda < \vartheta$, then there exists $\gamma \in \mathfrak{D}_\Sigma$ which has the property that if $\xi \le \sigma$ and $\xi \le \vartheta$ then $\xi \le \gamma$ and which has the property that if $\xi \le \gamma$ then $\xi \le \sigma$ and $\xi \le \vartheta$.

proof: Since $\lambda < \sigma$, then σ must be a Σ distance function. Since $\lambda < \vartheta$ then ϑ must also be a Σ distance function. For each $x \in X$ we define $\gamma(x)$ to be the minimum of $\sigma(x)$ and $\vartheta(x)$. It is not difficult to see that γ is also a Σ distance function. If $\xi < \sigma$ and $\xi < \vartheta$ then there exists ε_1 and ε_2 such that $0_\delta < \varepsilon_1$ and $0_\delta < \varepsilon_2$, and such that for any $x \in X$ and any $w, y, z \in Y$, if $\delta(w, y) < \xi(x)$ and $\delta(y, z) < \varepsilon_1$ then $\delta(w, z) < \sigma(x)$ and if

$\delta(w, y) < \xi(x)$ and $\delta(y, z) < \varepsilon_2$ then $\delta(w, z) < \vartheta(x)$. Let ε be the minimum of ε_1 and ε_2. For any $x \in X$ and any $w, y, z \in Y$, if $\delta(w, y) < \xi(x)$ and $\delta(y, z) < \varepsilon$, then $\delta(w, z) < \gamma(x)$. Hence, $\xi < \gamma$. Suppose, now, that $\xi < \gamma$. There exists ε such that for any $x \in X$ and any $w, y, z \in Y$, if $\delta(w, y) < \xi(x)$ and $\delta(y, z) < \varepsilon$ then $\delta(w, z) < \gamma(x)$. This, then, implies that $\delta(w, z) < \sigma(x)$ and $\delta(w, z) < \vartheta(x)$. Hence, $\xi < \sigma$ and $\xi < \vartheta$.

These lemmas permit us to establish the intended result:

Theorem 4: If $\mathfrak{F}$ is a collection of functions from a set X to a lower bound T_1 distance space $<Y, \delta, P>$, then $<\mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{P}_{\Sigma}>$ is a lower bound T_1 distance space.

proof: From lemma 1A, the system satisfies condition D_1 of definition 1. Lemma 2A implies that the system satisfies condition D_2. Lemma 3A implies that the system satisfies condition D_3. Lemma 4A implies that the system satisfies condition D_4. Lemma 5A implies that the system satisfies condition D_5. Hence, $<\mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{P}_{\Sigma}>$ is a distance space. The constant function $0_{\mathfrak{F}}(x) = 0_{\delta}$ is obviously a zero element and so the system is a zeroed distance space. Lemma 6A gives us that $<\mathfrak{F}, \delta_{\mathfrak{F}}, \mathfrak{P}_{\Sigma}>$ is a T_1 distance space and lemma 7A implies that it is a lower bound distance space.

With this distance function, the distance space also has the property that the evaluation function is continuous:

Theorem 5: Suppose that $<X, \rho, Q>$ is a distance space and that $<Y, \delta, P>$ is a summable lower bound T_1 distance space. If $\mathfrak{F}$ is a collection of continuous functions from $<X, \rho, Q>$ to $<Y, \delta, P>$, and if Σ is a collection of subsets of X as in definition 5, then the evaluation function $e : \mathfrak{F} \times X \longrightarrow Y$ defined by $e(f, x) = f(x)$ is continuous (as a function from the product of the distance spaces $<\mathfrak{F}, \delta_{\Sigma}, \mathfrak{P}_{\Sigma}>$ and $<X, \rho, Q>$ to the distance space $<Y, \delta, P>$.)

proof: For a given $x \in X$ and $\varepsilon \in P^*$, if $\delta(\ f(x), f(x)\ ) < \varepsilon$, then, since $<Y, \delta, P>$ is summable, there exists some $\mu \in P$ such that $f(x) \in \cup \{ N_\mu(z) : z \in N_\mu(f(x)) \} \subseteq N_\varepsilon(f(x))$. Since f is a continuous function, there exists some $\upsilon \in Q$ such that $x \in N_\upsilon(x)$ and such that $z \in N_\upsilon(x)$ implies $f(z) \in N_\mu(f(x))$. Choose an element $S \in \Sigma$ which is a neighborhood of x. Define σ to be the function $\tau_{\mu S}(x)$. There exists some $\psi \in P$, such that $x \in N_\psi(x) \subseteq S$. Let φ denote the minimum of υ and ψ. For any $g \in N_\sigma(f)$ and any $z \in N_\varphi(x)$, we have that $\delta(f(x), f(z)) < \mu$ and, since $z \in S$, we also have that $\delta(f(z), g(z)) < \mu$. Hence $\delta(f(x), g(z)) < \varepsilon$, and so the evaluation function is continuous.

Theorems 2, 3, 4 and 5 establish distance functions (and thus topologies) on function spaces and the relationship with the evaluation functions on these function spaces. The most common traditional technique for assigning a topology to a collection of functions is, in the case of locally compact spaces, the compact open topology. It seems natural, then, to ask how the compact open topology is related to the structures of theorems 2 and 4. This, however, requires an additional distance property for the spaces.

Definition 6: A distance space $<X, \delta, P>$ will be said to be *d-bounded* if for any $x \in X$ and any $\varepsilon \in P$, if $\delta(x, x) < \varepsilon$, there exists some $\gamma \in P$ such that if $\delta(x, y) < \gamma$ and $\delta(x, z) < \gamma$ then $d(y, z) < \varepsilon$.

We note that it is easily shown that for any topological space $<X, \mathcal{T}>$, the image $F_{TD}(<X, \mathcal{T}>)$ is d-bounded. If $<X, \mathcal{T}>$ is an arbitrary R_0 space, the image $Z_{TD}(<X, \mathcal{T}>)$ might NOT be d-bounded. (This is particularly easy to see in the case where $<X, \mathcal{T}>$ is an infinite cofinite space.) If, however, $<X, \mathcal{T}>$ is an R_0 space, then $Z_{TD}(<X, \mathcal{T}>)$ IS d-bounded.

Theorem 6: Suppose that $<X, \rho, Q>$ is a summable zeroed distance space, that $<Y, \delta, P>$ is a d-bounded summable lower bound T_1 distance space and that $\mathfrak{F}$ is the collection of all continuous functions from $<X, \rho, Q>$ to $<Y, \delta, P>$. If the image under F_{DT} of

$\langle X, \rho, Q\rangle$ is a locally compact space and if Σ is the collection of compact subsets of X, then the topology on $\mathfrak{F}$ induced by $\delta_{\mathfrak{F}} : \mathfrak{F} \times \mathfrak{F} \longrightarrow \mathfrak{D}^*_{\Sigma}$ is the compact-open topology.

proof: Denote $\mathcal{T}(\delta)$ the topology on $\mathfrak{F}$ induced by $\delta_{\mathfrak{F}}$ as indicated in the theorem. From theorem 5 we know that when $\mathfrak{F}$ is given this topology, the evaluation is a continuous function on the product $\mathfrak{F} \times X$. It is well known (see, for example, [3]) that the compact-open topology is the smallest (i.e. coarsest) topology on $\mathfrak{F}$ for which the evaluation function is continuous. Hence, the compact-open topology must be contained in $\mathcal{T}(\delta)$. Suppose, now, that $f \in U \in \mathcal{T}(\delta)$. There must, then, exist some element $A \in \Sigma$ (i.e. some compact subset A of X) and some $\varepsilon \in P$, such that $f \in N_{\alpha}(f) \subseteq U$, where $\alpha = \tau_{\varepsilon A}$. Since $\langle Y, \delta, P\rangle$ is summable, for each $y \in Y$ there exists $\gamma_y \in P$ such that $0_{\delta} < \gamma_y$ and such that $\cup \{ N_{\gamma_y}(z) : z \in N_{\gamma_y}(y) \} \subseteq N_{\varepsilon}(y)$. Since $\langle Y, \delta, P\rangle$ is d-bounded, there exists ζ_y such that $0_{\delta} < \zeta_y$ and such that $\delta(y, z) < \zeta_y$ and $\delta(y, w) < \zeta_y$ imply that $\delta(z, w) < \gamma_y$. Since f is continuous, for each $x \in A$, there exists some open neighborhood V_x of x such that the closure $\overline{V}_x$ is compact and such that the image $f[\overline{V}_x] \subseteq N_{\zeta_{f(x)}}(f(x))$. The collection $\{ V_x : x \in A \}$ is an open cover of the compact subspace A of X and so contains a finite subcover. Let $\mathfrak{A}$ be a finite subset of A such that $\{ V_x : x \in \mathfrak{A} \}$ covers A. For any compact subset B of X and any open subset W of Y we denote by $[B : W]$ the collection $\{ g \in \mathfrak{F} : g[B] \subseteq W \}$. (These are, of course, the subbase elements from which we construct the compact open topology.) It is clear that $f \in \cap \{ [\overline{V}_x : N_{\zeta_{f(x)}}(f(x))] : x \in \mathfrak{A} \}$, an open subset of $\mathfrak{F}$ (as a topological space with the compact open topology.) For any $g \in \cap \{ [\overline{V}_x : N_{\zeta_{f(x)}}(f(x))] : x \in \mathfrak{A} \}$ and any $z \in A$ we know that there is some $x \in \mathfrak{A}$ such that $z \in V_x$. Since $z \in V_x$ and $g \in [\overline{V}_x : N_{\zeta_{f(x)}}(f(x)]$ we have that $\delta(f(x), g(z)) < \zeta_{f(x)}$. Since $z \in V_x$ we have that $\delta(f(x), f(z)) < \zeta_{f(x)}$. Hence $\delta(f(z), g(z)) < \gamma_{f(x)}$ and so $\delta_{\mathfrak{F}}(f, g) < \alpha$. Thus $\delta_{\mathfrak{F}}$ neighborhoods are open in the compact open topology on $\mathfrak{F}$.

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