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EXTENSION OF EMBEDDINGS OF WALLMAN REMAINDERS

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Abstract: In this paper we show that the embedding of a Wallman remainder need not be a Wallman extendible function. Even if the embedding is Wallman extendible, it need not be uniquely extendible. We show, however, that if the space X is Hausdorff and if the embedding of WX\X in WX is Wallman extendible, then the extension must be unique. Further, if X is regular and if the embedding of WX\X in WX is Wallman extendible, then this embedding is a WC function.

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A natural way to consider a compactification αX of a topological space X is as the union of two spaces, one space being the original space X and the other being the remainder, i.e. the points $\alpha X \setminus X$ which are added in order to make the resulting union be compact. From this point of view, there are two very natural questions, what kind of spaces can be so joined and how do the two spaces fit together. For any given type of compactification, a common variant of the first question is the problem of determining precisely which spaces can occur as remainders for the compactification.

It is well known (see [2] for example) that the class of Stone-Čech remainders is the class of all Tychonov spaces. That is to say that for any Tychonov space X, there exists some space Y such that X is homeomorphic to the remainder $\beta Y \setminus Y$. An analogous result for the Wallman remainder was established in [1]; given any T_1 space X there exists some T_1 space Y such that X is homeomorphic to the Wallman remainder $WY \setminus Y$. Thus the question of which spaces can be remainders is settled for the case of the Wallman compactification.

Although every continuous function from a Tychonov space into a compact Hausdorff space has a unique Stone-Čech extension, the same is not true for Wallman extensions of continuous functions from T_1 spaces to compact T_1 spaces. Thus, in investigating how T_1 spaces and their Wallman remainders fit together, it seems most reasonable to inquire whether the natural embedding of a remainder $WX \setminus X$ in the compactification WX must be a Wallman extendible function, or whether there might exist a

space X for which the natural embedding of $WX \setminus X$ in WX does not have a Wallman extension. In this paper we modify the construction presented in [1] in such a way as to permit the construction of spaces having the property that the embeddings of their Wallman remainders are not Wallman extendible functions, thus establishing that Wallman remainders need not be "Wallman embedded" in the compactifications from which they derive.

Having established that the embedding of a Wallman remainder need not be Wallman extendible, the question remains of how imposition of higher separation properties on the underlying space might affect the extendibility of the embedding function. In this paper we will consider the embedding of the Wallman remainders of T_2 and T_3 spaces. We will show that if a space X is Hausdorff and if the embedding of $WX \setminus X$ in WX is Wallman extendible, then it is uniquely extendible. We will also show that if X is regular and if the embedding of $WX \setminus X$ in WX is Wallman extendible, then the embedding must be a WC function. (see [5])

Recall that for any T_1 space X, the Wallman compactification WX consists of the collection:

$\{\mu : \mu$ is an ultrafilter in the lattice of all closed subsets of X$\}$

with the topology generated by the collection:

$\{ C_X(A) = \{ \mu \in WX : A \in \mu \} : A$ is a closed subset of X $\}$

as a base for the closed sets.

With the topology defined above, WX is a compact T_1 space and is Hausdorff if and only if X is normal. The function e_X from X to WX defined by $e_X(x) = \{A : A$ closed in X and $x \in A\}$ is a dense embedding. It is common practice, when no ambiguity can result, to ignore the distinction between X and its image, the subspace $e_X[X]$ of WX. We note that for any closed subset A of X, the closure of A in WX is $C_X(A)$, and that if A is compact then A is equal to $C_X(A)$. Further, if A is a closed subset of X and is contained in a compact subset K of WX, then $C_X(A) \subseteq K$. If $f:X \longrightarrow Y$ is a continuous function, then a Wallman extension of f is a continuous function $f^*: WX \longrightarrow WY$ such that the composition $e_Y \cdot f$

equal to $f^* \cdot e_X$. Unlike the case for the Stone-Čech compactification, there are many continuous functions which have no Wallman extensions. Further, there also exist functions with more than one Wallman extension, i.e. functions with non-unique Wallman extensions (see [5]).

As the Wallman compactification is only defined for T_1 spaces, we will consider only spaces which satisfy this separation axiom, and, thus, dispense with repeating the condition over and over again throughout the paper.

An infinite cardinal is said to be regular provided that its set of ordinal predecessors contains no cofinal subset of smaller cardinality. For any cardinal α we define L_α and E_α to be, respectively, the collection of all ordinal numbers less than α and the collection of all ordinal numbers less than or equal to α. The sets L_α and E_α are given the order topology and it is easily seen that L_α is a subspace of the compact Hausdorff space E_α. Further, if α is an uncountable regular cardinal, then, any pair of disjoint closed subsets of L_α, has the property that at least one of the sets is compact. From this it is immediate that WL_α is equal to E_α.

With the foregoing definitions, we can begin construction of a space having an arbitrarily chosen space as its Wallman remainder. Given an arbitrary space X, choose α, a regular cardinal greater than the cardinality of the topology on X. Let X^ω denote the one point cofinite extension of X obtained by adding one cofinite point ω to the space X. Define S_X to be the subspace of the product $E_\alpha \times X^\omega$ consisting of $L_\alpha \times X^\omega$ together with the point (α, ω).

<u>Theorem 1</u>: The compactification WS_X is homeomorphic to $E_\alpha \times X^\omega$ and the remainder $WS_X \backslash S_X$ is homeomorphic to X.

proof: Suppose that μ is an element of $WS_X \backslash S_X$. It is not difficult to show (see for example [1]) that there exists some x_μ in X such that $L_\alpha \times \{x_\mu\}$ is an element of μ. In addition, for each element $x \in X$, there exists precisely one element (μ_x) of $WS_X \backslash S_X$ which contains the set $L_\alpha \times \{x\}$. Hence the function ι from X to $WS_X \backslash S_X$ defined by $\iota(x) = \mu_x$ is both one to one and onto. If a point μ_x is contained in an open subset U of WS_X, then there is some closed subset F_U of S_X such that μ_x is not in $C_X(F_U)$ (i.e. some element of μ_x is disjoint from F_U) and $WS_X \backslash C(F_U)$ is contained in U. It is almost immediate, then, that there is some element γ_U of L_α and some open subset V_U of X such that the product $\{ z \in L_\alpha : z > \gamma_U \} \times V_U$ is contained in $S_X \backslash F_U$. Clearly V_U is a neighborhood of x contained in the inverse image $\iota^{-1}[U]$, and so the function ι is continuous. Similarly, if U is an open subset of X, we will denote by U_L the (clearly open) subset $L_\alpha \times U$ of S_X. It is easily seen that the image $\iota[U]$ is $WS_X \backslash C_X(S_X \backslash U_L)$, an open subset of $WS_X \backslash S_X$. Hence the function ι is a homeomorphism from X onto the remainder $WS_X \backslash S_X$.

For any space X, we will denote by RX the subspace $WX \backslash X$ of the Wallman compactification WX, and by γ_{RX} the embedding of RX in WX. We will be dealing with extensions of γ_{RX} rather extensively and the following result will prove useful.

<u>Lemma 1</u>: If $\delta : WRX \longrightarrow WX$ is a Wallman extension of γ_{RX} and if μ is any element of $WRX \backslash RX$, then $\delta(\mu) \in X$.

proof: Suppose that ν is any element of RX. Since $\mu \neq \nu$, there is some element $F_\nu \in \mu$ such that $\nu \notin F_\nu$. Since RX is a subspace of WX, there is some closed subset G_ν of WX such that F_ν is equal to the intersection $G_\nu \cap RX$. If $\delta(\mu) = \nu$, then the inverse image $\delta^{-1}[G_\nu]$ is a compact subset of WRX which contains F_ν but does not contain μ. As noted above, however, $C_{RX}(F_\nu) \subseteq \delta^{-1}[G_\nu]$ and, by definition, $\mu \in C_{RX}(F_\nu)$. Thus, $\delta(\mu)$ cannot be any element of RX, and so, must be an element of X.

Theorem 1, above, is the same as (and the construction is similar to the construction for) the primary result of [1]. The reason for presenting this alternative construction is that its simplicity permits us to show that in many cases the embedding of the remainder is not a Wallman extendible function, a result that would be quite difficult to obtain using the relatively complex construction of [1].

<u>Theorem 2</u>: If X is not a W-complete space, (see [3]) then the embedding γ_X from X to WS_X is not Wallman extendible.

proof: Suppose that the function γ_X has a Wallman extension ι^*. Since $\gamma_X[X]$ is contained in the closed subset $\{\alpha\} \times X^\omega$ of $E_\alpha \times X^\omega$, it follows that $\iota^*[WX]$ is also contained in $\{\alpha\} \times X^\omega$. (Since $\iota^{*-1}[\{\alpha\} \times X^\omega]$ is a closed subset of WX containing all of X, it must contain all of WX.) Hence, ι^* must carry each point of WX\X to the point (α, ω), which is a closed subset of WS_X, and so the inverse image $\iota^{*-1}(\alpha, \omega)$ is closed in WX. From [3], the only spaces with the property that WX\X is closed in WX are the W-complete spaces.

Since many (in fact most) spaces are not W complete, there are many examples of spaces (the spaces S_X for X not W-complete) for which the embedding of the Wallman remainder is not a Wallman extendible function.

We have now established that many embedding functions γ_{RX} are not Wallman extendible. This does not, however, indicate whether extensions, when they exist, must be unique. A cursory examination of the proof of lemma 1 might even lead one to conjecture that extendible embeddings must be WI functions. In fact, however, it is easily shown that such extensions need not be unique.

<u>Theorem 3</u>: If an embedding γ_{RX} is Wallman extendible, then the extension need not be unique.

proof: Let X denote the space $\mathbb{Q} \cup \{a,b\}$ with topology generated by the open subsets of the rational numbers $\mathbb{Q}$, together with the sets $\{ (\mathbb{Q} \cap U) \cup \{a\} : U$ is an open neighborhood of $\pi \}$ and the sets $\{ (\mathbb{Q} \cap U) \cup \{b\} : U$ is an open neighborhood of $\pi \}$. It is clear that both WX\{a} and WX\{b} are compact Hausdorff spaces (each being the Wallman compactification of a metric space.) Thus

either of the functions γ_{RX}: RX → WX\{a} and γ_{RX} : RX → WX\{b} has a Wallman extension. These extensions are Wallman extensions of the embedding γ_{RX}: RX → WX, but the first must carry some point of WRX onto b and the second must carry some point of WRX onto a, and so they cannot be the same.

It is not entirely coincidental that the above example involved a non-Hausdorff space.

<u>Theorem 4</u>: If X is a Hausdorff space and if the embedding γ_{RX}: RX → WX has a Wallman extension γ^*_{RX}: WRX → WX, then the extension is unique.

proof: Suppose that the extension γ^*_{RX} is not unique. Then there must exist another Wallman extension δ : WRX → WX of γ_{RX}. Since δ and γ^*_{RX} are distinct functions, there is some $\mu \in$ WRX such that $\delta(\mu) \neq \gamma^*_{RX}(\mu)$. The point μ cannot be in RX, and so from lemma 1, $\delta(\mu)$ and $\gamma^*_{RX}(\mu)$ are distinct elements of X. Since X is Hausdorff, these two points have disjoint open neighborhoods. Suppose that U and V are disjoint open sets of X, that $\delta(\mu) \in U$ and that $\gamma^*_{RX}(\mu) \in V$. Because $WX = C_X(X\backslash U) \cup C_X(X\backslash V)$, at least one of the sets $RX \cap C_X(X\backslash U)$ and $RX \cap C_X(X\backslash V)$ is an element of μ. Assume that $RX \cap C_X(X\backslash U) \in \mu$. The inverse image $\delta^{-1}[C_X(X\backslash U)]$ is a closed subset of WRX which contains the set $RX \cap C_X(X\backslash U)$ but not μ. However, $\mu \in C_{RX}(RX \cap C_X(X\backslash U))$, the set which must be contained in any closed (hence compact) set containing $RX \cap C_X(X\backslash U)$.

We note that it is not necessary that a space X be Hausdorff in order that the embedding γ_{RX} have a unique extension. As a cpunterexample, one need only consider the disjoint union of a noncompact normal space and an infinite cofinite space. We also note that we have, at present, no examples of Hausdorff spaces X for which the embedding γ_{RX} is not extendible. (The spaces S_X, constructed having remainders with non-extendible embeddings, are not Hausdorff.)

We now turn our attention to conditions implying the extendibility of the embedding γ_{RX}. We will show that if X is regular and if RX is Hausdorff, then γ_{RX} is extendible. We will first establish some preliminary results and notation.

Lemma 2: If X is a Hausdorff space and if μ is any element of WRX, then the intersection $\cap \{ cl_{WX}(A) : A \in \mu \}$ contains exactly one element.

proof: Since $\{ cl_{WX}(A) : A \in \mu \}$ is a collection of closed subsets of WX having the finite intersection property, it is clear that the intersection $\cap \{ cl_{WX}(A) : A \in \mu \}$ is nonempty. If the intersection $\cap \{ A : A \in \mu \}$ is nonempty, then μ must contain an element which is a singleton, and the closure of this set in WX can contain only one element. Suppose, then, that $\cap \{ A : A \in \mu \}$ is empty. For each $y \in RX$, there is some $F_y \in \mu$ such that $y \notin F_y$. Since F_y is closed in the subspace RX of WX, there is some closed subset G_y of WX such that $F_y = G_y \cap RX$. Clearly then, the intersection $\cap \{ cl_{WX}(A) : A \in \mu \} \subseteq G_y$ which does not contain the point y. From this we can conclude that if $\cap \{ A : A \in \mu \}$ is empty, then $\cap \{ cl_{WX}(A) : A \in \mu \} \subseteq X$. Suppose now that x and y are distinct elements of X. Since X is Hausdorff, there exist disjoint open neighborhoods U_x and U_y of x and y respectively. It is clear then, that $C_X(X\backslash U_x)$ and $C_X(X\backslash U_y)$ are closed subsets of WX whose union is all of WX. Since μ is an ultrafilter, at least one of the sets $RX \cap C_X(X\backslash U_x)$ and $RX \cap C_X(X\backslash U_y)$ is an element of μ. If $RX \cap C_X(X\backslash U_x)$ is an element of μ, then $\cap \{cl_{WX}(A) : A \in \mu \}$ is contained in $C_X(X\backslash U_x)$ which does not contain the point x. (Similarly for y.) Hence, the intersection $\cap \{ cl_{WX}(A) : A \in \mu \}$ cannot contain two points of X, and so must contain exactly one element.

The result in lemma 2 permits us to define a function ζ_X from WRX to WX. This function ζ_X is, in fact, the only possible candidate to be the Wallman extension of γ_{RX}. If ζ_X is continuous, then it is clearly a Wallman extension for γ_{RX}. If ζ_X is not continuous, then γ_{RX} has no Wallman extension.

Lemma 3: Suppose that $f : X \longrightarrow Y$ is continuous and that for each $\mu \in WX$ the intersection $\cap \{ C_Y(cl_Y(f[A])) : A \in \mu \}$ is a singleton. If the function f has a Wallman extension f^*, then for each $\mu \in WX$, the image $f^*(\mu) \in \cap \{ C_Y(cl_Y(f[A])) : A \in \mu \}$.

proof: Suppose that $f^*(\mu) \notin \cap \{ C_Y(cl_Y(f[A])) : A \in \mu \}$. Then there is some $A \in \mu$ such that $f^*(\mu) \notin C_Y(cl_Y(f[A]))$. Hence, the inverse image $f^{*-1}[C_Y(cl_Y(f[A]))]$ is a closed subset of WX containing A. Any closed subset of WX containing A must contain $C_X(A)$, and, therefore, μ, thus contradicting either the closure of the inverse image or the fact that $f^*(\mu) \notin C_Y(cl_Y(f[A]))$.

We now turn our attention to the Wallman remainders of regular spaces.

Lemma 4: If X is a regular space, if $\zeta_X(\mu) \in X$ and if U is an open subset of WX containing $\zeta_X(\mu)$, then there exists an open neighborhood $V_{\mu,U}$ of μ contained in $\zeta_X^{-1}[U]$.

proof: By the regularity of X, there exists an open subset V of X such that $\zeta_X(\mu) \in V \subseteq cl_X(V) \subseteq U$. Let A_V denote the closed subset $C(X\backslash V)$ of WX. It is clear that $RX \cap A_V$ is not an element of μ, since if it were, then $\zeta_X(\mu)$ would be an element of $cl_{WX}(RX \cap A_V)$ which is contained in A_V and implies $\zeta_X(\mu) \notin V$. Thus $\mu \in WRX \setminus C_{RX}(RX \cap A_V)$. If ν is any element of $WRX \setminus C_{RX}(RX \cap A_V)$ then there is some element $F_\nu \in \nu$ disjoint from $RX \cap A_V$. By the definition of ζ_X, the image $\zeta_X(\nu)$ is contained in $cl_{WX}(F_\nu)$ which must be contained in $cl_{WX}(V) \subseteq U$. Thus $V_{\mu,U} = WRX \setminus C_{RX}(RX \cap A_V)$ is an open neighborhood of μ contained in $\zeta_X^{-1}[U]$.

Proposition 1: If X is a regular space, then the function ζ_X is a closed function.

proof: Suppose that A is a closed subset of WRX. Then there is some filter $\mathfrak{F}_A$ of closed subsets of RX such that $A = \cap \{ C_{RX}(B) : B \in \mathfrak{F}_A \}$. If $\mu \in cl_{WX}(\zeta[A]) \setminus \zeta_X[A]$, then either $\mu \in RX$ or $\mu \in X$. In the case that $\mu \in RX$, since $\mu \notin A$, there must be some $B_\mu \in \mathfrak{F}_A$ such that $\mu \notin B$. There is some closed subset G_B

of WX such that $B = RX \cap G_B$, and so, by the definition of the function ζ_X, we can conclude that $cl_{WX}(\zeta_X[A]) \subseteq cl_{WX}(\zeta_X[B]) \subseteq G_B$ and $\mu \in G_B$. Hence, if there is such a point μ, it must be an element of X. Consider now the collection $\{ B \cap cl_{WX}(V) : B \in \mathfrak{F}_A,$ V a neighborhood of $\mu \}$. Either this collection has finite intersection property or there is some $B_\mu \in \mathfrak{F}_A$ and some neighborhood V of μ such that $cl_{WX}(V)$ is disjoint from B_μ. If the collection has finite intersection property, then it is contained in some element $\tau \in WRX$. Since τ contains each B in $\mathfrak{F}_A$, it is an element of $\cap \{ C_{RX}(B) : B \in \mathfrak{F}_A \}$, which is equal to A. If $\zeta_X(\tau) \neq \mu$, then there exists some element $D \in \tau$ such that $\mu \notin cl_{WX}(D)$. From [4], there exist disjoint open sets U_D and V_D in WX such that $\mu \in U_D$ and $cl_{WX}(D) \subseteq V_D$. Since $cl_{WX}(U_D) \cap RX$ must be an element of τ, such disjoint open sets cannot exist, and so $\zeta_X(\tau)$ would be equal to μ which would contradict the assumption that $\mu \in cl_{WX}(\zeta[A]) \setminus \zeta_X[A]$. Thus, if such an element exists, then there is some $B_\mu \in \mathfrak{F}_A$ and some neighborhood V of μ such that $cl_{WX}(V)$ is disjoint from B_μ. This, however, implies that $cl_{WX}(\zeta_X[A]) \subseteq WX \setminus V$, and, hence, that $\mu \notin cl_{WX}(\zeta_X[A])$, and so, that no such μ can exist. Hence, the image $\zeta_X[A]$ must be closed.

This proposition has an immediate corollary:

<u>Corollary</u>: If X is regular and if the embedding γ_{RX} is Wallman extendible, then γ_{RX} is a WC function (see [5].)

This brings us to the our previously announced result.

<u>Theorem 5</u>: If X is a regular space and if RX is a Hausdorff space, then the embedding function γ_{RX} is a WC function.

proof: With the above corollary, the only thing we need prove is that RX being Hausdorff implies that ζ_X is a continuous function. Suppose, then, that $\zeta_X(\mu) \in U$ an open subset of WX. If $\zeta_X(\mu) \in X$, then (from [4]) for each $\nu \in WX\setminus U$ there exist disjoint open neighborhoods U_ν and V_ν of $\zeta_X(\mu)$ and ν respectively. The collection $\{ V_\nu : \nu \in WX\setminus U \}$ is an open cover of the compact set $WX\setminus U$ and thus contains a finite subcover $\{ V_\nu : \nu \in \mathfrak{F} \}$. The

intersection $U_{\mathfrak{F}} = \cap \{ U_\nu : \nu \in \mathfrak{F} \}$ is an open neighborhood of μ disjoint from $V_{\mathfrak{F}} = \cup \{ V_\nu : \nu \in \mathfrak{F} \}$. From the definition of the function ζ_X, we know that $RX \cap (WX\setminus U_{\mathfrak{F}})$ is not an element of μ, and so there must be some element $A \in \mu$ disjoint from $RX \cap (WX\setminus U_{\mathfrak{F}})$. Then $WRX\setminus C_{RX}(WX\setminus U_{\mathfrak{F}})$ is an open subset of WRX containing μ. For any $\nu \in WRX\setminus C_{RX}(WX\setminus U_{\mathfrak{F}})$, there will be some $A_\nu \in \nu$ disjoint from $RX \cap C_{RX}(WX\setminus U_{\mathfrak{F}})$. The image $\gamma_{RX}[A_\nu]$ is contained in $U_{\mathfrak{F}}$, and so $cl_{WX}(\gamma_{RX}[A_\nu])$ must be contained in $WX\setminus V_{\mathfrak{F}}$, which, in turn, is contained in U. Hence μ has an open neighborhood in WRX, $(WRX\setminus C_{RX}(WX\setminus U_{\mathfrak{F}}))$, which is contained in $\zeta_X^{-1}(U)$, and, thus, ζ_X is continuous.

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