On Semigroup rings which are Marot Rings

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In this note we obtain a necessary and sufficient condition for a semi group ring to be a Marot Ring. In fact we have proved all commutative semi group ring is a Marot Ring. For more properties of semi group rings please refer [1].

The author in [2] calls a commutative ring with identity to be a Marot ring if each regular ideal of R is generated by regular element of R. By a regular element of R the author means a non-zero divisor of the ring R. He calls an ideal containing regular elements to be a regular ideal. For more properties about Marot rings please refer [2].

Throughout this paper S denotes a commutative semigroup and K a commutative ring. KS the semigroup ring of S over K.

Theorem 1. KS is a Marot ring with no divisors of zero if and only if S is an ordered commutative semigroup with no zero divisors and S has no elements of finite order and K is an integral domain.

Proof. Suppose KS is a Marot ring with no divisors of zero, since KS is commutative so is S and K as \( S \subseteq KS \) and \( K \subseteq KS \). Further both S and K cannot have divisors of zero.

Conversely if S is a commutative ordered semigroup with no divisors of zero and K and integral domain, clearly KS is a commutative domain hence a Marot ring.

Proposition 2. Let K be a field. S a commutative semigroup having no proper zero divisors but element of finite order. Then KS is a Marot Ring.

Proof. The semigroup ring KS has nontrivial divisors of zero (for if S has an element of finite order \( s^n = 1 \ (s \neq 1) \), \( \mod (s-1) \ (s^{n+1} + s^{n+1} + \ldots + 1) = 0 \)).

To show KS is a Marot ring we need only show (1) if I is a regular ideal generated by
a regular element then I has no nontrivial divisors of zero (ii) if I is a regular ideal generated by a divisor of zero then I has no nontrivial regular element.

Proof of (i) Suppose I is generated by \( \alpha \) a regular element in KS. If possible let \( \beta \in I \) such that \( \beta \gamma = 0 \) (\( \beta \neq 0 \), \( \gamma \neq 0 \)) that is \( \beta \) is a non-trivial divisor of zero. Now \( \beta \in I \) and \( \alpha \) generates I so \( \beta = \sum \alpha \delta_i \) or

\[
\beta \gamma = \sum \alpha \delta_i \gamma = \sum \alpha \gamma \delta_i = 0
\]

That \( \alpha \) is a divisor of zero a contradiction. Hence I cannot contain divisors of zero.

Proof of (ii). Suppose I be a regular ideal of KS, but be generated by a zero divisor \( \alpha \in I \). I is regular so I has regular elements also let \( \beta \) be a regular element of I.

\[
\beta = \sum \alpha \delta_i,
\]

we have \( \alpha \gamma = 0 \) as \( \alpha \) is a divisor of zero. So

\[
\beta \gamma = \sum \alpha \delta_i \gamma = \sum \alpha \gamma \delta_i = 0
\]

implying \( \beta \) is also a divisor of zero a contradiction to our assumption \( \beta \) is a regular element of I. So I cannot contain regular element when I is generated by a zero divisor. Thus KS is a Marot Ring.

Proposition 3. Let \( K \) be a field, KS the semi group ring of \( S \) over \( K \) be a Marot ring with divisors of zero. Then \( S \) is a semi group either having elements of finite order or a semi group having divisors of zero or both.

Proof. \( K \subseteq KS \) and KS is a Marot ring with divisors of zero and \( K \) is a field so \( S \) has elements of finite order or \( S \) has zero divisors or both.

Proposition. Let KS be a Marot ring with divisor of zero and \( S \) be a ordered semigroup without divisors of zero then \( K \) has proper divisors of zero.

Proof. Since \( K \subseteq KS \) and KS is commutative and as KS is a Marot ring, \( K \) is a commutative structure. Given \( S \) is ordered with no divisors of zero. But given KS has divisors of zero, so to prove K has divisor of zero.

\[
suppose \ a\beta = 0 \ where \ a = \sum_{i=1}^{n} a_i s_i \ and
\]

\[
\beta = \sum_{j=1}^{m} b_j h_j \ where \ a_i, \ b_j \in K (a_i = 0, b_j = 0) and s_1, s_2, \ldots , s_n
\]

and \( h_1, h_2, \ldots , h_m \) are respectively mutually distinct elements of \( S \). To prove \( a_i h_j = 0 \) for all \( i = 1, 2, \ldots , n \) and \( j = 1, 2, \ldots , m \). If \( a = n = 1 \), nothing to prove. Suppose \( n \geq 2 \), \( m \geq 2 \). As \( S \) is ordered and \( s_1, s_2, \ldots , s_n \) and \( h_1, h_2, \ldots , h_m \) are mutually distinct, we may assume \( s_1 < s_2 < \ldots < s_n \), \( h_1 < h_2 < \ldots < h_m \). We have

\[
(1) \ a\beta = \sum a_i b_j s_i h_j = 0 \ and
\]

1 \( i \leq n \)

1 \( j \leq m \)

\( s_j h_j \) is the 'smallest among' \( s_j h_j \)' i.e., we have

\( s_j h_j < s_j h_j \) for any \( i, j \) with \( i < j, 1 < k \). Thus we should have \( a_i b_j = 0 \).

To simplify the further expression of our proof, we shall use the following expressions in pairs of indices \( i, j \), \( i', j' \) ... where

\[ i, i' \ldots \in \{1, 2, \ldots , n\}, j, j' \in \{1, 2, \ldots , m\} \].

These two pairs are ordered according to the 'magnitudes' of \( s_i h_j, s_j h_j, \ldots \); we shall say namely \( (i, j) \) is smaller than \( (i', j') \) and write \( (i, j) < (i', j') \) when \( s_i h_j < s_j h_j ; (i, j) \) is called equivalent to \( (i', j') \), written \( (i, j) = (i', j') \) when \( s_i h_j = s_j h_j \). From \( i < i' \) follows obviously \( (i, j) < (i', j) \), and from \( (i, j) < (i', j') \), \( (i', j') \) follows \( (i, j) < (i', j') \). We shall prove \( a_i b_j = 0 \) following 'the magnitudes' of \( (i, j) \) beginning from the smallest pair \( (1, 1) \). A pair \( (i, j) \) will be called
settled, if \( a_i b_j = 0 \) has been proved. Thus \((1,1)\) is settled, and in proving
\[ a_{i_0} b_{j_0} = 0 \]
for a fixed pair \((i_0, j_0)\), we can obviously assume that all
\((i, j)\) are settled for \((i, j) < (i_0, j_0)\). Let \((i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)\) be
the set of all unsettled pairs which are equivalent to \((i_0, j_0)\). From (1) follows.

\[
(2) \quad a_{i_k} b_{j_k} + a_{i_p} b_{j_p} = 0
\]

We have nothing more to prove if \( p = 1 \). So let \( p > 2 \) and \( i_1 < i_2 < \ldots < i_p \).
Then we have for \( K > 2 \) \((i_1, j_k) < (i_k, j_k) = (i_0, j_0)\) so that \((i_1, j_k)\) is settled
by our assumption and \( a_{i_k} b_{j_k} = 0 \) whence follows \( b_{j_k} a_{i_k} = 0 \) as \( K \) is commutative.

Multiplying (2) by \( a_{i_k} \) from right, we obtain \( a_{i_k} b_{j_k} = 0 \) i.e., \((i_1, j_1)\) is settled
and we can proceed further.

**Theorem 5.** The semigroup ring \( KS \) is a Marot ring with nontrivial divisors of zero if and
only if \((1) \) \( S \) is a finite commutative semigroup without divisors of zero and \( K \) is a field,
or \((2) \) \( S \) is a ordered semigroup without divisors of zero and \( K \) is a commutative ring with
divisors of zero or \((3) \) \( S \) is any commutative semigroup without divisors of zero and \( K \) ring
any with divisors of zero.

**Proof.** Follows from the above three propositions.

**Problem.** If \( S \) is a commutative semigroup and \( K \) a commutative ring with unit. Can \( RS \) have
nontrivial regular ideals?

**REFERENCES**
