Extremal graphs for $\alpha$-index

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Abstract. Let $N(G)$ be the number of vertices of the graph $G$. Let $P_i(B_i)$ be the tree obtained of the path $P_i$ and the trees $B_1, B_2, ..., B_l$ by identifying the root vertex of $B_i$ with the $i$-th vertex of $P_i$. Let $V_n = \{ P_i(B_i) : N(P_i(B_i)) = n; N(B_i) \geq 2; l \geq m \}$. In this paper, we determine the tree that has the largest $\alpha$-index among all the trees in $V_n$.

Keywords: Caterpillar, diameter, distance, index, tree.

MSC2010: 05C50, 05C76, 15A18, 05C12, 05C75.

Grafos extremales para $\alpha$-índice

Resumen. Sea $N(G)$ el número de vértices del grafo $G$. Sean $P_i(B_i)$ los árboles obtenidos del camino $P_i$ y los árboles $B_1, B_2, ..., B_l$, identificando el vértice raíz de $B_i$ con el $i$-ésimo vértice de $P_i$. Sea $V_n = \{ P_i(B_i) : N(P_i(B_i)) = n; N(B_i) \geq 2; l \geq m \}$. En este artículo determinamos el árbol que tiene el $\alpha$-índice más grande entre todos los árboles en $V_n$.

Palabras clave: Oruga, diámetro, distancia, índice, árbol.

1. Introduction

Let $G$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is $d(v)$ or simply $d_v$. We denote by $N(G)$ the number of vertices of the graph $G$. A graph $G$ is bipartite if there exists a partitioning of $V(G)$ into disjoint, nonempty sets $V_1$ and $V_2$ such that the end vertices of each edge in $G$ are in distinct sets $V_1, V_2$. In this case $V_1, V_2$ are referred as a bipartition of $G$. A graph $G$ is a complete bipartite graph if $G$ is bipartite with bipartition $V_1$ and $V_2$, where each vertex in $V_1$ is connected to all the vertices in $V_2$. If $G$ is a complete bipartite graph and $N(V_1) = p$ and $N(V_2) = q$, the graph $G$ is written as $K_{p,q}$. The Laplacian matrix of $G$ is the $n \times n$ matrix $L(G) = D(G) - A(G)$, where $A(G)$ and $D(G)$ are the matrices adjacency and diagonal of vertex degrees of $G$ (7), (8), and (11), respectively. It is well known that $L(G)$ is a positive semi-definite matrix and that $(0, e)$ is an eigenpair of $L(G)$ where $e$ is the...
all ones vector. The matrix $Q(G) = A(G) + D(G)$ is called the signless Laplacian matrix of $G$ (see [4], [5], and [6]). The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of $G$, respectively. The matrices $Q(G)$ and $L(G)$ are positive semi-definite, (see [20]). The spectra of $L(G)$ and $Q(G)$ coincide if and only if $G$ is a bipartite graph, (see [2], [4], [7], and [8]). The largest eigenvalue $\mu_1$ of $L(G)$ is the Laplacian index of $G$, the largest eigenvalue $q_1(G)$ of $Q(G)$ is known as the signless Laplacian index of $G$ and the largest eigenvalue $\lambda_1(G)$ of $A(G)$ is the adjacency index or index of $G$ [3].

In [12], it was proposed to study the family of matrices $A_\alpha(G)$ defined for any real number $\alpha \in [0,1]$ as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

Since $A_0(G) = A(G)$ and $2A_{1/2}(G) = Q(G)$, the matrices $A_\alpha(G)$ can underpin a unified theory of $A(G)$ and $Q(G)$. In this paper, the eigenvalues of the matrices $A_\alpha(G)$ are called the $\alpha$-eigenvalues of $G$. We write $\rho_\alpha(G)$ for the spectral radii of the matrices $A_\alpha(G)$ and are called the $\alpha$-indices of $G$. The $\alpha$-eigenvalue set of $G$ is called $\alpha$-spectrum of $G$. The spectrum of a matrix $M$ will be denoted by $\text{Sp}(M)$.

Let $[l]$ denote the set $\{1,2,...,l\}$. Given a rooted graph, define the level of a vertex to be equal to its distance to the root vertex increased by one. A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree. Throughout this paper, $\{B_i : i \in [l]\}$ is a set of generalized Bethe trees. Let $T_l$ be a path of $l$ vertices.

In this paper, we study the tree $P_l\{B_i : i \in [l]\}$ obtained from $P_l$ and $B_1, B_2, ... , B_l$, by identifying the root vertex of $B_i$ with the $i$-th vertex of $P_l$ where each $B_i$ has order greater than or equal to 2. For brevity, we write $P_l(B_i)$ instead of $P_l\{B_i : i \in [l]\}$. Let

$$V^m_n = \{P_l(B_i) : N(P_l(B_i)) = n; N(B_i) \geq 2; l \geq m\}.$$

![Figure 1](http://example.com/figure1.png)

**Figure 1.** The complete caterpillar $P_n(K_{1,2,1,1,1})$.

In a graph, a vertex of degree at least 2 is called an internal vertex, a vertex of degree 1 is a pendant vertex and any vertex adjacent to a pendant vertex is a quasi-pendant vertex. We recall that a caterpillar is a tree in which the removal of all pendant vertices and incident edges results in a path. We define a complete caterpillar as a caterpillar in which each internal vertex is a quasi-pendant vertex.

A complete caterpillar $P_l(K_{1,p_1})$ is a graph obtained from the path $P_l$ and the stars $K_{1,p_1}, ... , K_{1,p_l}$ by identifying the root of $K_{1,p_i}$ with the $i$-th vertex of $P_l$ where $p_i \geq 1$ for all $i \in [l]$ (see Fig. 1 for an example). Let $q \in [l]$. Let $A_q$ be the complete caterpillar $P_l(K_{1,p_q})$, where $p_q = n - 2l + 1$ and $p_i = 1$ for all $i \neq q$.

Let $T_{n,d}$ be the class of all trees on $n$ vertices and diameter $d$. Let $P_n$ be a path on $m$ vertices and $K_{1,p}$ be a star on $p + 1$ vertices.

In [19] the authors prove that the tree in $T_{n,d}$ having the largest index is the caterpillar $P_{d,n-d}$ obtained from $P_{d+1}$ on the vertices 1, 2, ..., $d + 1$ and the star $K_{1,n-d-1}$ identifying the root of $K_{1,n-d-1}$ with the vertex $\frac{d+1}{2}$ of $P_{d+1}$. In [10], for $3 \leq d \leq n - 4$, the first
\( \left\lceil \frac{n}{d} \right\rceil + 1 \) indices of trees in \( T_{n,d} \) are determined. In [9], for \( 3 \leq d \leq n - 3 \), the first Laplacian spectral radii of trees in \( T_{n,d} \) are characterized. In [14] the authors present some extremal results about the spectral radius \( \rho_{\alpha}(G) \) of \( A_{\alpha}(G) \) that generalize previous results about \( \rho_{\alpha}(G) \) and \( \rho_{1/2}(G) \). In [23], the authors give three edge graft transformations on \( A_{\alpha} \)-spectral radius. As applications, we determine the unique graph with maximum \( A_{\alpha} \)-spectral radius among all connected graphs with diameter \( d \), and determine the unique graph with minimum \( A_{\alpha} \)-spectral radius among all connected graphs with given clique number. In [13] the authors give several results about the \( A_{\alpha} \)-matrices of trees. In particular, it is shown that if \( T_{\Delta} \) is a tree of maximal degree \( \Delta \), then the spectral radius of \( A_{\alpha}(T_{\Delta}) \) satisfies the tight inequality

\[
\rho(A(T_{\Delta})) < \alpha\Delta + 2(1 - \alpha)\sqrt{\Delta - 1}.
\]

The complete caterpillars were initially studied in [17] and [18]. In particular, in [17] the authors determine the unique complete caterpillars that minimize and maximize the algebraic connectivity (second smallest Laplacian eigenvalue) among all complete caterpillars on \( n \) vertices and diameter \( m + 1 \). Below we summarize the result corresponding to the caterpillar attaining the largest algebraic connectivity.

**Theorem 1.1** ([17] Theorems 3.3 and 3.6.). Among all caterpillars on \( n \) vertices and diameter \( m + 1 \), the largest algebraic connectivity is attained by the caterpillar \( A_{\left\lceil \frac{n}{d} \right\rceil} \).

**Theorem 1.2** (Abreu, Lenes, Rojo [1]). Let \( \alpha = 0, 1/2 \). Let \( G \) be a complete caterpillars on \( n \) vertices and diameter \( m + 1 \). Then,

\[
\rho_{\alpha}(G) \leq \rho_{\alpha}(A_{\left\lceil \frac{n}{d} \right\rceil}),
\]

with equality if, and only if, \( G \cong A_{\left\lceil \frac{n}{d} \right\rceil} \).

Numerical experiments suggest us that \( A_{\left\lceil \frac{n}{d} \right\rceil} \) is also the tree attaining the largest \( \alpha \)-index in the class \( V^m_n \). In this paper we prove that this conjecture is true; we come up with a bound for the whole family \( A_{\alpha}(G) \), which implies the result of Abreu, Lenes, and Rojo. This is organized as follows. In Section 2, we introduce trees obtained of the path \( P_1 \) and the trees \( B_1, B_2, ..., B_l \) by identifying the root vertex of \( B_i \) with the \( i \)-th vertex of \( P_l \) and give a reduction procedure for calculating their \( \alpha \)-spectra, thereby extending the main results of [15]. In the Section 3, we determine the graph that maximize the \( \alpha \)-index in \( V^m_n \). We finish the section maximizing the \( \alpha \)-index among all the unicyclic connected graphs on \( n \) vertices.

### 2. The \( \alpha \)-Eigenvalues of \( P_1(B_l) \)

Given a generalized Bethe tree \( B_l \) with \( k_i \) levels and an integer \( j \in [k_i] \), we write \( n_{i,k_i-j+1} \) for the number of vertices at level \( j \) and \( d_{i,k_i-j+1} \) for their degree. In particular, \( d_{i,1} = 1 \) and \( n_{i,k_i} = 1 \). Further, for any \( j \in [k_i - 1] \), let \( m_{i,j} = n_{i,j}/n_{i,j+1} \). Then, for any \( j \in [k_i - 2] \), we see that

\[
n_{i,j} = (d_{i,j+1} - 1)n_{i,j+1},
\]

and, in particular,

\[
n_{i,k_i} = d_{i,k_i} = m_{i,k_i-1}.
\]
For $i \in [l]$, it is worth pointing out that $m_{i,1}, \ldots, m_{i,k_i-1}$ are always positive integers, and that $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,k_i}$. We label the vertices of $P_l(B_i)$ as in [15]. (See figure 2).

Recall that the Kronecker product $C \otimes E$ of two matrices $C = (c_{i,j})$ and $E = (e_{i,j})$ of sizes $m \times m$ and $n \times n$, is an $mn \times mn$ matrix defined as $C \otimes E = (c_{i,j}E)$.

Two basic properties of $C \otimes E$ are the identities

$$(C \otimes E)^T = C^T \otimes E^T$$

and

$$(C \otimes E)(F \otimes H) = (CF \otimes EH),$$

which hold for any matrices of appropriate sizes.

We write $I_l$ for the identity matrix of order $l$ and $j_l$ for the column $l$-vector of ones. For $i \in [l]$, let $s_i = \sum_{j=1}^{k_i-2} n_{i,j}$ and $D_i$ be the matrix of order $s_i \times l$ defined by

$$D_i(p,q) = \begin{cases} 1, & \text{if } q = i \text{ and } s_i + 1 \leq p \leq s_i + n_{i,k_i-1}, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $\beta = 1 - \alpha$, and assume that $P_l(B_i)$ is a tree labeled as described above. It is not hard to see that the matrix $A_\alpha(P_l(B_i))$ can be represented as a symmetric block tridiagonal matrix

$$
\begin{bmatrix}
X_1 & 0 & \cdots & 0 & \beta D_1 \\
0 & X_2 & \cdots & \beta D_2 \\
& \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & 0 & \vdots & \vdots \\
0 & 0 & X_l & \beta D_l \\
\beta D_1^T & \beta D_2^T & \cdots & \beta D_l^T & X_{l+1}
\end{bmatrix},
$$

where, for $i \in [l]$, the matrix $X_i$ is the block tridiagonal matrix:

$$
\begin{bmatrix}
\gamma_{i,1}I_{n_{i,1}} & \beta I_{m_{i,2}} \otimes j_{n_{i,1}}^T \\
\beta I_{m_{i,2}} \otimes j_{m_{i,1}}^T & \gamma_{i,2}I_{n_{i,2}} & \beta I_{m_{i,3}} \otimes j_{n_{i,2}}^T \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \beta I_{m_{i,k_i-1}} \otimes j_{m_{i,k_i-2}}^T \\
& & & \ddots & \beta I_{m_{i,k_i-1}} \otimes j_{m_{i,k_i-2}}^T \\
& & & & \gamma_{i,k_i-1}I_{n_{i,k_i-1}} \\
\end{bmatrix},
$$

REVISTA INTEGRACIÓN, TEMAS DE MATEMÁTICAS
and

\[
X_{i+1} = \begin{bmatrix}
\gamma_{1,k_1} + \alpha & \beta & \beta & \cdots & \beta \\
\beta & \gamma_{2,k_2} + 2\alpha & \beta & \cdots & \beta \\
\beta & \beta & \gamma_{3,k_3} + 2\alpha & \beta & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\beta & \beta & \beta & \cdots & \gamma_{l,k_l} + \alpha
\end{bmatrix},
\]

where

\[\gamma_{i,j} = \alpha d_{i,j}.\]

Let’s define the polynomials \(P_0(\lambda), P_1(\lambda), \ldots, P_l(\lambda)\) and \(P_{i,j}(\lambda)\) for \(i \in [l]\) and \(j \in [k_i]\) as follows:

**Definition 2.1.** For \(i \in [l]\) and \(j \in [k_i]\), let

\[\gamma_{i,j} = \alpha d_{i,j}.\]

For \(i \in [l]\), let

\[P_{i,0}(\lambda) = 1, P_{i,1}(\lambda) = \lambda - \alpha,
\]

and for \(i \in [l]\) and \(j = 2, 3, \ldots, k_i - 1\), let

\[P_{i,j}(\lambda) = (\lambda - \gamma_{i,j})P_{i,j-1}(\lambda) - \beta^2 m_{i,j-1}P_{i,j-2}(\lambda).\] (1)

Moreover, let

\[P_1(\lambda) = (\lambda - \gamma_{1,k_1} - \alpha)P_{1,k_1-1}(\lambda) - \beta^2 n_{1,k_1-1}P_{1,k_1-2}(\lambda),
\]

\[P_l(\lambda) = (\lambda - \gamma_{l,k_l} - \alpha)P_{l,k_l-1}(\lambda) - \beta^2 n_{l,k_l-1}P_{l,k_l-2}(\lambda),\]

and

\[P_i(\lambda) = (\lambda - \gamma_{i,k_i} - 2\alpha)P_{i,k_i-1}(\lambda) - \beta^2 n_{i,k_i-1}P_{i,k_i-2}(\lambda),\] (2)

for \(i = 2, 3, \ldots, l - 1\).

**Theorem 2.2.** The characteristic polynomial \(\phi(\lambda)\) of \(A_\alpha(P_l(B_i))\) satisfies

\[\phi(\lambda) = P(\lambda) \prod_{i=1}^{m} \prod_{j=1}^{k_i-1} P_{n_{i,j}-n_{i,j+1}}(\lambda),\] (3)

where

\[
P(\lambda) = \begin{vmatrix}
P_1(\lambda) & -\beta P_{1,k_1-1}(\lambda) \\
-\beta P_{2,k_2-1}(\lambda) & \cdots & \beta P_{l-1,k_{l-1}-1}(\lambda) \\
\cdots & \cdots & \cdots \\
-\beta P_{l,k_l-1}(\lambda) & \cdots & \beta P_{1,k_1-1}(\lambda)
\end{vmatrix}.
\]

*Proof.* Write \(|A|\) for the determinant of a square matrix \(A\). To prove 3, we shall reduce \(\phi(\lambda) = |\lambda I - A_\alpha(P_l(B_i))|\) to the determinant of an upper triangular matrix. For a start,
note that
\[
\phi(\lambda) = \begin{bmatrix}
X_1(\lambda) & 0 & \cdots & 0 & -\beta D_1 \\
0 & X_2(\lambda) & \cdots & -\beta D_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & X_l(\lambda) & -\beta D_l \\
-\beta D_1^T & -\beta D_2^T & \cdots & -\beta D_l^T & X_{l+1}(\lambda)
\end{bmatrix},
\]
where, for \(i \in [l]\), the matrix \(X_i(\lambda)\) given by,
\[
\begin{bmatrix}
P_{i,1}(\lambda)I_{n_{i,1}} & -\beta I_{n_{i,2}} \otimes \mathbf{J}_{m_{i,1}} \\
-\beta I_{n_{i,2}} \otimes \mathbf{J}_{m_{i,1}}^T (\lambda - \gamma_{i,2}) I_{n_{i,2}} & -\beta I_{n_{i,3}} \otimes \mathbf{J}_{m_{i,2}} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-\beta I_{n_{i,k_i-1}} \otimes \mathbf{J}_{m_{i,k_i-2}}^T (\lambda - \gamma_{i,k_i-1}) I_{n_{i,k_i-1}} & -\beta I_{n_{i,k_i-1}} \otimes \mathbf{J}_{m_{i,k_i-2}} \\
\end{bmatrix},
\]
and
\[
X_{l+1}(\lambda) = \begin{bmatrix}
\lambda - \gamma_{i,k_i} - \alpha & -\beta \\
-\beta & \lambda - \gamma_{i,k_i} - 2\alpha & -\beta \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\lambda - \gamma_{i-1,k_i-1} - 2\alpha & -\beta & \lambda - \gamma_{i,k_i} - \alpha
\end{bmatrix}.
\]
Let \(\lambda \in \mathbb{R}\) be such that \(P_{i,j}(\lambda) \neq 0\) for any \(i \in [l]\) and \(j \in [k_i - 1]\); set \(P_{i,j} = P_{i,j}(\lambda)\). For each \(i \in [l]\) and for all \(j \in [k_i - 2]\), multiplying the \(j\)-th row of \(X_i(\lambda)\) inserted in \(\phi(\lambda)\) by \(\frac{\beta P_{i,j}}{P_{i,j}} \otimes \mathbf{J}_{m_{i,j}}\) and add it to the next row. Since
\[
\lambda - \gamma_{i,j+1} - \frac{\beta^2 m_{i,j} P_{i,j-1}}{P_{i,j}} = \frac{(\lambda - \gamma_{i,j+1}) P_{i,j} - \beta^2 m_{i,j} P_{i,j-1}}{P_{i,j}} = \frac{P_{i,j+1}}{P_{i,j}},
\]
we obtain,
\[
\phi(\lambda) = \begin{bmatrix}
Y_1(\lambda) & 0 & \cdots & 0 & -\beta D_1 \\
0 & Y_2(\lambda) & \cdots & -\beta D_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & Y_l(\lambda) & -\beta D_l \\
0 & 0 & \cdots & 0 & Y_{l+1}(\lambda)
\end{bmatrix},
\]
where, for \(i \in [l]\), the matrix \(Y_i(\lambda)\) is given by
\[
\begin{bmatrix}
P_{i,1}I_{n_{i,1}} & -\beta I_{n_{i,2}} \otimes \mathbf{J}_{m_{i,1}} \\
\frac{P_{i,2}}{P_{i,1}} I_{n_{i,2}} & -\beta I_{n_{i,3}} \otimes \mathbf{J}_{m_{i,2}} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-\beta I_{n_{i,k_i-1}} \otimes \mathbf{J}_{m_{i,k_i-2}} & \frac{P_{i,k_i-1}}{P_{i,k_i-2}} I_{n_{i,k_i-1}}
\end{bmatrix}
\]
[Revista Integración, temas de matemáticas]
and
\[
Y_{i+1}(\lambda) = \begin{bmatrix}
\frac{P_1}{P_{i,k_i-1}} & -\beta & \frac{P_2}{P_{i,k_i-1}} & -\beta \\
-\beta & \frac{P_2}{P_{i,k_i-1}} & -\beta & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\frac{P_{i-1}}{P_{i-1,k_i-1}} & -\beta & \frac{P_{i-1}}{P_{i-1,k_i-1}} & -\beta & \frac{P_i}{P_{i,k_i-1}}
\end{bmatrix}.
\]

Thereby,
\[
\phi(\lambda) = \prod_{i=1}^{l+1} |Y_i(\lambda)|
\]
\[
= |Y_{i+1}(\lambda)| \prod_{i=1}^l \left( \frac{P_{i,2}}{P_{i,1}} \right)^{n_{i,2}} \cdots \left( \frac{P_{i,k_i-1}}{P_{i,k_i-2}} \right)^{n_{i,k_i-1}}  \prod_{i=1}^l P_{i,k_i-1}^{n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}},
\]
where
\[
|Y_{i+1}(\lambda)| = \frac{1}{\prod_{i=1}^l P_{i,k_i-1}} \begin{vmatrix}
P_1 & -\beta P_{i,k_i-1} \\
-\beta P_{i,k_i-1} & P_2 & -\beta P_{i,k_i-1} \\
& \ddots & \ddots & \ddots \\
P_{i-1,k_i-2} & -\beta P_{i-1,k_i-1} & P_{i-1} & -\beta P_{i-1,k_i-1} & P_i
\end{vmatrix}.
\]

Hence
\[
|\lambda - A_\alpha(P(B_i))| = P(\lambda) \prod_{i=1}^l \prod_{j=1}^{n_{i,k_i-1}} P_{i,j}^{n_{i,j,k_i-1}(\lambda)}.
\]

Thus, the equality (3) is proved whenever \( P_{i,j}(\lambda) \neq 0 \) for any \( i \in [l] \) and \( j \in [k_i - 1] \).
Since for any \( i \in [l] \) and \( j \in [k_i - 1] \) the polynomials \( P_i(\lambda) \) have finitely many roots,
the equality (3) is verified for infinitely many value of \( \lambda \). The proof is complete.

**Definition 2.3.** For \( i \in [l] \) and \( j \in [k_i - 1] \), let \( T_{i,j} \) be the \( j \times j \) leading principal submatrix of the \( k_i \times k_i \) symmetric tridiagonal matrix
\[
T_i = \begin{bmatrix}
\frac{\alpha d_{i,1}}{\beta d_{i,2} - 1} & \beta \sqrt{d_{i,2} - 1} \\
\beta \sqrt{d_{i,2} - 1} & \frac{\alpha d_{i,2}}{\beta d_{i,2} - 1} \\
& \ddots & \ddots & \ddots \\
& \beta \sqrt{d_{i,k_i - 1} - 1} & \frac{\alpha d_{i,k_i - 1}}{\beta d_{i,k_i - 1} - 1} & \beta \sqrt{d_{i,k_i - 1} - 1} & \frac{\alpha d_{i,k_i - 1}}{\beta d_{i,k_i - 1} - 1} \\
& & \beta \sqrt{d_{i,k_i - 1} - 1} & \frac{\alpha d_{i,k_i - 1}}{\beta d_{i,k_i - 1} - 1} & \beta \sqrt{d_{i,k_i - 1} - 1} & \frac{\alpha d_{i,k_i} + \gamma_{i,k_i}}{\beta d_{i,k_i} + \gamma_{i,k_i}}
\end{bmatrix},
\]
where \( \beta = 1 - \alpha \), \( c = 2 \) for \( i \in [l - 1] \) and \( c = 1 \) for \( i = 1 \) and \( i = l \).
Since $d_s > 1$ for all $s = 2, \ldots, j$, each matrix $T_j$ has nonzero codiagonal entries and it is known that its eigenvalues are simple. Using the well known three-term recursion formula for the characteristic polynomials of the leading principal submatrices of a symmetric tridiagonal matrix and the formulas (1) and (2), one can easily prove the following assertion:

**Lemma 2.4.** Let $\alpha \in [0, 1)$. Then

$$|\lambda I - T_{i,j}| = P_{i,j}(\lambda)$$

and

$$|\lambda I - T_i| = P_i(\lambda),$$

for any $i \in [l]$ and $j \in [k_i - 1]$.

Let $\tilde{A}$ be the matrix obtained from a matrix $A$ by deleting its last row and last column. Moreover, for $i, j \in [r]$, let $E_{i,j}$ be the $k_i \times k_j$ matrix with $E_{i,j}(k_i, k_j) = 1$ and zeroes elsewhere. We recall the following Lemma.

**Lemma 2.5 ([16]).** For $i, j \in [r]$, let $C_i$ be a matrix of order $k_i \times k_i$ and $\mu_{i,j}$ be arbitrary scalars. Then,

$$\begin{vmatrix}
C_1 & \mu_{1,2}E_{1,2} & \cdots & \mu_{1,r-1}E_{1,r-1} & \mu_{1,r}E_{1,r} \\
\mu_{2,1}E_{2,1} & C_2 & \cdots & \cdots & \mu_{2,r-1}E_{2,r-1} & \mu_{2,r}E_{2,r} \\
\mu_{3,1}E_{3,1} & \mu_{3,2}E_{3,2} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & C_{r-1} & \mu_{r-1,r}E_{r-1,r} \\
\mu_{r,1}E_{r,1} & \mu_{r,2}E_{r,2} & \cdots & \cdots & \cdots & C_r
\end{vmatrix} = 
\begin{vmatrix}
|C_1| & \mu_{1,2} & \cdots & \mu_{1,r-1} & \mu_{1,r} \\
\mu_{2,1} & |C_2| & \cdots & \cdots & \mu_{2,r} \\
\mu_{3,1} & \mu_{3,2} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & C_{r-1} \\
\mu_{r,1} & \mu_{r,2} & \cdots & \cdots & \cdots \\
\end{vmatrix}.
$$

From now on, for $i \in [l - 1]$, by $F_i$ we denote the matrix of order $k_i \times k_i+1$ whose entries are 0, except for the entry $F_i(k_i, k_i+1) = 1$.

**Lemma 2.6.** Let $r = \sum_{i=1}^{l} k_i$. Let $M(P_i(B_i))$ be the symmetric matrix of order $n \times n$ defined by

$$\begin{bmatrix}
T_1 & \beta F_1 \\
\beta F_1^T & T_2 & \ddots \\
& \ddots & \ddots & \beta F_{l-1} \\
& & \beta F_{l-1}^T & T_l
\end{bmatrix}.
$$

Then,

$$|\lambda I - M(P_i(B_i))| = P(\lambda).$$

[Revista Integración, temas de matemáticas]
Proof. The characteristic polynomial of the matrix $M(P_i(B_i))$ is given by

$$\begin{vmatrix} \lambda I - T_1 & -\beta F_1 \\ -\beta F_1^T & \lambda I - T_2 & \ddots \\ & \ddots & \ddots & -\beta F_{l-1} \\ & & -\beta F_{l-1}^T & \lambda I - T_l \end{vmatrix}.$$  

From Lemma 2.5, we have that $|\lambda I - M(P_i(B_i))|$ is given by

$$\begin{vmatrix} |\lambda I - T_1| & -\beta |\lambda I - T_1| \\ -\beta |\lambda I - T_2| & |\lambda I - T_2| & \ddots & \ddots \\ & \ddots & \ddots & -\beta |\lambda I - T_{l-1}| \\ & & -\beta |\lambda I - T_{l-1}| & |\lambda I - T_{l-1}| \\ & & & -\beta |\lambda I - T_l| & |\lambda I - T_l| \end{vmatrix}.$$  

Since $\lambda I - T_i = \lambda I - T_{i,k_i - 1}$ for $i \in [l]$, by Lemma 2.4, the proof is complete.  

Theorem 2.2, Lemma 2.4, Lemma 2.6, and the interlacing property for the eigenvalues of hermitian matrices yield the following summary statement:

**Theorem 2.7.** Let $\alpha \in [0,1)$. Then:

1. the $\alpha$-spectrum of $P_i(B_i)$ is

$$\bigcup_{i=1}^{l} \bigcup_{j=1}^{k_i - 1} \text{Sp}(T_{i,j}) \cup \text{Sp}(M(P_i(B_i)));$$

2. the multiplicity of each eigenvalue of $T_{i,j}$ as an $\alpha$-eigenvalue of $P_i(B_i)$ is $n_{i,j} - n_{i,j+1}$, if $i \in [l]$ and $j \in [k_i - 1]$, and is 1 if $i \in [l]$ and $j = k_i$;

3. $\rho_{\alpha}(P_i(B_i))$ is the largest eigenvalue of $M(P_i(B_i))$;

4. $\rho_{\alpha}(P_i(B_i)) > \alpha$.

### 3. The $\alpha$-index of graphs

In Theorem 2.7, we characterize the $\alpha$-eigenvalues of the trees $P_i(B_i)$ obtained from path $P_i$ and the generalized Bethe trees $B_1, B_2, \ldots, B_l$ obtained identifying the root vertex of $B_i$ with the $i$-th vertex of $P_i$. This is the case for the caterpillars $P_i(K_{1,p_i})$ in which the path is $P_i$ and each star $K_{1,p_i}$ is a generalized Bethe tree of 2 levels. From Theorem 2.7, we get
Lemma 3.1. Let $\alpha \in [0, 1)$. Then:

1. the $\alpha$-spectrum of $P_l(K_1, p_i)$ is formed by $\alpha$ with multiplicity $\sum_{i=1}^l p_i - l$, and the eigenvalues of the $2l \times 2l$ irreducible nonnegative matrix

$$M(P_l(K_1, p_i)) = \begin{bmatrix} T(p_1) & \beta E & \beta E & \ldots & \beta E \\ \beta E & S(p_2) & \beta E & \ldots & \beta E \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta E & \beta E & \ldots & \beta E & T(p_l) \end{bmatrix},$$

where

$$T(x) = \begin{bmatrix} \alpha & \beta \sqrt{E} \\ \beta \sqrt{E} & \alpha(x+1) \end{bmatrix}, E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; S(x) = T(x) + \alpha E,$$

2. $\rho_\alpha(P_l(K_1, p_i))$ is the largest eigenvalue of $M(P_l(K_1, p_i))$;
3. $\rho_\alpha(P_l(K_1, p_i)) > \alpha$.

Let $t(\lambda, x)$ and $s(\lambda, x)$ be the characteristic polynomials of the matrices $T(x)$ and $S(x)$, respectively. That is,

$$t(\lambda, x) = \lambda^2 - \alpha(x+2)\lambda + \alpha^2(x+1) - \beta^2 x$$

and

$$s(\lambda, x) = \lambda^2 - \alpha(x+3)\lambda + \alpha^2(x+2) - \beta^2 x.$$

Then,

$$s(\lambda, x) - t(\lambda, x) = \alpha(\alpha - \lambda).$$

The notation $|A|_l$ will be used to denote the determinant of the matrix $A$ of order $l \times l$.

The next result is an immediate consequence of the Lemma 2.5.

Lemma 3.2. The characteristic polynomial of $M(P_l(K_1, p_i))$ is

$$det\begin{bmatrix} t(\lambda, p_1) & \beta(\alpha - \lambda) \\ \beta(\alpha - \lambda) & s(\lambda, p_2) & \beta(\alpha - \lambda) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta(\alpha - \lambda) & t(\lambda, p_l) \end{bmatrix}.$$
Let $\phi_q(\lambda)$ be the characteristic polynomial of $M(A_q)$, then,

$$
\phi_q(\lambda) = |\lambda I - M(A_q)|.
$$

**Lemma 3.3.** Let $\alpha \in [0, 1)$. Then

$$
\phi_q(\lambda) - \phi_{q+1}(\lambda) = (a - 1)(\alpha - 2\alpha + 1)(\beta(\lambda - \alpha))^{q-1}\left[ar_{m-2q}(\lambda) + \beta^2(\lambda - \alpha)r_{l-2q-1}(\lambda)\right]
$$

for all $q \in \left\lceil \frac{l+1}{2} \right\rceil - 1$, where $l \geq 3$.

**Proof.** By Lemma 3.2, the $(\lambda, a)$-entry of $\phi_q(\lambda) = |\lambda I - M(A_q)|$ is $t(\lambda, a)$ if $q = 1$ and $s(\lambda, a)$ if $q \neq 1$. Let $E_i \cong P_i(K_{1, p_i})$, where $p_i = 1$ for all $i \in [l]$. Let $\varphi_s(\lambda) = |\lambda I - M(E_s)|$. From Lemma 3.2, we have

$$
\varphi_s(\lambda) = \begin{vmatrix}
 t(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\

& \ddots & \ddots \\
& & s(\lambda, 1) & \beta(\alpha - \lambda) \\
& & \beta(\alpha - \lambda) & t(\lambda, 1)
\end{vmatrix}_q.
$$

Since $r_0(\lambda) = 1$, $r_1(\lambda) = t(\lambda, 1)$ and

$$
\begin{vmatrix}
 s(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\

& \ddots & \ddots \\
& & s(\lambda, 1) & \beta(\alpha - \lambda) \\
& & \beta(\alpha - \lambda) & t(\lambda, 1)
\end{vmatrix}_q
$$

for $q = 2, \ldots, \left\lceil \frac{l+1}{2} \right\rceil$; then, expanding along the first row, we obtain

$$
r_q(\lambda) = s(\lambda, 1)r_{q-1}(\lambda) - \beta^2(\lambda - \alpha)^2 r_{q-2}(\lambda). \quad (4)
$$

Since $s(\lambda, x) = t(\lambda, x) + \alpha(\alpha - \lambda)$, by linearity on the first column, we have

$$
r_q(\lambda) = \begin{vmatrix}
 t(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\

& \ddots & \ddots \\
& & s(\lambda, 1) & \beta(\alpha - \lambda) \\
& & \beta(\alpha - \lambda) & t(\lambda, 1)
\end{vmatrix}_q + \alpha(\alpha - \lambda)r_{q-1}(\lambda).
$$

Then,

$$
r_q(\lambda) = \varphi_q(\lambda) + \alpha(\alpha - \lambda)r_{q-1}(\lambda).
$$

Let $q \in \left\lceil \frac{l+1}{2} \right\rceil - 1$. We search for the difference $\phi_q(\lambda) - \phi_{q+1}(\lambda)$. We recall that $(q, q)$-entry of $\phi_q(\lambda) = |\lambda I - M(A_q)|$ is $t(\lambda, a)$ if $q = 1$ and $s(\lambda, a)$ if $q \neq 1$. Since
\[ t(\lambda, a) = t(\lambda, 1) + (1 - a)(\alpha \lambda - 2\alpha + 1) \quad \text{and} \quad s(\lambda, a) = s(\lambda, 1) + (1 - a)(\alpha \lambda - 2\alpha + 1), \]

by linearity on the \( q \)-th column, we have

\[
\phi_q(\lambda) = \begin{vmatrix}
  t(\lambda, 1) & \beta(\alpha - \lambda) & \beta(\alpha - \lambda) \\
  \beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
  \vdots & \ddots & \ddots \\
  \vdots & \ddots & \ddots \\
  \beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
  \beta(\alpha - \lambda) & t(\lambda, 1) & \beta(\alpha - \lambda)
\end{vmatrix}
\]

\[ + (1 - a)(\alpha \lambda - 2\alpha + 1) \begin{vmatrix}
  r_{q-1}(\lambda) & 0 \\
  0 & r_l(\lambda)
\end{vmatrix}. \]

The \((q + 1, q + 1)\)-entry of the determinant of order \( l \) on the second right hand of (5) is \( s(\lambda, 1) \), and since \( s(\lambda, 1) = s(\lambda, a) + (a - 1)(\alpha \lambda - 2\alpha + 1) \), by linearity on the \((q + 1)\)-th column, we obtain

\[
\phi_{q+1}(\lambda) + (1 - a)(\alpha \lambda - 2\alpha + 1) \begin{vmatrix}
  r_{q}(\lambda) & 0 \\
  0 & r_l(\lambda)
\end{vmatrix}.
\]

Thereby,

\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) = (1 - a)(\alpha \lambda - 2\alpha + 1) \begin{vmatrix}
  r_{q-1}(\lambda) & 0 \\
  0 & r_l-1(\lambda)
\end{vmatrix} + (a - 1)(\alpha \lambda - 2\alpha + 1) \begin{vmatrix}
  r_{q}(\lambda) & 0 \\
  0 & r_l-1(\lambda)
\end{vmatrix}.
\]

Thus,

\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) = (a - 1)(\alpha \lambda - 2\alpha + 1)[r_q(\lambda)r_{m-q} - r_{q-1}(\lambda)r_{m-q-1}].
\]

Applying the recurrence formula (4) to \( r_q(\lambda) \) and \( r_{l-q}(\lambda) \), we obtain

\[
r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_l(\lambda) = [s(\lambda, 1)r_{q-1}(\lambda) - \beta^2(\lambda - \alpha)^2r_{q-2}(\lambda)]r_{l-q-1}(\lambda)
\]

\[- r_q(\lambda){s(\lambda, 1)r_{l-q-1}(\lambda) - \beta^2(\lambda - \alpha)^2r_{l-q-2}(\lambda)}].
\]

Then,

\[
r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_l(\lambda) = \beta^2(\lambda - \alpha)^2[r_q(\lambda)r_{l-q-2}(\lambda) - r_{q-2}(\lambda)r_{l-q-1}(\lambda)].
\]

By repeated applications of this process, we conclude that

\[
r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_l(\lambda) = [\beta(\lambda - \alpha)]^{2(q-1)}[r_1(\lambda)r_{l-2q} - r_{l-2q+1}].
\]
Hence,
\[
\begin{align*}
  r_q(\lambda) & r_{l-2q-1}(\lambda) - r_{q-1}(\lambda) r_{l-q}(\lambda) \\
  &= [\beta (\lambda - \alpha)]^{2q-1} [t(\lambda, 1) r_{l-2q}(\lambda) - s(\lambda, 1) r_{l-2q}(\lambda) + \beta^2 (\lambda - \alpha)^2 r_{l-2q-1}(\lambda)] \\
  &= [\beta (\lambda - \alpha)]^{2q-1} [\alpha (\lambda - \alpha) r_{l-2q}(\lambda) + \beta^2 (\lambda - \alpha)^2 r_{l-2q-1}(\lambda)] \\
  &= [\beta (\lambda - \alpha)]^{2q-1} [\alpha r_{l-2q}(\lambda) + \beta^2 (\lambda - \alpha) r_{l-2q-1}(\lambda)].
\end{align*}
\]

Thus,
\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) = (a - 1)(\alpha \lambda - 2\alpha + 1)[\beta (\lambda - \alpha)]^{2q-1} [\alpha r_{l-2q}(\lambda) + \beta^2 (\lambda - \alpha) r_{l-2q-1}(\lambda)].
\]

Let \( \rho(A) \) be the spectral radius of the square matrix \( A \). From Perron-Frobenius’s Theory for nonnegative matrices [22], if \( A \) is a nonnegative irreducible matrix then \( A \) has a unique eigenvalue equal to its spectral radius and it increases whenever any entry of it increases. Hence, we have the next result.

**Lemma 3.4 ([21]).** If \( A \) is a nonnegative irreducible matrix and \( B \) is any principal submatrix of \( A \), then \( \rho(B) < \rho(A) \).

Let \( C_{n,l} \) be the class of all complete caterpillars on \( n \) vertices and diameter \( l + 1 \). A special subclass of \( C_{n,l} \) is \( A_{n,l} = \{A_1, A_2, \ldots, A_l\} \), where \( A_q \cong P(K_{1, p_q}) \in C_{n,l} \), with \( p_i = 1 \) for \( i \neq q \) and \( p_q = n - 2l + 1 \). Since \( A_q \) and \( A_{q+1} \) are isomorphic caterpillars for all \( q \in \left[\frac{n}{2l} \right] \), the next theorem gives a total ordering in \( A_{n,l} \) by the \( \alpha \)-index.

**Theorem 3.5.** Let \( \alpha \in [0, 1) \). Then
\[
\rho_\alpha(A_q) < \rho_\alpha(A_{q+1})
\]
for all \( q \in \left[\frac{n}{2l} \right] - 1 \), where \( l \geq 3 \).

**Proof.** Let \( l \geq 3 \). Let \( q \in \left[\frac{n}{2l} \right] - 1 \). Let \( \phi_q(\lambda) \) and \( \phi_{q+1}(\lambda) \) be the characteristic polynomials of degrees \( 2l \) of the matrices \( M(A_q) \) and \( M(A_{q+1}) \), respectively. The matrices \( M(A_q) \) and \( M(A_{q+1}) \) are nonnegative irreducible matrices, then its spectral radii are simple eigenvalues.

Let
\[
\rho_\alpha(A_q) = \mu_1 > \mu_2 \geq \cdots \geq \mu_{2l}
\]
and
\[
\rho_\alpha(A_{q+1}) = \gamma_1 > \gamma_2 \geq \cdots \geq \gamma_{2l}
\]
be the eigenvalues of the matrices \( M(A_q) \) and \( M(A_{q+1}) \), respectively. By Lemma 3.3, we have
\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) = \prod_{j=1}^{2l} (\lambda - \mu_j) - \prod_{j=1}^{2l} (\lambda - \gamma_j)
\]
\[
= (a - 1)(\alpha \lambda - 2\alpha + 1)[\beta (\lambda - \alpha)]^{2q-1}
\]
\[
\times [\alpha r_{l-2q}(\lambda) + \beta^2 (\lambda - \alpha) r_{l-2q-1}(\lambda)].
\]

Vol. 38, No 1, 2020]
We recall that \( r_{t-2q}(\lambda) \) and \( r_{t-2q-1}(\lambda) \) are the characteristic polynomials of the matrices \( M(\tilde{E}_{t-2q+1}) \) and \( M(\tilde{E}_{t-2q}) \) whose spectral radii are \( \rho(M(\tilde{E}_{t-2q+1})) \) and \( \rho(M(\tilde{E}_{t-2q})) \), respectively. The matrices \( M(\tilde{E}_{t-2q+1}) \) and \( M(\tilde{E}_{t-2q}) \) are principal submatrices of \( M(A_q) \).

By Lemma 3.4, \( \rho(M(\tilde{E}_{t-2q+1})) < \rho_\alpha(A_q) \) and \( \rho(M(\tilde{E}_{t-2q})) < \rho_\alpha(A_q) \). Hence, \( r_{t-2q}(\rho_\alpha(A_q)) > 0 \) and \( r_{t-2q-1}(\rho_\alpha(A_q)) > 0 \). We claim that \( \rho_\alpha(A_q) < \rho_\alpha(A_{q+1}) \).

Suppose \( \rho_\alpha(A_q) \geq \rho_\alpha(A_{q+1}) \). Then \( \rho_\alpha(A_q) \geq \gamma_j \) for all \( j \). Taking \( \lambda = \rho_\alpha(A_q) \) in (6), we obtain

\[
-\varphi_{q+1}(\rho_\alpha(A_q)) = -\prod_{j=1}^{2l}(\rho_\alpha(A_q) - \gamma_j) \\
= (a - 1)(\alpha \rho_\alpha(A_q) - 2\alpha + 1)(\beta(\rho_\alpha(A_q) - \alpha))^{2q-1} \\
\ast [\alpha r_{t-2q}(\rho_\alpha(A_q)) + \beta^2(\rho_\alpha(A_q) - \alpha)r_{t-2q-1}(\rho_\alpha(A_q))].
\]

By Lemma 3.1, \( \rho_\alpha(A_q) > \alpha \). Then \( \alpha \rho_\alpha(A_q) - 2\alpha + 1 > 0 \). Thus,

\[
0 \geq -\prod_{j=1}^{2l}(\rho_\alpha(A_q) - \gamma_j) \\
= (a - 1)(\alpha \rho_\alpha(A_q) - 2\alpha + 1)(\beta(\rho_\alpha(A_q) - \alpha))^{2q-1} \\
\ast [\alpha r_{t-2q}(\rho_\alpha(A_q)) + \beta^2(\rho_\alpha(A_q) - \alpha)r_{t-2q-1}(\rho_\alpha(A_q))]] \\
> 0.
\]

which is a contradiction. The proof is complete.

\[\Box\]

Lemma 3.6 ([7]). Let \( A \) be a nonnegative symmetric matrix and \( x \) be a unit vector of \( \mathbb{R}^n \). If \( \rho(A) = x^T Ax \), then \( Ax = \rho(A)x \).

![Figure 3. Graphs G and Gu with s = 3.](image-url)
Let $N_G(v)$ be the vertex set adjacent to $v$ in $G$.

**Lemma 3.7 ([23]).** Let $\alpha \in [0, 1)$. Let $G$ be a connected graph and $\rho_\alpha(G)$ be the $\alpha$-index of $G$. Let $u, v$ be two vertices of $G$. Suppose $v_1, v_2, ..., v_s$, are some vertices of $N_G(v) - (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, ..., x_n)$ is the Perron’s vector of $A_\alpha(G)$, where $x_i$ corresponds to the vertex $v_i$ for $i \in [s]$. Let

$$G_u \cong G - vv_1 - \cdots - vv_s + uv_1 + \cdots + uv_s$$

(as shown in Fig. 3). If $x_u \geq x_v$, then $\rho_\alpha(G) < \rho_\alpha(G_u)$.

An immediate consequence of Lemma 3.7 is

**Theorem 3.8.** Let $T \in V_n^m$. Then

$$\rho_\alpha(T) \leq \rho_\alpha(A_{\left\lceil \frac{m+1}{2} \right\rceil}),$$

where $A_{\left\lceil \frac{m+1}{2} \right\rceil} \in A_{n,m}$. For $\alpha \in [0, 1)$, the bound (7) is attained if and only if, $T \cong A_{\left\lceil \frac{m+1}{2} \right\rceil}$. For $\alpha = 1$, the bound (7) is attained if and only if, $T \cong A_k$, where $k = 2, ..., \left\lfloor \frac{m+1}{2} \right\rfloor$ and $m \geq 3$ or $T \cong A_{\left\lceil \frac{m+1}{2} \right\rceil}$, where $m = 2$.

**Proof.** Let $\alpha \in [0, 1)$. Let $T \cong P_i(B_i) \in V_n^m$. Let $x_1, x_2, ..., x_l$ be the vertices of the path $P_i$ in the tree $T$. Let $B_i$ be a tree with $k_i$ levels for all $i \in [l]$. Suppose $T$ has the largest $\alpha$-index in $V_n^m$.

Suppose $k_i > 2$ for some $2 \leq i \leq l - 1$. Let $u_1, ..., u_{s_i}$ be all the vertices in the second level of $B_i$; we can assume without loss of generality that $u_{s_i}$ is an internal vertex. Let $w_1, ..., w_{r_i}$ be all the vertices of $N_G(u_{s_i}) - \{x_i\}$. Let

$$T_{x_i} \cong T - u_{s_i}w_1 - \cdots - u_{s_i}w_{r_i} + x_iw_1 + \cdots + x_iw_{r_i},$$

and

$$T_{u_{s_i}} \cong T - x_{i-1}x_i - x_{i+1}x_i - u_1x_i - \cdots - u_{s_i-1}x_i + x_{i-1}u_{s_i} + x_{i+1}u_{s_i} + u_1u_{s_i} + \cdots + u_{s_i-1}u_{s_i}.$$  

By Lemma 3.7, $\rho_\alpha(T_{x_i}) > \rho_\alpha(T)$ or $\rho_\alpha(T_{u_{s_i}}) > \rho_\alpha(T)$. Moreover, $\rho_\alpha(T_{x_i}) \in V_n^m$ and $\rho_\alpha(T_{u_{s_i}}) \in V_n^m$, which is a contradiction. If $i = 1$ or $i = l$, we reason analogously. Then, $k_i = 2$ for all $i \in [l]$. This is,

$$T \cong P_i(K_{1,p_i}).$$

By reasoning analogously we can verify that

$$T \in A_{n,m}.$$  

Let $m \geq 3$. By Theorem 3.5,

$$\rho_\alpha(A_1) < \rho_\alpha(A_2) < \cdots < \rho_\alpha(A_{\left\lceil \frac{m+1}{2} \right\rceil}).$$

Then the largest $\alpha$-index is attained by $A_{\left\lceil \frac{m+1}{2} \right\rceil}$. For $m = 2$ the result is immediate.

Let $\alpha = 1$; then $A_\alpha = D$, where $D$ is the diagonal matrix of vertex degrees. Let $T \in V_n^m$.

Let $m = 3$; then the maximum degree of $T$ is less than or equal to $n - 2l + 3$. Then, $\rho_\alpha(T) \leq n - 2l + 3 \leq \rho_\alpha(A_k)$ for all $k = 2, ..., \left\lceil \frac{m+1}{2} \right\rceil$. For $m = 2$ is result is immediate.
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Extremal graphs for $\alpha$-index


