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# Topological Spaces as Pseudo-Distance Spaces*

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ABSTRACT

In this paper we provide a generalized definition of distance and show that, with this definition, any topological space can be generated by (distance derived) neighborhoods in exactly the same manner as metric topologies.

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It is quite common to motivate the study of general topology as a generalization of the study of metric spaces, and, thus, to consider the concept of a neighborhood as being related to, or motivated by, the idea of "closeness". This relationship, however, between "closeness" and the open sets of a general topological space is not a particularly obvious one. The commonly used statement "two points are *close* to each other if *most* of the neighborhoods of one are also neighborhoods of the other" is patent nonsense. Even in cases where closeness can be defined with precision (metrizable spaces) the collection of neighborhoods of one point which contain another and the collection of neighborhoods which do not contain the other will typically have the same cardinality.

In this paper we provide a generalized definition of *distance* and show that, with this definition, any topological space can be generated by (distance derived) neighborhoods in exactly the same manner as metric topologies. In this setting, $R_0$ spaces (see [1]) and regular spaces have natural characterizations, as does the compact-open topology. In the setting of $T_1$ spaces, there is a natural analog to uniform continuity, a condition which, if the distance function is defined correctly, can be satisfied by any continuous function. (This paper extends work originated by Hajek, Perlis and Wilson, see [2].)

Recall (for example from [4]) that a metric space is a set M together with a distance function m, from the product $M \times M$ to the (nonnegative) real numbers R, having the properties:

$M_1$. $m(a,b)=0$ if and only if $a=b$.
$M_2$. $m(a,b)=m(b,a)$ for all $a,b \in M$.
$M_3$. $m(a,b)+m(b,c) \geq m(a,c)$ for all $a,b,c \in M$

We can identify any metric space $\langle M, m\rangle$ in a natural manner with a topological space $\langle M, \mathcal{T}_m\rangle$ by defining the c neighborhood $N_c(x)$ of a point x to be the set $\{y \in M : m(x,y) < c\}$. The collection $\{ N_c(x) : x \in M, c > 0 \}$ forms a base for a topology $\mathcal{T}_m$ on M. This topology is called, quite naturally, the metric



topology on M. This identification of the metric space $\langle M, m\rangle$ with the topological space $\langle M, \mathcal{T}_m\rangle$ induces a functor from the category of metric spaces to the category of (metrizable) topological spaces.

A natural way to generalize the idea of metric spaces would seem to be by replacing the real number system R with some less restrictive structure. If we are to retain a concept of neighborhood similar to that obtained from a metric, some kind of order relation is necessary. Thus, we must consider, at an absolute minimum, a partially ordered set as a replacement for R.

Definition 1: By a distance space we will mean a set Y together with a function $\delta$ from $Y \times Y$ to a partially ordered set P such that:

$D_1$. for any $x, y \in Y$, if $\delta(x,y) < p \in P$, then $\delta(x,x) < p$ and $\delta(y,y) < p$.
$D_2$. $\delta(x,y) = \delta(y,x)$ for all $x, y \in Y$
$D_3$. if $\delta(x,y) < \sigma$, then there exists some $\mu \in P$ such that $\delta(y,y) < \mu$ and such that $\delta(y,z) < \mu$ implies $\delta(x,z) < \sigma$.
$D_4$. If $\delta(x,y) < \mu$ and $\delta(x,y) < \nu$, then there exists some $\sigma \in P$ such that $\delta(x,y) < \sigma$, $\sigma \leq \mu$ and $\sigma \leq \nu$.
$D_5$. For any $x, y \in Y$, there exists some $p \in P$ such that $\delta(x,y) < p$.

The partially ordered set P will be called a distance set for Y and the function $\delta$ will be called a distance function. We will denote by $N_c(x)$ the collection $\{ y \in Y : \delta(x,y) < c \}$. A set $N_c(x)$ is said to be a distance neighborhood (or a $\delta$ neighborhood) of x. Please note that distance neighborhoods may be empty. (For alternative treatments see [5], [7], [8] and [10] among many others.)

An arbitrary partially ordered set need not contain any minimum or zero element. Hence we could not require $\delta(x,x)$ be zero. Bearing this in mind, condition $D_1$ is a reasonable analog of $M_1$. Condition $D_2$ is, of course, exactly the same as $M_2$. Finally, condition $D_3$ takes the place of $M_3$. A distance set P

has not been required to have any type of algebraic structure, and so we cannot formulate a true triangle inequality. Condition $D_3$ is about as close as we can come without any form of addition, and, as we stated earlier, it will be adequate to establish a connection between distance spaces and topological spaces similar to that between metric spaces and metrizable spaces. Conditions $D_4$ and $D_5$ serve to eliminate trivial examples. Metric analogs of $D_4$ and $D_5$ would be trivial consequences of the totally ordered and unbounded characteristics of R.

The following lemma will be useful in establishing the connection between distances and topologies mentioned above.

Lemma 1: Suppose that $\langle X, \delta, P\rangle$ is a distance space, that $\delta(x,z) < \mu$ and that $\delta(y,z) < \nu$. Then there exists some $\sigma \in P$ such that $\delta(z,z) < \sigma$ and such that $N_\sigma(z) \subseteq N_\mu(x) \cap N_\nu(y)$.
proof: Under the conditions of the lemma, condition $D_3$ implies that there exist $\rho$, $\tau \in P$ such that $\delta(z,z) < \rho$, $\delta(z,z) < \tau$, $N_\rho(z) \subseteq N_\mu(x)$ and $N_\tau(z) \subseteq N_\nu(y)$. From condition $D_4$ we can conclude that there exists some $\sigma \in P$ such that $\delta(z,z) < \sigma$, such that $\sigma < \rho$ and such that $\sigma < \tau$.

This approach to generalized concepts of distance has more than just academic interest. One possible application lies in the field of computer science. A parallel multi-processor system might consist of a number of small processors connected on a high speed 'belt' allowing them to work jointly on parts of the same problem. (For a description of such a system see [9]). Each processor would have its own (privately accessible) memory, but might also have joint access to other memory in common with other processors. In this context, it would make good sense to say that two processors are "near" if they share access to large amounts of memory. One might also say that two processors are "farther apart" if they share less memory or share none, but each share access to some memory with a third processor. These "distances" clearly relate to the efficiency of information transfer within the system. The results of this paper provide a method for formalizing these ideas of distance in a way which would bring a large body of knowledge



(all of topology) to bear on the problems of information transfer within a parallel processing system.

In a hypertext system (see [6]), the difficulty in transitioning from one point to another in a document depends on both the number of transitions and the complexity of the decisions to be made at each of the various transition points. Since number and complexity are distinct types of entity whose relation is not easily ascertainable, a partially ordered set is a natural candidate for measuring *distance* (in the sense of difficulty in transition) between points in a hypertext system.

If we are to generalize the idea of metric spaces, we will also certainly have to consider the idea of a continuous function. Recall that if $\langle M_1, m_1\rangle$ and $\langle M_2, m_2\rangle$ are metric spaces, then a function $f : M_1 \rightarrow M_2$ is said to be (metric) continuous at a point $x \in M_1$ if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that $m_1(x,y) < \delta$ implies that $m_2(f(x),f(y)) < \varepsilon$. The function f is said to be a continuous function from the metric space $M_1$ to the metric space $M_2$.

This definition has a perfectly natural analog in the setting of distance spaces:

Definition 2: If $\langle D_1, \delta_1, P_1\rangle$ and $\langle D_2, \delta_2, P_2\rangle$ are distance spaces, and f is a function from $D_1$ to $D_2$, then f is said to be (*distance*) continuous provided that for each $x \in D_1$ and for each $\varepsilon \in P_2$, if $\delta_2(f(x), f(x)) < \varepsilon$ then there exists some $\tau \in P_1$ such that:

a) $\delta_1(x, x) < \tau$
b) if $\delta_1(x, y) < \tau$ then $\delta_2(f(x), f(y)) < \varepsilon$

There are two primary attributes of *continuous* functions that we would like to have carry over from metric spaces to distance spaces, the fact that the composition of continuous functions is continuous and the fact that they correspond (in a very natural way) to (*topologically*) continuous functions. These properties are established in the following two theorems.

**Theorem 1:** If f is a continuous function from the distance space $\langle D_1, \delta_1, P_1 \rangle$ to the distance space $\langle D_2, \delta_2, P_2 \rangle$, and if g is a continuous function from $\langle D_2, \delta_2, P_2 \rangle$ to the distance space $\langle D_3, \delta_3, P_3 \rangle$, then the composition $g \cdot f$ is a continuous function from $\langle D_1, \delta_1, P_1 \rangle$ to $\langle D_3, \delta_3, P_3 \rangle$.

**proof:** Suppose x is an element of $D_1$, that $\varepsilon$ is an element of $P_3$ and that $\delta_3( g( f(x) ), g( f(x) ) ) < \varepsilon$. Since the function g is continuous, there exists some $\sigma \in P_2$ such that $\delta_2( f(x), f(x) ) < \sigma$ and such that $\delta_2(y, f(x) ) < \sigma$ implies that $\delta_3( g(y), g( f(x) ) ) < \varepsilon$. Since the function f is also continuous, there exists some $\tau \in D_1$ such that $\delta_1( x, x ) < \tau$ and such that $\delta_1(x, z) < \tau$ implies that $\delta_2( f(x), f(z) ) < \sigma$, which, in turn, implies that $\delta_3( g( f(x) ), g( f(z) ) ) < \varepsilon$. Hence, the composition $g \cdot f$ is continuous.

The result of theorem 1 can be rephrased in a meaningful way:

**Corollary 1.1:** The distance spaces and the distance continuous functions form the object class and morphism class for a category (see [3].)

We will refer to this category, the category of all distance spaces and all continuous functions, as *DST*.

**Theorem 2:** Suppose that $\langle X, \delta, P \rangle$ is a distance space. Then the collection $\{ N_\varepsilon(x) : x \in X, \varepsilon \in P \}$ is a base for a topology $\mathcal{T}(\delta)$ on X.

**proof:** From [4], in order to show that a collection $\mathcal{U}$ of subsets of X forms a base for some topology on X, it is necessary only to show that if a point $x \in X$ is contained in two elements $S_1$ and $S_2$ of $\mathcal{U}$, then there must exist some $S_3 \in \mathcal{U}$ such that $x \in S_3$ and $S_3 \subseteq S_1 \cap S_2$. Suppose, then, that x is contained in both $N_\gamma(y)$ and $N_\eta(z)$. This means that $\delta(x,y) < \gamma$ and $\delta(x,z) < \eta$. By lemma 1 there exists some $\lambda \in P$ such that $\delta(x,x) < \lambda$ and such that $N_\lambda(x) \subseteq N_\gamma(y) \cap N_\eta(z)$.



Theorem 2 provides a relation between objects of *DST* and the objects of *TOP*. This naturally suggests the possibility that this relation might induce a relation between the morphisms in these categories.

**Proposition 1:** A *SET* morphism $f : X \longrightarrow Y$ is a *DST* morphism from the distance space $\langle X, \delta, P \rangle$ to the distance space $\langle Y, \chi, Q \rangle$ if and only if it is a *TOP* morphism from $\langle X, \mathcal{T}(\delta) \rangle$ to $\langle Y, \mathcal{T}(\kappa) \rangle$.

**proof:** Suppose f is a *DST* morphism from $\langle X, \delta, P \rangle$ to $\langle Y, \kappa, Q \rangle$. Let U be an element of $\mathcal{T}(\kappa)$ and suppose that $f(x) \in U$. From the definition of $\mathcal{T}(\kappa)$, there is some $\varepsilon \in Q$ such that $f(x) \in N_\varepsilon( f(x) ) \subseteq U$. From the (distance) continuity of the function f, there exists some $\tau \in P$ such that $x \in N_\tau(x) \subseteq f^{-1}[U]$, and so the function f is continuous in the sense of general topology. We suppose, next, that f is a *TOP* morphism from $\langle X, \mathcal{T}(\delta) \rangle$ to $\langle Y, \mathcal{T}(\kappa) \rangle$. For any $x \in X$ and any $\varepsilon \in Q$, if $\kappa( f(x), f(x) ) < \varepsilon$, then $f(x) \in N_\varepsilon( f(x) ) \in \mathcal{T}(\kappa)$. By the (topological) continuity of f, we have that $f^{-1}[N_\varepsilon( f(x) ]$ is an element of $\mathcal{T}(\delta)$. Hence, by the definition of $\mathcal{T}(\delta)$, there exists some $\gamma \in P$ such that $x \in N_\gamma(x) \subseteq f^{-1}[N_\varepsilon(f(x)]$, and thus f is a *DST* morphism.

**Theorem 3:** The relation $\langle X, \delta, P \rangle \longrightarrow \langle X, \mathcal{T}(\delta) \rangle$ determines a functor from *DST* to *TOP*. (We will denote this functor DT.)

**proof:** From theorem 2, the above relation does associate each *DST* object with a *TOP* object, and, from proposition 1, each *DST* morphism can be associated (in a natural way) with a *TOP* morphism. This association of morphisms clearly respects composition, and so the relation from *DST* to *TOP* is a functor.

The result in Theorem 2 should not have been any great surprise. It might, however, be more unexpected that all topologies can be constructed in this fashion. Our first step in establishing this result will be the following definitions.

**Definition 3:** Suppose that $\langle X, \mathcal{T} \rangle$ is an arbitrary topological space. We will denote by $\mathcal{P}(X, \mathcal{T})$ the collection of all subsets of X with the partial order obtained by defining $A < B$ provided that B is open and $A \subset B$. Finally, we define a function $\delta_X$ from $X \times X$ to $\mathcal{P}(X, \mathcal{T})$ by defining $\delta_X(x, y)$ to be the set $\{x, y\}$.

With the above definitions, the next result is exactly what is to be expected.

**Lemma 2:** For any topological space $\langle X, \mathcal{T} \rangle$ the set $\mathcal{P}(X, \mathcal{T})$, is a distance set, for X and the function $\delta_X$ is a distance function on X. Hence, $\langle X, \delta_X, \mathcal{P}(X, \mathcal{T}) \rangle$ forms a distance space.
proof: $\mathcal{P}(X, \mathcal{T})$ is, by construction, a partially ordered set, and so, is a legitimate candidate to be a distance set. For any $x, y \in X$, if the image $\delta_X(x, y) = \{x, y\} < U \in \mathcal{P}(X, \mathcal{T})$, then U is an open subset of X and $\{x, y\} \subseteq U$. This implies that $\delta_X(x, x) = \{x\} < U$ and that $\delta_X(y, y) = \{y\} < U$. Hence $\delta_X$ satisfies condition $D_1$. For any two points x, y in X, the image $\delta_X(x, y) = \{x, y\}$ is equal to $\{y, x\} = \delta_X(y,x)$. Thus, $\delta_X$ satisfies condition $D_2$. If $\delta_X(x,y) < c \in P(X, \mathcal{T})$, then c must be an open subset of X containing both x and y. Clearly then, $y \in N_c(y) \subseteq N_c(x)$, and so $\delta_X$ satisfies condition $D_3$. Now, if $\delta_X(x, y) < \gamma$ and $\delta_X(x, z) < \eta$, then both $\gamma$ and $\eta$ must be open subsets of X, and the intersection $\lambda = \gamma \cap \eta$ must also be an open subset of X which contains the point x. Therefore $\delta_X(x, x) < \lambda$. Further, $\delta_X(x, w) < \lambda$ implies that $w \in \lambda$, and, therefore, that $\delta_X(y, w) = \{y, w\} \subset \gamma$ and $\delta_X(z, w) = \{z, w\} \subset \eta$. This, then, implies that $\delta_X$ satisfies condition $D_4$. The set X, being an open subset of the space X, is an element of $\mathcal{P}_{\mathcal{T}}$, and for any $x, y \in X$, the image $\delta_X(x,y) = \{x, y\}$ is contained in X, and so the function $\delta_X$ satisfies condition $D_5$, and, satisfying all five conditions, is a distance function on the space X.

Now that we have a way of generating distance spaces from topological spaces, it is natural to ask whether this too gives us a functor.



**Theorem 4:** The relation $\langle X, \mathcal{T} \rangle \longrightarrow \langle X, \delta_X, \mathcal{P}(X, \mathcal{T}) \rangle$ induces a functor from *TOP* to *DST*. (We will denote this functor TD.)
proof: Suppose that f is a continuous function from topological space $\langle X, \mathcal{T} \rangle$ to topological space $\langle Y, \mathcal{S} \rangle$. For any $x \in X$ and any $c \in \mathcal{P}(Y, \mathcal{S})$, if $\delta_Y( f(x), f(x) ) < c$, then $f(x) \in c$ and $c \in \mathcal{S}$, since elements of $\mathcal{S}$ are the only elements of $\mathcal{P}(Y, \mathcal{S})$ which dominate anything (i.e. if $\alpha < \beta$, for any $\alpha$, then $\beta \in \mathcal{S}$.) Since f is topologically continuous, we know that $x \in f^{-1}[c] \in \mathcal{T}$. Then, if $\delta_X(x, y) = \{x, y\} \subseteq f^{-1}[c]$, we can conclude that $\{f(x), f(y)\} \subseteq c$. Hence, the function f is a morphism in *DST*.

We now have a functor, DT, from *DST* to *TOP* and another, TD, from *TOP* to *DST*. Our next problem is to determine the relationship between these functors.

**Theorem 5:** If X is any topological space, then the topology on X induced by the $\delta_X$ neighborhoods coincides with the original topology on X.
proof: If $N_c(x)$ is nonempty, then c must be a nonempty open subset of X, since these are the only elements of $P_X$ which strictly dominate other points of $P_X$, and we have that $x \in c$. In this case, it is clear that $N_c(x) = c$. Thus any $\delta_X$ neighborhood is open in the original topology on X. Further, if U is a nonempty open subset of X, then for any $v \in U$, the set $N_U(v)$ is equal to U. Hence, open sets are $\delta_X$ neighborhoods and $\delta_X$ neighborhoods are open sets.

Theorem 5 provides us with the previously announced result, that any topological space can be derived from a distance space. It also lets us describe *TOP* as a subcategory of *DST*.

**Corollary 5.1:** The composition of the functors DT • TD is the identity functor on *TOP*.

**Corollary 5.2:** *DST* contains a full subcategory, TD[*TOP*], which is isomorphic to *TOP*.

Having identified *TOP* as being equivalent to (isomorphic to) a subcategory of *DST*, it seems only natural to investigate the relationship between this subcategory and the category *DST*.

Lemma 3: For an arbitrary distance space $\langle X, \delta, P\rangle$, the identity function $\iota_X : X \longrightarrow X$ determines a *DST* morphism $\iota_X : \langle X, \delta, P\rangle \longrightarrow \langle X, \delta_X, \mathcal{P}(X, \mathcal{T}_\delta)\rangle$
proof: Clearly the only thing to be proved is that $\iota_X$ is a *DST* continuous function from $\langle X, \delta, P\rangle$ to $\langle X, \delta_X, \mathcal{P}(X, \mathcal{T}_\delta)\rangle$. Suppose that $\mu \in \mathcal{T}_\delta$ and that $\delta_X(x,x) < \mu$. By the definition of $\mathcal{P}(X, \mathcal{T}_X)$, this implies that $\mu \in \mathcal{T}_X$ and that $x \in \mu$. Since $\mathcal{T}_X$ is generated by $\delta$ neighborhoods, there is some $c \in P$ such that $\delta(x, y) < c$ implies that $y \in \mu$. This, in turn, implies that $\delta_X(x, y) = \{x, y\} < \mu$.

Lemma 4: Suppose that $\langle X, \delta, P\rangle$ is a distance space, that $\langle Y, \mathcal{L}\rangle$ is a topological space, that $\langle Y, \delta_Y, \mathcal{P}(Y, \mathcal{L})\rangle$ is the distance space associated with the space $\langle Y, \mathcal{L}\rangle$ and that $f : X \longrightarrow Y$ is a *DST* morphism from $\langle X, \delta, P\rangle$ to $\langle Y, \delta_Y, \mathcal{P}(Y, \mathcal{L})\rangle$. Then the function f is also a *DST* morphism from $\langle X, \delta_X, \mathcal{P}(X, \mathcal{T}_\delta)\rangle$ to $\langle Y, \delta_Y, \mathcal{P}(Y, \mathcal{L})\rangle$.
proof: Suppose that $c \in \mathcal{P}(Y, \mathcal{L})$ and that $\delta(f(x), f(x)) < c$. By the continuity of the function f from $\langle X, \delta, P\rangle$ to $\langle Y, \delta_Y, \mathcal{P}(Y, \mathcal{L})\rangle$, there exists some $\sigma \in P$ such that $\delta(x, x) < \sigma$ and such that $\delta(x, z) < \sigma$ implies that $\delta_Y(f(x), f(y)) < c$. This, then, implies that $x \in N_\sigma(x) \subseteq f^{-1}[N_c(f(x))]$. The open set $N_\sigma(x)$ is an element of $\mathcal{P}(X, \mathcal{T}_\delta)$ and the $N_\sigma(x)$ neighborhood of x contains x. Further, if $\delta_X(x, z) < \sigma$, then $\delta_Y(f(x), f(y)) < c$, and so the function f from $\langle X, \delta_X, \mathcal{P}(X, \mathcal{T}_\delta)\rangle$ to $\langle Y, \delta_Y, \mathcal{P}(Y, \mathcal{L})\rangle$ is a *DST* morphism.

Theorem 6: The full subcategory TD[*TOP*] of *DST* is epireflective and is isomorphic to *TOP*.
proof: It was noted in corollary 5.2, above, that TD[*TOP*] is a full subcategory which is isomorphic to *TOP*. Lemma 4 establishes that this subcategory is reflective. Thus, it remains to show that the function $\iota_X$ of lemma 3 is an epimorphism in *DST*. This, however, follows immediately from the fact that the function $\iota_X$ is an epimorphism in *SET*.



In fact an even stronger result is true concerning the relation between *DST* and *TOP*. The epireflelection from theorem 6 is, in fact, an isomorphism.

Lemma 5: For an arbitrary distance space $\langle X, \delta, P\rangle$, the identity function $\tau_X : X \longrightarrow X$ determines a *DST* morphism $\tau_X : \langle X, \delta_X, \mathcal{P}(X, \mathcal{T}_\delta)\rangle \longrightarrow \langle X, \delta, P\rangle$
proof: If $\delta(x, x) < c \in P$, then $x \in N_c(x) \in \mathcal{T}_\delta$. Hence $\delta_X(x, x) < N_c(x)$ and, clearly, for any $y \in X$, if $\delta_X(x, y) < N_c(x)$ then $\delta(x, y) < c$. Thus $\tau_X$ is a *DST* morphism.

This yields the following theorem.

Theorem 7: The isomorphism equivalence classes of *DST* form a category isomorphic to *TOP*.

In many categories it is common to consider isomorphic objects to be completely equivalent. It is not entirely clear that this is appropriate in categories related to distance spaces. The set {n :n a natural number} and the set {1/n :n a natural number} both inherit a metric from the real numbers. As metric spaces, they are isomorphic but the first has the property that if it is the domain of a function to a metric space, then the function is uniformly continuous. The second space does not have this property. Thus, in a setting in which it is possible to talk about uniform continuity, isomorphic distance spaces cannot be considered completely equivalent.

At this point we would like to introduce some terminology and some basic results for future reference.

Definition 4: If $\langle X, \delta, P\rangle$ is a distance space, then an element $p \in P$ will be said to be a *distance* element if $p \in \delta[X \times X]$. An element $p \in P$ will be said to be a *measurement* element provided that for some $x, y \in X$, the distance $\delta(x, y) < p$. The distance space will be called a *distinguishing* space if each element of P is a distance element or a measurement element and if for each

pair p and q of measurement elements of P, there exists some $x \in X$ such that $N_p(x) \neq N_q(x)$.

**Lemma 6:** Every distance space is isomorphic to a distinguishing space (with a maximum distance element greater than any distance element.)

**proof:** Suppose that $\langle X, \delta, P \rangle$ is a distance space. Let $P'$ denote the set obtained by removing from P all elements which are neither distance elements nor measurement elements and by adding one element greater than any other element of P. Let Q denote the quotient obtained by identifying the measurement elements of $P'$ which generate identical neighborhood families in X. It is obvious that $\langle X, \delta, Q \rangle$ is a distance space and that the identity function on X determines an isomorphism from $\langle X, \delta, P \rangle$ to $\langle X, \delta, Q \rangle$.

Among the reasons for introducing the above lemma is the identification of products in the category *DST*.

**Definition 5:** Suppose that for each element $\lambda$ of a set $\Lambda$, $\langle X_\lambda, \delta_\lambda, P_\lambda \rangle$ is a distance space. From lemma 6 above, for each $\lambda$ there exists a distinguishing space $\langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda \rangle$ in which the distance set contains a maximum element $m_\lambda$ greater than any distance element, and there exists a *DST* isomorphism $f_\lambda : \langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda \rangle \longrightarrow \langle X_\lambda, \delta_\lambda, P_\lambda \rangle$. We define $\Pi_\Lambda Y_\lambda$ to be the Cartesian product of the sets $Y_\lambda$, we define $\Pi_\Lambda \mathcal{P}_\lambda$ to be the Cartesian product of the sets $\mathcal{P}_\lambda$ and we define a function $\Pi_\Lambda \gamma_\lambda$ from $(\Pi_\Lambda Y_\lambda) \times (\Pi_\Lambda Y_\lambda)$ to $\Pi_\Lambda \mathcal{P}_\lambda$ by defining $\Pi_\Lambda \gamma_\lambda( \langle y_\lambda \rangle, \langle z_\lambda \rangle )$ to be $\langle \gamma_\lambda(y_\lambda, z_\lambda) \rangle$. Finally, we define a partial order on $\Pi_\Lambda \mathcal{P}_\lambda$ by defining $\langle p_\lambda \rangle$ to be less than $\langle q_\lambda \rangle$ provided that $p_\lambda < q_\lambda$ for each $\lambda \in \Lambda$, and that $q_\lambda = m_\lambda$ for all but finitely many $\lambda$ in $\Lambda$.

It is immediate that $\langle \Pi_\Lambda Y_\lambda, \Pi_\Lambda \gamma_\lambda, \Pi_\Lambda \mathcal{P}_\lambda \rangle$ is a distance space. t is our intention to establish that this is a product in *DST*. ur next lemma will be to this end.



**Lemma 7:** Suppose that $\langle Z, \varphi, Q \rangle$ is a distance space, and that $\{ \langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda \rangle : \lambda \in \Lambda \}$ is a collection of distinguishing spaces, in each of which, the distance set $\mathcal{P}_\lambda$ contains an element $m_\lambda$ greater than any distance $\gamma_\lambda(r, s)$. Suppose also that $\langle \Pi_\Lambda Y_\lambda, \Pi_\Lambda \gamma_\lambda, \Pi_\Lambda \mathcal{P}_\lambda \rangle$ is as defined in definition 5, and that for each $\lambda \in \Lambda$, we have a *DST* morphism $\psi_\lambda$ from $\langle Z, \varphi, Q \rangle$ to $\langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda \rangle$. Then the function $\Pi_\Lambda \psi_\lambda$ from Z to $\Pi_\Lambda Y_\lambda$ is a *DST* morphism.

**proof:** Let z be an element of Z, and let $\varepsilon$ be an element of $\Pi_\Lambda \mathcal{P}_\lambda$ and suppose that $\Pi_\Lambda \gamma_\lambda( \Pi_\Lambda \psi_\lambda(z) , \Pi_\Lambda \psi_\lambda(z) ) < \varepsilon$. As $\varepsilon$ is a measurement element of $\Pi_\Lambda \mathcal{P}_\lambda$, it must be of the form $\langle \varepsilon_\lambda \rangle$ with $\varepsilon_\lambda = m_\lambda$ except for $\lambda$ in some finite subset $\xi \subseteq \Lambda$. Then for each $\lambda$, we have that $\gamma_\lambda( \psi_\lambda(z) , \psi_\lambda(z) ) < \varepsilon_\lambda$. By the continuity of the function $\psi_\lambda$, there exists some $\tau_\lambda \in Q$ such that $\varphi(z, z) < \tau_\lambda$ and such that $\varphi(w, z) < \tau_\lambda$ implies that $\gamma_\lambda( \psi_\lambda(w), \psi_\lambda(z) ) < \varepsilon_\lambda$. The collection $\{ \tau_\lambda : \lambda \in \xi \}$ is a finite subset of $\Lambda$, and $\varphi(z, z)$ is less than each element. Hence, by $D_4$, there is some $\tau_0 \in Q$ such that $\varphi(z, z) < \tau_0 \leq \tau_\lambda$ for each $\lambda$ in $\xi$. Then if $\varphi(w, z) < \tau_0$, it is clear that $\gamma_\lambda( \psi_\lambda(w), \psi_\lambda(z) ) < \varepsilon_\lambda$ for each $\lambda \in \Lambda$, and, hence, $\Pi_\Lambda \gamma_\lambda( \Pi_\Lambda \psi_\lambda(w), \Pi_\Lambda \psi_\lambda(z) ) < \varepsilon$.

**Lemma 8:** Suppose that $\{ \langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda \rangle : \lambda \in \Lambda \}$ is a collection of distinguishing spaces, in each of which, the distance set $\mathcal{P}_\lambda$ contains an element $m_\lambda$ greater than any distance $\gamma_\lambda(r, s)$. Suppose also that $\langle \Pi_\Lambda Y_\lambda, \Pi_\Lambda \gamma_\lambda, \Pi_\Lambda \mathcal{P}_\lambda \rangle$ is as defined in definition 5. Then each projection $\pi_\lambda$ from $\Pi_\Lambda Y_\lambda$ to $Y_\lambda$ is a *DST* morphism.

**proof:** Suppose that $\langle y_\lambda \rangle \in \Pi_\Lambda Y_\lambda$, that $\varepsilon \in \mathcal{P}_\sigma$ and that $\gamma_\sigma( \pi_\sigma( \langle y_\lambda \rangle ), \pi_\sigma( \langle y_\lambda \rangle ) ) < \varepsilon$. We can, then, define an element $\zeta = \langle z_\lambda \rangle \in \Pi_\Lambda \mathcal{P}_\lambda$ by defining $z_\lambda = m_\lambda$ for all $\lambda$ except $\sigma$, and by defining $z_\sigma = \varepsilon$. Clearly, then, if $\Pi_\Lambda \gamma_\lambda(\langle y_\lambda \rangle, \langle w_\lambda \rangle) < \zeta$, then $\gamma_\sigma( \pi_\sigma(\langle y_\lambda \rangle), \pi_\sigma(\langle w_\lambda \rangle) ) = \gamma_\sigma( y_\sigma, w_\sigma ) < \varepsilon$.

**Theorem 8:** If $\{ \langle X_\lambda, \delta_\lambda, P_\lambda \rangle : \lambda \in \Lambda \}$ is a set of distance spaces, then $\langle \Pi_\Lambda Y_\lambda, \Pi_\Lambda \gamma_\lambda, \Pi_\Lambda \mathcal{P}_\lambda \rangle$ (as defined in definition 5) is the product, in *DST*, of the collection $\{ \langle X_\lambda, \delta_\lambda, P_\lambda \rangle : \lambda \in \Lambda \}$.

**proof:** Suppose that $\langle Z, \xi, Q \rangle$ is a distance space and that for each $\lambda \in \Lambda$, we have a *DST* morphism $\varphi_\lambda : \langle Z, \xi, Q \rangle \longrightarrow \langle X_\lambda, \delta_\lambda, P_\lambda \rangle$. Since $\langle X_\lambda, \delta_\lambda, P_\lambda \rangle$ is isomorphic to $\langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda \rangle$, each $\varphi_\lambda$ can be

written as the composition $f_\lambda \cdot \psi_\lambda$ for a *DST* morphism $\psi_\lambda$ from $\langle Z, \xi, Q\rangle$ to $\langle Y_\lambda, \gamma_\lambda, \mathcal{P}_\lambda\rangle$. Each $\psi_\lambda$ can be written as the composition of the *DST* morphisms $\pi_\lambda$ and $\Pi_\Lambda \psi_\lambda$. Hence, each $\varphi_\lambda$ can be written as the composition of $f_\lambda \cdot \pi_\lambda$ and $\Pi_\Lambda \psi_\lambda$.

Above, we have demonstrated that any topology can be considered to be generated by a generalized idea of "distance" between points. The generalization, however, might be considered a bit extreme. The distance sets (arbitrary partially ordered sets) lack many (in fact most) of the characteristics which we normally associate with distances. It would seem natural, then, to inquire whether we might utilize something with more appealing characteristics.

One of the characteristics most noticeably lacking from our distance definition is the idea of a zero distance. The idea that the distance between one point and itself should be different from the distance between another point and itself, is not very appealing.

Definition 6: If $\langle X, \delta, \mathcal{P}\rangle$ is a distance space, then $\delta$ will be said to be a zeroed distance function and $\langle X, \delta, \mathcal{P}\rangle$ will be called a zeroed distance space, provided that for any two points $x,y \in X$, the images $\delta(x,x)$ and $\delta(y,y)$ are the same element, $0(\mathcal{P})$ of $\mathcal{P}$. The full subcategory of *DST* consisting of zeroed distance spaces will be referred to as *ZDST*. (Please note that a zeroed distance space will be *DST* isomorphic to distance spaces which are not zeroed distance spaces.)

The reason for designating the point $\delta(x,x)$ of $\mathcal{P}$, the distance set, as a zero should be obvious. Our next project will be the investigation of the relation between *DST* and *ZDST*.



Lemma 9: Suppose that $\langle X, \delta, P\rangle$ is a distance space, that $\langle Y, \gamma, Q\rangle$ is a zeroed distance space and that f is a *DST* morphism from $\langle X, \delta, P\rangle$ to $\langle Y, \gamma, Q\rangle$. If we define $\gamma_f$ to be the function from $X \times X$ to Q defined by $\gamma_f(x_1, x_2) = \gamma(\, f(x_1), f(x_2)\,)$, then $\langle X, \gamma_f, Q\rangle$ is a zeroed distance space and the identity function on X provides a *DST* morphism from $\langle X, \delta, P\rangle$ to $\langle X, \gamma_f, Q\rangle$.

proof: It is clear that Q is a partially ordered space and that $\gamma_f$ is a function from $X \times X$ to Q. For any two points $x_1$ and $x_2$ of X, if $\gamma_f(x_1, x_2) < c \in Q$, then, since $\gamma(\, f(x_1), f(x_2)\,) < c$ and since the function $\gamma$ satisfies condition $D_1$, $\gamma(\, f(x_1), f(x_1)\,) < c$ and $\gamma(\, f(x_1), f(x_2)\,) < c$. Hence we can conclude that $\gamma_f$ also satisfies condition $D_1$. Since the function $\gamma$ satisfies condition $D_2$, we know that $\gamma(\, f(x_1), f(x_2)\,) = \gamma(\, f(x_2), f(x_1)\,)$, and so $\gamma_f$ satisfies condition $D_2$. If $\gamma_f(x_1, x_2) < c \in Q$, then, since the function $\gamma$ satisfies condition $D_3$, there exists some $\mu \in Q$ such that $\gamma(\, f(x_2), f(x_2)\,) < \mu$ and such that $\gamma(\, f(x_2), y\,) < \mu$ implies $\gamma(\, f(x_1), y\,) < c$. Obviously, then, $\gamma_f(x_2, x_2) < \mu$ and $\gamma_f(x_2, z) < \mu$ implies $\gamma_f(x_1, z) < c$, and, hence, the function $\gamma_f$ satisfies condition $D_3$. If $\gamma_f(x_1, x_2) < \sigma$ and if $\gamma_f(x_1, x_2) < \tau$, then, as $\gamma$ satisfies $D_4$, there exists some $c \in Q$ such that $c < \sigma$, such that $c < \tau$ and such that $\gamma(\, f(x_1), f(x_2)\,) < c$. Thus $\gamma_f$ satisfies condition $D_4$. For any $x_1$ and $x_2$ in X, there is, by $D_5$, some $c \in Q$ such that $\gamma(\, f(x_1), f(x_2)\,) < c$, and so $\gamma_f$ also satisfies condition $D_5$. Therefore $\langle X, \gamma_f, Q\rangle$ is a distance space. Since $\langle Y, \gamma, Q\rangle$ is a zeroed distance space, for any $x_1$ and $x_2$ in X, $\gamma_f(x_1, x_1) = \gamma_f(x_2, x_2)$ and $\langle X, \gamma_f, Q\rangle$ is also a zeroed distance space. If $\gamma_f(x, x) < c \in Q$, then, since f is a *DST* morphism there exists some $\mu \in P$ such that $\delta(x, x) < \mu$ and such that $\delta(x, z) < \mu$ implies that $\gamma_f(x, y) < c$, and, hence, that the identity function on X is a *DST* morphism from $\langle X, \delta, P\rangle$ to $\langle X, \gamma_f, Q\rangle$.

With the above constructions, we will be able to establish an initial relation between *DST* and *ZDST*.

**Theorem 9:** *ZDST* is an epireflective subcategory of *DST*.
**proof:** Suppose that $< X, \delta, P >$ is an arbitrary distance space. Let $\{ (\gamma_\lambda, Q_\lambda) : \lambda \in \Lambda \}$ be the collection (easily seen to be a set) of all pairs $(\gamma_\lambda, Q_\lambda)$ such that $<X, \gamma_\lambda, Q_\lambda>$ is a distinguishing zeroed distance space with a maximum distance element $m_\lambda$, and such that the identity function $\iota_X$ on X defines a *DST* morphism $\iota_\lambda : <X, \delta, P> \longrightarrow <X, \gamma_\lambda, Q_\lambda>$. It is easily established that the product $<\Pi_\Lambda X, \Pi_\Lambda \gamma_\lambda, \Pi_\Lambda Q_\lambda>$ is a zeroed distance space and that the diagonal function $\Delta : X \longrightarrow X \times X$ induces a *DST* morphism from $<X, \delta, P>$ to $<\Pi_\Lambda X, \Pi_\Lambda \gamma_\lambda, \Pi_\Lambda Q_\lambda>$. We will denote by $\gamma_\Delta$ the function from $X \times X$ to $\Pi_\Lambda Q_\lambda$ which carries $(x_1, x_2)$ to $<\gamma_\lambda(x_1, x_2)>$. From lemma 9, $<X, \gamma_\Delta, \Pi_\Lambda Q_\lambda>$ is a zeroed distance space and the identity function $\iota_X : X \longrightarrow X$ induces a *DST* morphism $\iota_\Delta$ from $<X, \delta, P>$ to $<X, \gamma_\Delta, \Pi_\Lambda Q_\lambda>$. Suppose, now, that $g : <X, \delta, P> \longrightarrow <Z, \zeta, \mathcal{P}>$ is a *DST* morphism and that $<Z, \zeta, \mathcal{P}>$ is a zeroed distance space. The function g can be written as the composition $g_1 \cdot g_2$, where $g_2 : <X, \delta, P> \longrightarrow <X, \zeta_g, \mathcal{P}>$ and $g_1 : <X, \zeta_g, \mathcal{P}> \longrightarrow <Z, \zeta, \mathcal{P}>$. The function $g_2$, in turn, can be written as the composition $g_3 \cdot \iota_\beta$, where $g_3$ is an isomorphism from $<X, \gamma_\beta, Q_\beta>$ to $<X, \zeta_g, \mathcal{P}>$ and where $<X, \gamma_\beta, Q_\beta>$ is a distinguishing zeroed distance space with a maximum distance element $m_\beta$. Finally, the function $\iota_\beta$ is the composition $\pi_\beta \cdot \Delta \cdot \iota_\Delta$. Thus, the *DST* morphism g is equal to the composition of the *ZDST* morphism $(g_1 \cdot g_3 \cdot \pi_\beta \cdot \Delta)$ and the *DST* morphism $\iota_\Delta$. Since the function $\iota_\Delta : X \longrightarrow X$ is an epimorphism in *SET*, the *DST* morphism $\iota_\Delta : <X, \delta, P> \longrightarrow <X, \gamma_\Delta, \Pi_\Lambda Q_\lambda>$ is a *DST* epimorphism.

A natural question to ask at this point would be whether all topological spaces can be obtained as zeroed distance spaces (that is, are images, under the functor DT from theorem 3, of objects from *ZDST*.) The answer, as the next theorem shows, is no. There are topologies which cannot be generated by zeroed distance functions. In fact, the spaces whose topologies can be obtained from zeroed distances are precisely the $R_0$ spaces. (From [1] a topological space X is said to be an $R_0$ space provided that for



any two points x, y ∈ X, the sets $\overline{\{x\}}$ and $\overline{\{y\}}$ are either equal or disjoint.)

**Theorem 10:** If $<X, \delta, \mathcal{P}>$ is a zeroed distance space, then $<X, \mathcal{T}(\delta)>$ is an $R_0$ space.
**proof:** Suppose that two points x, y ∈ X do not have identical neighborhood families (in the topology $\mathcal{T}(\delta)$.) Then one of the points (assume that it is x) has a neighborhood U which does not contain the other. Hence there must exist some neighborhood of the form $N_\epsilon(z)$ which contains x and does not contain y. This implies that there is some $\gamma \in \mathcal{P}$ such that $x \in N_\gamma(x) \subseteq N_\epsilon(z)$. Now, as $\delta$ is zeroed and $\delta(x,x) < \gamma$, it follows that $\delta(y,y) < \gamma$, and so $N_\gamma(y)$ is an open neighborhood of y. If it were the case that $\delta(x, y) < \gamma$, then y would be an element of $N_\gamma(x)$, and therefore an element of U. By hypothesis, $y \notin U$, and so $N_\gamma(x)$ does not contain y and $N_\gamma(y)$ does not contain x. We conclude, then, that the closure of $\{x\}$ is contained in $X \setminus N_\gamma(y)$ and the closure of $\{y\}$ is contained in $X \setminus N_\gamma(x)$, and, hence, that the closure of any singleton consists of precisely those points having exactly the same neighborhoods.

The following result follows immediately from the previous theorem.

**Corollary 10.1:** If $< X, \delta, P >$ is a zeroed distance space, and if $< X, \mathcal{T}(\delta) >$ is a $T_0$ space then $< X, \mathcal{T}(\delta) >$ is a $T_1$ space.

The previous result demonstrates that zeroed distance spaces produce $R_0$ spaces. It does not, however, prove that all $R_0$ spaces are produced by zeroed distance spaces. We will now show that all $R_0$ spaces are, in fact, images of zeroed distance spaces, that is, the image DT[*ZDST*] is equal to the subcategory *REGO* of *TOP*.

**Definition 7:** Suppose that $< X, \mathcal{T} >$ is an arbitrary topological space. We define the set $Z_\mathcal{T}$ to be the collection of all symmetric subsets of $X \times X$ containing the diagonal $\Delta_X = \{(x,x) : x \in X\}$. The set $Z_\mathcal{T}$ is partially ordered by the relation $p_1 < p_2$ provided that $p_1 \subset p_2$ and $p_2$ is open in the product $X \times X$.

We next define the function $\zeta_{\mathcal{T}}$ from $X \times X$ to $Z_{\mathcal{T}}$ by: $\zeta_{\mathcal{T}}(x, y) = \Delta_X \cup \{ (x, y), (y, x) \}$.

The following lemmas should simplify the proof of intended result.

Lemma 10: If $< X, \mathcal{T} >$ is an $R_0$ space, and if $\zeta_{\mathcal{T}}(x, y) < \sigma \in Z_{\mathcal{T}}$, then there exists some $\mu \in Z_{\mathcal{T}}$ such that $\zeta_{\mathcal{T}}(y, y) < \mu$ and such that $\zeta_{\mathcal{T}}(y, w) < \mu$ implies that $\zeta_{\mathcal{T}}(x, w) < \sigma$ (i.e. $\zeta_{\mathcal{T}}$ satisfies condition $D_3$.)
proof: If $\zeta_{\mathcal{T}}(x, y) < \sigma \in Z_{\mathcal{T}}$, then $\sigma$ must be a symmetric open subset of $X \times X$ containing $\Delta_X$, and so for each $z \in X$, there is some open neighborhood $U_z$ of $z$ such that $U_z \times U_z \subseteq \sigma$. Since X is assumed to be an $R_0$ space, we can assume that either $U_x = U_y$ or $x \notin U_y$ and $y \notin U_x$. For each additional point $z \in X$, we can assume that $U_z = U_x$, that $U_z = U_y$ or that neither x nor y is an element of $U_z$. In addition there must also exist open neighborhoods $V_x$ and $V_y$ of x and y respectively, such that $( V_x \times V_y ) \cup ( V_y \times V_x ) \subseteq \sigma$. If we let $W_x$ represent $V_x \cap U_x$ and we let $W_y$ represent $V_y \cap U_y$, and we let C denote the set $X \setminus \overline{\{x, y\}}$, then we can define an element $\mu$ of $Z_{\mathcal{T}}$ to be:
$(\cup\{U_z \times U_z : z \in C\}) \cup (W_x \times W_y) \cup (W_y \times W_x) \cup (W_x \times W_x) \cup (W_y \times W_y)$
From construction, it is clear that $\mu \leq \sigma$ and that $\zeta_X(y, y) < \mu$. For any $w \in X$, if $\zeta_X(y, w) < \mu$, then either $\{y, w\} \subseteq W_y \times W_y$, or $\{y, w\} \subseteq ( W_x \times W_y ) \cup ( W_y \times W_x )$. Thus, either $w \in W_y$, which implies that $\zeta_{\mathcal{T}}(x, w) < \sigma$, or $w \in W_x$, which also implies that $\zeta_X(x, w) < \sigma$. Hence, the function $\zeta_{\mathcal{T}}$ satisfies condition $D_3$.

Lemma 11: If X is any topological space, if $\zeta_{\mathcal{T}}(x, y) < \mu \in Z_{\mathcal{T}}$ and if $\zeta_{\mathcal{T}}(x, y) < \nu \in Z_{\mathcal{T}}$, then there exists some $\sigma \in Z_{\mathcal{T}}$ such that $\sigma \leq \mu$, such that $\sigma \leq \nu$, and such that $\zeta_{\mathcal{T}}(x, y) < \sigma$. (In other words, the function $\zeta_{\mathcal{T}}$ satisfies condition $D_4$.)
proof: Since $\zeta_{\mathcal{T}}(x, y) < \mu$ and $\zeta_{\mathcal{T}}(x, y) < \nu$, it follows that $\mu$ and $\nu$ are symmetric open subsets of $X \times X$ containing $\Delta_X$. Hence, the intersection $\sigma = \mu \cap \nu$ is also a symmetric open subset of $X \times X$ containing $\Delta_X$. Clearly $\zeta_{\mathcal{T}}(x, y) \subseteq \mu \cap \nu$, and so the function $\zeta_{\mathcal{T}}$ satisfies condition $D_4$.



Theorem 11: For any $R_0$ space $< X, \mathcal{T} >$, the function $\zeta_{\mathcal{T}}$, described above, is a zeroed distance function. Further, the topology $\mathcal{T}$ coincides with the topology $\mathcal{T}(\zeta_{\mathcal{T}})$ generated by the $\zeta_{\mathcal{T}}$ neighborhoods.
proof: The set $Z_{\mathcal{T}}$ is (clearly) partially ordered and $\zeta_{\mathcal{T}}$ is a function from $X \times X$ to $Z_{\mathcal{T}}$. If $\zeta_{\mathcal{T}}(x,y) < \mu \in Z_{\mathcal{T}}$ then $\mu$ must be a symmetric open subset of $X \times X$ containing $\Delta_X$. Thus, by definition, $\zeta_{\mathcal{T}}(x, x) = \zeta_{\mathcal{T}}(y, y) = \Delta_X \subseteq \mu$. Thus the function $\zeta_{\mathcal{T}}$ satisfies condition $D_1$. Clearly the two sets $\Delta_X \cup \{ (x, y), (y, x) \}$ and $\Delta_X \cup \{ (y, x), (x, y) \}$ are equal, and so $\zeta_{\mathcal{T}}(x, y)$ is equal to $\zeta_{\mathcal{T}}(y, x)$. Hence the function $\zeta_{\mathcal{T}}$ satisfies condition $D_2$. From lemmas 10 and 11 above, the function $\zeta_{\mathcal{T}}$ satisfies conditions $D_3$ and $D_4$. Finally, we need only note that for any two points $x, y \in X$, it is obvious that $\zeta_{\mathcal{T}}(x, y) < X \times X$ and $X \times X$ is a symmetric open set containing $\Delta_X$, and so the function $\zeta_{\mathcal{T}}$ also satisfies condition $D_5$. Therefore, $<X, \zeta_{\mathcal{T}}, Z_{\mathcal{T}}>$ is a distance space, and, since $\zeta_{\mathcal{T}}(x, x) = \Delta_X$ for any $x \in X$, a zeroed distance space. It remains, then, to show that the topology $\mathcal{T}(\zeta_{\mathcal{T}})$ is the same as the original topology $\mathcal{T}$ on X. For any $x \in X$ and any $c$ in $Z_{\mathcal{T}}$, either $c$ is not an open set containing $\Delta_X$, and hence, $N_c(x) = \emptyset$, or $c$ IS a symmetric open set containing $\Delta_X$. In this case there is some open neighborhood U of x such that $U \times U \subseteq c$. This would imply that $U \subseteq N_c(x)$, and, hence the $\zeta_{\mathcal{T}}$ neighborhood $N_c(x)$ is a $\mathcal{T}$ neighborhood of x. Suppose, now, that U is an arbitrary nonempty element of $\mathcal{T}$ and that $x \in U$. The set V, defined to be $X \setminus \overline{\{x\}}$, is clearly open in X. If we denote by $c$ the set $(U \times U) \cup (V \times V)$, then it should be clear that $c$ is symmetric, open in $X \times X$, and contains $\Delta_X$. It should also be clear that a point $y \in X$ is in $N_c(x)$ if and only if $(x, y) \in c$. This, however, will be the case precisely when $y \in U$. Thus every (topological) neighborhood in $<X, \mathcal{T}>$ is a $\zeta_{\mathcal{T}}$ neighborhood and every $\zeta_{\mathcal{T}}$ neighborhood is a $\mathcal{T}$ neighborhood, and, thus, the topologies coincide.

The result of theorem 11 immediately gives us the following:

Corollary 11.1: The association $\langle X, \mathcal{T} \rangle \longrightarrow \langle X, \zeta_{\mathcal{T}}, Z_{\mathcal{T}} \rangle$ induces a functor RZ from *REGO* into *ZDST*. Further, the composition DT • RZ is the identity functor on *REGO*.

Recalling that distance spaces with the same image under DT are isomorphic (in *DST*, and, hence, in the full subcategory *ZDST*) produces another related result:

Corollary 11.2: The category *REGO* is equivalent to the category of isomrphism equivalence classes of *ZDST*.

We earlier indicated that the concept uniform continuity has important consequences for our understanding of the relationships among the categories we are investigating. It is, in fact, precisely in the setting of zeroed distance spaces where this concept can be formulated.

Definition 8: Suppose that $\langle X, \delta_X, P_X \rangle$ and $\langle Y, \delta_Y, P_Y \rangle$ are zeroed distance spaces. A function f from X to Y will be said to be *uniformly continuous* (with respect to $\delta_X$ and $\delta_Y$) provided that for any $\varepsilon \in P_Y$, if $\varepsilon > 0_Y$, there exists some $\sigma > 0_X$ such that for any $x, y \in X$, if $\delta_X(x,y) < \sigma$ then $\delta_Y( f(x), f(y) ) < \varepsilon$.

In the case where the distances, $\delta_X$ and $\delta_Y$, are, in fact, metrics, the above definition reduces to precisely the standard definition of uniform continuity. This, then, provides us with numerous examples of continuous functions which are not uniformly continuous. It also provides us with examples of situations where a given topology can be generated by distinct distance functions, and in which a particular function may be uniformly continuous with respect to one of the distance functions, and not uniformly continuous with respect to the other.

A rather natural question to raise at this point would seem to be which continuous functions can be obtained as images (under the functor DT) of uniformly continuous functions.



Theorem 12: Suppose that $\langle X, \mathcal{T} \rangle$ and $\langle Y, \mathcal{Y} \rangle$ are $R_0$ spaces and that f is a *TOP* morphism from the space $\langle X, \mathcal{T} \rangle$ to the space $\langle Y, \mathcal{Y} \rangle$. Then, as a *ZDST* morphism, the function f from $\langle X, \zeta_{\mathcal{T}}, Z_{\mathcal{T}} \rangle$ to $\langle Y, \zeta_{\mathcal{Y}}, Z_{\mathcal{Y}} \rangle$ is uniformly continuous.

proof: Suppose that $0_Y < \varepsilon \in Z_{\mathcal{Y}}$. By the construction of the partial order on $Z_{\mathcal{Y}}$, the distance element $\varepsilon$ must be a symmetric open subset of $Y \times Y$ containing $\Delta_Y$. Then for each $y \in Y$, there must exist some open set $U_y \subseteq Y$, a neighborhood of y, such that $U_y \times U_y \subseteq \varepsilon$. Since f is a *DST* morphism, for each $x \in X$, there exists some open neighborhood $V_x$ of x such that $V_x \subseteq f^{-1}[U_{f(x)}]$. The set $\sigma = \cup \{ V_x \times V_x : x \in X \}$ is clearly an element of $Z_{\mathcal{T}}$. For any two points $w, z \in X$, if $\zeta_X(w,z) < \sigma$, then some $V_x$ contains both w and z, that is $(w,z) \in V_x \times V_x$ for some $x \in X$. This implies that both f(w) and f(z) are in $U_{f(x)}$ and, thus, that $\zeta_Y( f(w), f(z) ) < \varepsilon$.

This shows us that, if the distance functions are chosen correctly, any *REGO* morphism might be considered to be uniformly continuous. Every *ZDST* morphism in the full subcategory RZ[*REGO*] of *ZDST* is uniformly continuous. It is interesting to conjecture that objects of RZ[*REGO*] might have the compactness like property that any *ZDST* morphism with domain in RZ[*REGO*] be uniformly continuous. We are not certain whether this is true or not, but we can obtain the result by adding another condition, a condition which turns out to have interesting topological implications.

Definition 9: A distance space $\langle X, \delta, P \rangle$ (and its distance function $\delta$) will be said to be *divisible* provided that for any $x \in X$ and any $\varepsilon \in P$, if $\delta(x, x) < \varepsilon$ then there exists some $\sigma \in P$ such that $x \in \cup \{ N_\sigma(y) : y \in N_\sigma(x) \} \subseteq N_\varepsilon(x)$.

We note that by itself, divisibility is not a particularly strong condition. (For any topological space $\langle X, \mathcal{T} \rangle$, the distance space $\langle X, \delta_X, \mathcal{P}(X, \mathcal{T}) \rangle$ is easily seen to be divisible.) It is in the context of zeroed distance spaces that divisibility has interesting consequences.

Lemma 12: If a zeroed distance space $\langle X, \delta, P \rangle$ is divisible, then $\langle X, \mathcal{T}_\delta \rangle$, its image under the functor DT, is a regular space (that is, in the notation of [1], an $R_1$.)
proof: Suppose that x is an element of an open set $U \in \mathcal{T}_\delta$. Then there exists some $\varepsilon \in P$ such that $x \in N_\varepsilon(x) \subseteq U$. By the divisibility of $\delta$, there exists some $\sigma \in P$ such that $x \in N_\sigma(x)$ and such that $\cup \{ N_\sigma(y) : y \in N_\sigma(x) \} \subseteq N_\varepsilon(x)$. Then $z \in X \setminus U$ implies that $N_\sigma(x)$ and $N_\sigma(z)$ are disjoint, since if y were an element of the intersection, then z would be an element of $N_\sigma(y)$, but $N_\sigma(y)$ is contained in U, and thus could not contain z. Therefore the sets $N_\sigma(x)$ and $\cup \{ N_\sigma(z) : z \in X \setminus U \}$ are disjoint open sets, the first containing x and the second containing $X \setminus U$.

Lemma 13: If the topological space $\langle X, \mathcal{T} \rangle$ is regular, then its image, $\langle X, \zeta_\mathcal{T}, Z_\mathcal{T} \rangle$, in *ZDST* is divisible.
proof: Suppose that $x \in N_\varepsilon(x)$, for a given element $\varepsilon$ of $Z_\mathcal{T}$. By the definition of $Z_\mathcal{T}$, $\varepsilon$ is an open subset of $X \times X$ containing $\Delta_X$. Then there is some open neighborhood U of x in $\mathcal{T}$ such that $U \times U \subseteq \varepsilon$. By the regularity of $\langle X, \mathcal{T} \rangle$, there exists an open neighborhood V of x such that the closure $\overline{V}$ is contained in U. Let $W(\varepsilon)$ denote the following set:
$\varepsilon \cap ( (V \times V) \cup ( (U \setminus \{x\}) \times (U \setminus \{x\}) ) \cup ( (X \setminus \overline{V}) \times (X \setminus \overline{V}) )$.
Clearly $W(\varepsilon)$ is an element of $Z_\mathcal{T}$ and $x \in N_{W(\varepsilon)}(x)$. If $y \in N_{W(\varepsilon)}(x)$ then $y \in V$, which would imply that if $z \in N_{W(\varepsilon)}(y)$, then $z \in U$. Hence $\cup \{ N_{W(\varepsilon)}(y) : y \in N_{W(\varepsilon)}(x) \} \subseteq U \subseteq N_\varepsilon(x)$, and so $\langle X, \zeta_\mathcal{T}, Z_\mathcal{T} \rangle$ is divisible.

Lemmas 12 and 13 combine to give the following:

Theorem 13: The category *DIVZDST* of divisible zeroed distance spaces contains a full subcategory isomorphic to *REG1*, the category of regular topological spaces, and the category *REG1* is isomorphic to the category of isomorphism equivalence classes of *DIVZDST*.

A natural extension of the idea of divisibility is the following:



Definition 10: A zeroed distance space $\langle X, \delta, P \rangle$ will be said to be *uniformly divisible* provided that for any $\varepsilon > 0(P)$ there exists some $\sigma \in P$ such that $0(P) < \sigma < \varepsilon$ and such that for any $x \in X$ the union $\cup \{ N_\sigma(y) : y \in N_\sigma(x) \} \subseteq N_\varepsilon(x)$.

Uniform divisibility has a topological consequence similar to that of divisibility.

Lemma 14: If $\langle X, \delta, P \rangle$ is a uniformly divisible distance space, then the measurement elements of P determine a uniformity $\mathcal{U}$ on X and the topology $\mathcal{T}(\delta)$ is equivalent to the topology generated by the uniformity $\mathcal{U}$.
proof: Let $P_M$ denote the collection of measurement elements of P. For each $\varepsilon \in P_M$, let $U_\varepsilon$ denote $\{ (x, y) \in X \times X : \delta(x, y) < \varepsilon \}$. Define $\mathcal{U}$ to be the collection $\{ U_\varepsilon : \varepsilon \in P_M \}$. Then $\mathcal{U}$ is clearly a collection of subsets of $X \times X$ and each subset contains $\Delta_X$. From property $D_2$, each $U_\varepsilon$ is equal to $U_\varepsilon^{-1}$. Given any $\varepsilon \in P_M$, by the uniform divisibility of $\delta$, there exists some $\sigma \in P_M$ such that $\delta(x, y) < \sigma$ and $\delta(y, z) < \sigma$ imply $\delta(x, z) < \varepsilon$. Hence, the composition $U_\sigma \circ U_\sigma \subseteq U_\varepsilon$. Finally, from $D_4$, given any $U_\varepsilon$ and $U_\gamma$ in $\mathcal{U}$ there exists some $\sigma \in P$ such that $\delta(x, x) < \sigma$, such that $\sigma < \varepsilon$ and such that $\sigma < \gamma$. Thus $U_\sigma \in \mathcal{U}$ and $U_\sigma \subseteq U_\varepsilon \cap U_\gamma$, and so $\mathcal{U}$ is a base for a uniformity on X. It is easily seen that the family $\{ N_\varepsilon(x) : x \in X, \varepsilon \in P_M \}$ is a base for both the topology generated by $\mathcal{U}$ and for $\mathcal{T}(\delta)$.

Lemma 15: Suppose that $\mathcal{U}$ is a uniformity on X. Let $P_\mathcal{U}$ denote the collection of all symmetric subsets of $X \times X$ containing $\Delta_X$. The relation $A \leq B$ provided that $A \subseteq B \in \mathcal{U}$ is a partial order on $P_\mathcal{U}$. We define the function $\delta_\mathcal{U}(x, y) = \Delta_X \cup \{ (x, y), (y, x) \}$. The construct $\langle X, \delta_\mathcal{U}, P_\mathcal{U} \rangle$ is a uniformly divisible zeroed distance space. Further, the topology on X induced by the uniformity $\mathcal{U}$ is identical to $\mathcal{T}(\delta_\mathcal{U})$.
proof: It is clear that $P_\mathcal{U}$ is a partially ordered set, that $\delta_\mathcal{U}$ is a function from $X \times X$ to $P_\mathcal{U}$ which satisfies the conditions $D_1$, $D_2$, $D_3$, $D_4$ and $D_5$ (see the proof of theorem 11) and that $\delta_\mathcal{U}(x, x) = \Delta_X$ and so $\langle X, \delta_\mathcal{U}, P_\mathcal{U} \rangle$ is a zeroed distance space. It is immediate from definitions that the topology induced by $\mathcal{U}$ is equivalent to

$\mathcal{T}(\delta_{\mathcal{U}})$, and so it remains to show that $< X, \delta_{\mathcal{U}}, P_{\mathcal{U}} >$ is uniformly divisible. If $\Delta_X < c \in P_{\mathcal{U}}$, then $c \in \mathcal{U}$ by the definition of the partial order on $P_{\mathcal{U}}$. Since $\mathcal{U}$ is a uniformity, there exists some (symmetric) element $\sigma$ of $\mathcal{U}$ with the property that $\sigma \circ \sigma \leq c$. This is equivalent to saying that for any $x, y, z \in X$, if $\delta_{\mathcal{U}}(x, y) < \sigma$ and $\delta_{\mathcal{U}}(y, z) < \sigma$, then $\delta_{\mathcal{U}}(x, z) < c$. Hence, $< X, \delta_{\mathcal{U}}, P_{\mathcal{U}} >$ is uniformly divisible.

These results permit us to characterize yet another topological property in terms of distance properties.

Theorem 14: The category of completely regular spaces is equivalent to the category of isomorphism classes of *UNIFDIVZDST*, the full subcategory of *DST* whose objects are the uniformly divisible distance spaces.
proof: From lemma 14 we have that if <X, δ, P> is uniformly divisible, then its image under the functor DT is a uniformizable space. From [4] we know that any uniformizable space is completely regular. Thus the image under DT of *UNIFDIVZDST* consists of completely regular spaces. Again from [4], every completely regular space is uniformizable, and so, from lemma 15, is the image under DT of some uniformly divisible space.

We have now defined a reasonable extension of the concept of a distance on a set. This concept provides a natural construction of the category *TOP*, in much the same way that metrizable spaces are constructed from metrics. Thus, up to isomorphism, topological spaces can be considered to be distance spaces.

The addition of a natural condition, a zero element in the distance set, provides a category which yields an alternate characterization of the $R_0$ spaces (and as a natural extension, the $T_1$ spaces.) In this setting, it is possible to define an analogue of uniform continuity, and thus provides a generalization of the concept of a uniformity.

The addition of other natural conditions (divisibility and uniform divisibility) on the distance sets produces distance derived characterizations of regular spaces and of completely regular (i.e. uniformizable) spaces.




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