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On Semigroup rings which are Marot Rings

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In this note we obtain a necessary and sufficient condition for a semi group ring to be a Marot Ring. In fact we have proved all commutative semi group ring is a Marot Ring. Fore more properties of semi group rings please refer [1].

The author in [2] calls a commutative ring with identity to be a Marot ring if each regular ideal of R is generated by regular element of R. By a regular element of R the author means a non-zero divisor of the ring R. He calls an ideal containing regular elements to be a regular ideal. For more properties about Marot rings please refer [2].

Throughut this paper S denotes a commutative semigroup and K a commutative ring. KS the semigroup ring of S over K.

Theorem 1. KS is a Marot ring with no divisors of zero if and only if S is a ordered commutative semigroup with no zero divisors and S has no elements of finite order and K is an integral domain.

Proof. Suppose KS is a Marot ring with no divisors of zero, since KS is commutative so is S and K as $S \subseteq KS$ and $K \subseteq KS$. Further both S and K cannot have divisors of zero.

Conversely if S is a commutative ordered semigroup with no divisors of zero and K and integral domain, clearly KS is a commutative domain hence a Marot ring.

Proposition 2. Let K be a field. S a commutative semigroup having no proper zero divisors but element of finite order. Then KS is a Marot Ring.

Proof. The semigroup ring KS has nontrivial divisors of zero (for if S has an element of finite order

$$s^n = 1 \ (s \neq 1 \text{ or } = 0 \ , \ \ s \in S). \ So (s-1) \ (s^{n-1} + s^{n-2} + \dots + 1) = 0$$

To show KS is a Marot ring we need only show (i) if I is a regular ideal generated by a regular element then I has no nontrivial divisors of zero (ii) if I is a regular ideal generated by a divisor of zero then I has no nontrivial regular element.

Proof of (i) Suppose I is generated by α a regular element in KS. If possible let $\beta \in I$ such that $\beta\gamma = 0$ $(\beta \neq 0, \gamma \neq 0)$ that is β is a nontrivial divisor of zero. Now $\beta \in I$ and α generates I so $\beta = \Sigma \alpha \delta_i$ or

$$\beta\gamma = \sum_i \alpha \delta_i \gamma = \sum_i \alpha \gamma , \delta_i$$

$$= \alpha\gamma (\Sigma \delta_i) = \alpha (\gamma \Sigma \delta_i) = 0$$

That α is a divisor of zero a contradiction. Hence I cannot contain divisors of zero.

Proof of (ii). Suppose I be a regular ideal of KS, but be generated by a zero divisor $\alpha \in I$. I is regular so I has regular elements also lef β be a regular element of I.

$\beta = \sum_i \alpha \delta_i$, we have $\alpha \gamma = 0$ as α is a divisor of zero. So

$\beta \gamma = \Sigma \alpha \delta_i \gamma = \sum_i \alpha \gamma \delta_i = 0$ implying β is also a divisor of zero a contradiction to our assumption β is a regular element of I. So I cannot contain regular element when I is generated by a zero divisor. Thus KS is a Marot Ring.

Proposition 3. Let K be a field. KS the semi group ring of S over K be a Marot ring with divisors of zero. Then S is a semigroup either having elements of finite order or a semi group having divisors of zero or both.

Proof. $K \subseteq KS$ and KS is a Marot ring with divisors of zero and K is a field so S has elements of finite order or S has zero divisors or both.

Proposition. Lef KS be a Marot ring with divisor of zero and S be a ordered semigroup without divisors of zero then K has proper divisors of zero.

Proop. Since $K \subseteq KS$ and KS is commutative and as KS is a Marot ring, K is a commutative structure. Given S is ordered with no divisors of zero. But given KS has divisors of zero, so to prove K has divisor of zero.

suppose $\alpha \beta = 0$ where $\alpha = \sum_{i=1}^{n} a_i s_i$ and

$\beta = \sum_{j=1}^{m} b_j h_j$ where $a_i , b_j \in K$ $(a_i \neq 0, b_j \neq 0)$ and $s_1, s_2, \ldots, s_n$ and $h_1, h_2, \ldots, h_m$ are respectively mutually distinct elements of S. To prove $a_i b_i = 0$ for all $i = 1,2, \ldots, n$ and $j = 1,2, \ldots, m$.

If $m = n = 1$, nothing to prove. Suppose $n \geq 2$, $m \geq 2$. As S is orderd and $s_1, s_2, \ldots, s_n$ and $h_1, h_2, \ldots, h_m$ are mutually distinct, we may assume $s_1 < s_2 < \ldots < s_n$, $h_1 < h_2 < \ldots < h_m$. We have

(1) ... $\alpha\beta = \Sigma a_i b_j s_i h_j = 0$ and

$$1 \leq i \leq n$$
$$1 \leq j \leq m$$

$s_1 h_1$ is the 'smallest among $s_i h_j$' i.e., we have $s_1 h_1 < s_i h_j$ for any i,j with $1 < i$, $1 < k$. Thus we should have $a_1 b_1 = 0$.

To simplify the further description of our proof, we shall use the following expressions in pairs of indices (i,j), (i',j') ... where $i, i', \ldots \in \{1,2,\ldots,n\}$, $j, j' \in \{1,2,\ldots,m\}$. These nm pairs are ordered according to the 'magnitudes of' $s_i h_j$, $s_i h_{j'}$, ...; we shall say namely (i,j) is smaller than (i', j') and write $(i,j) < (i', j')$ when $s_i h_j < s_{i'} h_{j'}$; (i,j) is called equivalent to (i', j'), written $(i, j) \sim (i', j')$, when $s_i h_j = s_{i'} h_{j'}$. From $i < i'$ follows obviously $(i, j) < (i', j)$, and from $(i, j) < (i', j')$, $(i', j') \sim (i'', j'')$ follows $(i, j) < (i'', j'')$. We shall prove $a_i b_j = 0$ following 'the magnitudes' of (i,j) beginning from the smallest pair $(1,1)$. A pair (i, j) will be called settled, if $a_i b_j = 0$ has been proved. Thus (1,1) is settled, and in proving $a_{i_o} b_{j_o} = 0$ for a fixed pair (i_o, j_o), we can obviously assume that all (i, j) are settled for $(i, j) < (i_o, j_o)$. Let $(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p$ be the set of all unsettled pairs which are equivalent to (i_o, j_o). From (1) follows.

$$(2) \qquad a_{i_1} b_{j_1} + \qquad + a_{i_p} b_{j_p} = 0$$

We have nothing more to prove if $p = 1$. So let $p > 2$ and $i_1 < i_2 < \ldots < i_p$. Then we have for $K > 2$ $(i_1, j_K) < (i_K, j_K) - (i_o, j_o)$ so that (i_1, j_K) is settled by our assumption and $a_{i_1} b_{j_K} = 0$ whence follows $b_{K_j} a_{i_1} = 0$ as K is commutative.

Multiplying (2) by a_{i_1} from right, we ontain $a_{i_1} b_{j_1} = 0$ i.e., (i_1, j_1) is settled and we can proceed further.

Theorem 5. The semigroup ring KS is a Marot ring with nontrivial divisors of zero if and only if (i) S is a finite commutative semigroup without divisors of zero and K is a field, or (2) S is a ordered semigroup without divisors of zero and K is a commutative ring with divisors of zero or (3) S is any commutative semigroup without divisors of zero and K ring any with divisors of zero.

Proof. Follows from the above three propositions.

Problem. If S is a commutative semigroup and R a commutative ring with unit. Can RS have nontrivial regular ideals?

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