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Some topological properties of C-normality

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Abstract. A topological space X is C-normal if there exists a bijective function $f : X \to Y$, for some normal space Y, such that the restriction $f\restriction_C : C \to f(C)$ is a homeomorphism for each compact $C \subset X$. The purpose of this work is to extend the known classes of C-normal spaces and clarify the behavior of C-normality under several usual topological operations; in particular, it is proved that C-normality is not preserved under closed subspaces, unions, continuous and closed images, and inverse images under perfect functions. These results are used to answer some questions raised in [1], [2] and [6].
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Algunas propiedades topológicas de la C-normalidad

Resumen. Un espacio topológico X es C-normal si existe una función biyectiva $f : X \to Y$, para algún espacio normal Y, tal que la restricción $f\restriction_C : C \to f(C)$ es un homeomorfismo para cada compacto $C \subset X$. El propósito de este trabajo es extender las clases conocidas de los espacios C-normales y aclarar el comportamiento de C-normalidad bajo varias operaciones topológicas habituales; en particular, se demuestra que la normalidad C no se conserva bajo subespacios cerrados, uniones, imágenes continuas y cerradas e imágenes inversas bajo funciones perfectas. Estos resultados se utilizan para responder algunas preguntas planteadas en [1], [2] y [6].
Palabras clave: Normalidad, compacidad local, epi-normalidad, compacidad.

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1. Introduction

In 2012 Arhangel'skii proposed the study in General Topology of two variants of normality; C-normality and epi-normality. Years later AlZahrani and Kalantan published a study of the behavior of these two topological properties and their relations with other normal-type properties (see [1],[6]).

At the beginning of this work we present a systematic study of the classes C-$\mathcal{P}$ and epi-$\mathcal{P}$ of topological spaces. These classes are defined in a similar way to C-normality and epi-normality, but considering an arbitrary topological property $\mathcal{P}$ instead of normality. We show that the classes C-$\mathcal{P}$ and epi-$\mathcal{P}$ are hereditary (additive or productive) when $\mathcal{P}$ is hereditary (additive or productive, respectly). Then we apply these results to study C-normal spaces; we extend the known classes of C-normal spaces by showing that they include products of locally compact spaces and locally Lindelöf spaces. We also describe some specific examples. In [6] Saeed showed the existence of a Tychonoff space which is not C-normal; we use some spaces associated with such example to prove that C-normality is not preserved under closed subspaces, unions of subspaces, continuous and closed images, and perfect preimages. This shows that the categorical behavior of C-normality is very different from normality's categorical behavior, and answers some questions posed in [1]. We conclude the work comparing some characteristics of C-normality and epi-normality.

2. Notation

Throughout the text all spaces under consideration will be assumed to be Hausdorff. The symbol ω represents the first infinite ordinal and ω_1 is the first uncountable ordinal. The continuum is denoted by $\mathfrak{c}$. The set of natural numbers is denoted by $\mathbb{N}$ and the symbol $\mathbb{R}$ stands for the set of real numbers.

We say that a space X is a k-space if a set $U \subset X$ is open if, an only if, $U \cap C$ is open in C for every compact $C \subset X$. The space X is *locally Lindelöf* if for each point x in X there is a neighborhood U of x which is Lindelöf. The space X is *Urysohn* if for each pair of different points $x, y \in X$ there exist open sets $U, V \subset X$ satisfying $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Given a space X, we denote as $A(X)$ the *Alexandroff duplicate* $X \cup X'$ of X, where X' is a disjoint copy of X and there exist a bijective assignment $x \mapsto x'$ from X onto X'. Given a set $U \subset X$ we choose $U' = \{x'\}_{x \in U}$. The topology of $A(X)$ is defined as follows. All points of X' are isolated and a point $x \in X$ has as a basis of open neighborhoods the family of all sets of the form $U \cup U' \setminus \{x'\}$, where U is a open neighborhood of x in X.

All non stated concepts and notation can be understood as in [5].

3. The classes of epi-$\mathcal{P}$ and C-$\mathcal{P}$ spaces

The following notions describe two different ways in which we can extend the class of all topological spaces satisfying a given property.

Definition 3.1. Let $\mathcal{P}$ be a topological property.

- A topological space X is called *epi-*$\mathcal{P}$ if there exists a bijective continuous function $f : X \to Y$ for some space Y which satisfies $\mathcal{P}$.
- A topological space X is C-$\mathcal{P}$ if there exists a bijective function $f : X \to Y$, where Y has property $\mathcal{P}$, and $f \restriction_C : C \to f(C)$ is a homeomorphism for each compact $C \subset X$.

Given a topological property $\mathcal{P}$, since every bijective continuous function defined on a compact Hausdorff space is a homeomorphism onto its image, all epi-$\mathcal{P}$ spaces are C-$\mathcal{P}$. The other implication is not always true, for example when $\mathcal{P}$ coincides with normality (see Example 6.5). The following result gives us a condition under which these two notions are equivalent; the proof follows since for a k-space X a function $f : X \to Y$ is continuous if, and only if, $f \restriction_C$ is continuous for each compact $C \subset X$ (see [5, Theorem 3.3.21]).

Proposition 3.2. *If $\mathcal{P}$ is a topological property and X is a k-space, then X is C-$\mathcal{P}$ if, and only if, X is epi-$\mathcal{P}$.*

The classes C-$\mathcal{P}$ and epi-$\mathcal{P}$ can coincide, for example when a space X satisfies $\mathcal{P}$ if, and only if, every compact subset of X is metrizable.

If a topological property $\mathcal{P}$ implies another topological property $\mathcal{Q}$, then all epi-$\mathcal{P}$ (C-$\mathcal{P}$) spaces are epi-$\mathcal{Q}$ (C-$\mathcal{Q}$). Besides, if $\mathcal{P}$ and $\mathcal{Q}$ are different properties, the class of epi-$\mathcal{P}$ spaces and the class of epi-$\mathcal{Q}$ spaces can coincide; for example, when $\mathcal{Q}$ is the class of epi-$\mathcal{P}$ spaces the class of epi-$\mathcal{P}$ spaces coincides with the class of epi-$\mathcal{Q}$ spaces. Similarly, the classes C-$\mathcal{P}$ and C-$\mathcal{Q}$ can coincide, as we will show now.

Theorem 3.3. *If $\mathcal{P}$ is a topological property, the class of C-$\mathcal{P}$ spaces and the class of C-(C-$\mathcal{P}$) spaces coincide.*

Proof. It is sufficient to prove that every C-(C-$\mathcal{P}$) space is C-$\mathcal{P}$. Suppose that there exists a bijective function $f : X \to Y$, where Y is C-$\mathcal{P}$ and $f \restriction_C : C \to f(C)$ is a homeomorphism for each compact subspace $C \subset X$; we shall prove that X is C-$\mathcal{P}$. As Y is C-$\mathcal{P}$, there exists a space Z with property $\mathcal{P}$ and a bijective function $g : Y \to Z$ such that $g \restriction_D : D \to g(D)$ is a homeomorphism for each compact subspace $D \subset Y$. We claim that $g \circ f$ witnesses that X is C-$\mathcal{P}$. Indeed, let $C \subset X$ compact. Since $f \restriction_C : C \to f(C)$ is a homeomorphism, the space $D = f(C)$ is compact. It follows that $g \restriction_D : D \to g(D)$ is a homeomorphism. Thus $(g \circ f) \restriction_C = (g \restriction_D) \circ (f \restriction_C) : C \to g \circ f(C)$ is a homeomorphism and, since Z has property $\mathcal{P}$, we conclude that X is C-$\mathcal{P}$. ☑

In what follows we will analyze some properties of the classes *epi-*$\mathcal{P}$ and C-$\mathcal{P}$ inherited from the property $\mathcal{P}$.

Theorem 3.4. *If a property $\mathcal{P}$ is hereditary, then the classes C-$\mathcal{P}$ and epi-$\mathcal{P}$ are closed under arbitrary subspaces.*

Proof. We will show the case of C-$\mathcal{P}$ spaces; the proof for the epi-$\mathcal{P}$ spaces is similar. Let A be a subset of X. As X is a C-$\mathcal{P}$ space, there exists a bijective function $f : X \to Y$, where Y has property $\mathcal{P}$, such that $f \restriction_C : C \to f(C)$ is a homeomorphism for each compact subspace $C \subset X$. Since the property $\mathcal{P}$ is hereditary, the space $f(A)$ has property $\mathcal{P}$. It is clear that $f\restriction_A : A \to f(A)$ is bijective. Since any compact subspace of A is compact in X, the restriction $(f\restriction_A)\restriction_C = f\restriction_C$ is a homeomorphism for each compact subspace $C \subset A$. Thus A is C-$\mathcal{P}$. ☑

Theorem 3.5. *If κ is a cardinal and $\mathcal{P}$ is a κ-productive property, then the classes C-$\mathcal{P}$ and epi-$\mathcal{P}$ are closed under products of κ-factors.*

Proof. We will prove the result for the class C-$\mathcal{P}$; the case of the class epi-$\mathcal{P}$ is similar. Let $\{X_s\}_{s\in S}$ be a family of C-$\mathcal{P}$ spaces where S has cardinality κ. For each $s \in S$ let $f_s : X_s \to Y_s$ be a bijective function for some Y_s with property $\mathcal{P}$ such that $f_s\restriction_{C_s} : C_s \to f_s(C_s)$ is a homeomorphism for each compact subspace $C_s \subset X_s$. Note that $f = \prod_{s\in S} f_s : \prod_{s\in S} X_s \to \prod_{s\in S} Y_s$ is bijective. Besides, as $\mathcal{P}$ is a κ-productive property, it follows that $\prod_{s\in S} Y_s$ has property $\mathcal{P}$. Given a compact set $C \subset \prod_{s\in S} X_s$, notice that $D = \prod_{s\in S} \pi_s(C)$ is compact, and so the function $f\restriction_D = \prod_{s\in S} f_s\restriction_{\pi_s(C)} = D \to f(D)$ is a homeomorphism; consequently, $f\restriction_C : C \to f(C)$ also is a homeomorphism. Thus, the product $\prod_{s\in S} X_s$ is C-$\mathcal{P}$. ☑

Theorem 3.6. *If $\mathcal{P}$ is a κ-aditive property, then the classes C-$\mathcal{P}$ and epi-$\mathcal{P}$ are closed under disjoint sums of κ-factors.*

Proof. We will prove the result for the class of C-$\mathcal{P}$ spaces, the other case is similar. Let $\{X_s\}_{s\in S}$ be a family of spaces C-$\mathcal{P}$ where $|S| = \kappa$. For each $s \in S$, let $f_s : X_s \to Y_s$ be a bijective function for some space Y_s with property $\mathcal{P}$ such that $f_s\restriction_{C_s} : C_s \to f_s(C_s)$ is a homeomorphism for each compact subspace $C_s \subset X_s$. As $\mathcal{P}$ is a κ-additive property, it follows that $\bigoplus_{s\in S} Y_s$ has property $\mathcal{P}$. Besides, the function $\bigoplus_{s\in S} f_s : \bigoplus_{s\in S} X_s \to \bigoplus_{s\in S} Y_s$ is bijective. Now let $C \subset \bigoplus_{s\in S} X_s$ be a compact space, then the set $S_0 = \{s \in S : C \cap X_s \neq \emptyset\}$ is finite and $C_s = C \cap X_s$ is compact for each $s \in S_0$. Then $(\bigoplus_{s\in S} f_s)\restriction_C = \bigoplus_{s\in S_0} f_s\restriction_{C_s}$ is a homeomorphism, because $f_s\restriction_{C_s}$ is a homeomorphism for each $s \in S_0$. Thus, the disjoint sum $\bigoplus_{s\in S} X_s$ is C-$\mathcal{P}$. ☑

By an argument similar to the one used in the proof of Theorem 3.5 we can prove the following result.

Proposition 3.7. *Consider two properties $\mathcal{P}$ and $\mathcal{Q}$ such that $X \times Y$ has $\mathcal{P}$ when X has $\mathcal{P}$ and Y has $\mathcal{Q}$. Then $X \times Y$ is C-$\mathcal{P}$ when X is C-$\mathcal{P}$ and Y is C-$\mathcal{Q}$.*

Theorem 3.8. *Let $\mathcal{P}$ be a property preserved under Alexandroff duplicates; then $A(X)$ is C-$\mathcal{P}$ (epi-$\mathcal{P}$) whenever X is C-$\mathcal{P}$ (epi-$\mathcal{P}$).*

Proof. We will show the case of C-$\mathcal{P}$ spaces; the case for the epi-$\mathcal{P}$ spaces is similar. Let X be a C-$\mathcal{P}$ space; then, there exists a space Y with property $\mathcal{P}$ and a bijective function $f : X \to Y$ such that $f\restriction_C : C \to f(C)$ is a homeomorphism for each compact subspace $C \subset X$. Consider the Alexandroff duplicated $A(X)$ and $A(Y)$ of X and Y, respectly. Since Y has $\mathcal{P}$, the space $A(Y)$ also has $\mathcal{P}$. Define $F : A(X) \to A(Y)$ by $F(x) = f(x)$ and $F(x') = f(x)'$ for each $x \in X$, the natural function induced by f. Notice that F is a bijective function. Let $C \subset A(X)$ be a compact subspace. We shall prove that $F\restriction_C : C \to F(C)$ is a homeomorphism. Let $p : A(X) \to X$ be the function given by $p(x) = p(x') = x$, for each $x \in X$. Observe that p is continuous. For the compact set

$D = p(C)$ we have that $g = f \restriction_D$ is bijective and continuous. It is easy to verify that the natural function $G : A(D) \to A(g(D))$ induced by g, given by $G(x) = g(x)$ and $G(x') = g(x)'$ for each $x \in D$, also is bijective and continuous. As $A(D)$ is compact, the function G is a homeomorphism. We know that $C \subset A(D) \subset A(X)$, thus $F \restriction_C = G \restriction_C$ also is a homeomorphism. ☑

We consider now the following well known construction. Let X be an arbitrary space. Take $kX = X$. Define a topology on kX as follows. A set of kX is open if, and only if, its intersection with any compact subspace C of X is open in C. Then the space kX endowed with this topology is a k-space, has exactly the same compact subspaces that X, and induces the same topology that X on these compact subspaces. From these observations it is easy to conclude the following.

Proposition 3.9. *Let $\mathcal{P}$ be a topological property. A space X is C-$\mathcal{P}$ if, and only if, kX is C-$\mathcal{P}$.*

4. C-normal spaces

In this text we will be particularly interested in C-normality and some related properties. Notice that all epi-normal spaces, all C-compact spaces and all C-metrizable spaces are C-normal. We will provide another classes of spaces which are C-normal.

As is stated in Exercise 3.3.D from [5], every locally compact space is epi-compact, so we can apply Theorem 3.5 to obtain the following corollary.

Corollary 4.1. *If $\{X_s\}_{s\in S}$ is a family of locally compact spaces, then the product $\prod_{s\in S} X_s$ is epi-compact.*

Example 4.2. *The space of real numbers $\mathbb{R}$ is locally compact, because of Corollary 4.1 the product $\mathbb{R}^S$ is C-normal, for any set S. Moreover, if $\mathbb{S}$ is the Sorgenfrey line, then $\mathbb{S}^S$ admits a bijective continuous function onto $\mathbb{R}^S$, so we can apply Theorem 3.3 to see that $\mathbb{S}^S$ is C-normal. However, $\mathbb{R}^S$ is not normal when the set S is not countable (see [5, Exercise 2.3.E]) and $\mathbb{S}^S$ is not normal when S has at least two elements (see [5, Example 2.3.12]).*

Now we will deal with a notion more general than locally compactness, local Lindelöfness, in order to get more examples of C-normal spaces.

Theorem 4.3. *If X is regular and locally Lindelöf, then X is epi-Lindelöf.*

Proof. We must prove that X admits a bijective and continuous function onto a Lindelöf space. Let $Y = X \cup \{y\}$ where $y \notin X$. We define a topology in Y in the following way. The topology of Y is the minimal topology on Y which satisfies the following conditions:

1. It contains the topology of X.

2. It contains each set $U \subset Y$ such that $y \in U$ and whose complement $Y \setminus U$, is closed in X and has a neighborhood in X whose closure in X has the Lindelöf property.

As X is regular and locally Lindelöf, the space Y is T_1. We will verify now that Y is regular. Given $A \subset X$, along this proof $\overline{A}$ always refers to the closure of A in X. Take a point $x \in Y$ and a neighborhood U of x in Y. If $x \neq y$, since X is regular, we can suppose that $U \subset X$ and $\overline{U}$ is Lindelöf. By the regularity of X, there exists an open neighborhood V of x such that $x \in V \subset \overline{V} \subset U$. Notice that $\overline{V}$ is closed in Y. If $x = y$ we can suppose that $F = Y \setminus U$ is closed in X and has a neighborhood V in X whose closure $\overline{V}$ in X has the Lindelöf property. As $\overline{V}$ is normal, there exists an open set W in X such that $F \subset W \subset \overline{W} \subset V \subset \overline{V}$. It follows that $\overline{W}$ is closed in Y, and if $O = Y \setminus \overline{W}$, then $y \in O \subset \{y\} \cup \overline{O \setminus \{y\}} \subset Y \setminus W \subset Y \setminus F = U$, where $\{y\} \cup \overline{O \setminus \{y\}}$ is the closure of O in Y. Thus, the space Y is regular.

It is easy to verify that Y is Lindelöf. Fix $x \in X$ and consider the space Z which is obtained from Y identifying the points x and y, and let $q : Y \to Z$ be the quotient function associated with this identification. As Y is normal and q only identifies a closed set, the space Z is regular. Since q is continuous, the space Z is Lindelöf. Finally, is clear that the function $q \restriction_X : X \to Z$ is bijective, and hence this function witnesses that X is epi-Lindelöf. ☑

We now describe some examples of locally Lindelöf spaces, and hence C-normal spaces, which are neither locally compact nor normal.

Example 4.4. *Let X be a locally compact not normal space and let Y be a Lindelöf not locally compact space. Consider the space $X \times Y$. Note that $X \times Y$ is not normal, because it has a closed subspace homeomorphic to X which is not normal. Observe that $X \times Y$ is not locally compact, because it has a closed subspace homeomorphic to Y which is not locally compact. However, the product $X \times Y$ is locally Lindelöf, because the product of a compact space and a Lindelöf space is always Lindelöf. Thus, Theorem 4.3 implies that $X \times Y$ is C-normal. As a particular case, we can take X as the deleted Tychonoff plank and Y as the Sorgenfrey line.*

Example 4.5. *Consider the following variant of a Ψ-space. Let $\mathcal{A}$ be a maximal family of uncountable subsets of ω_1 such that $A \cap B$ is countable for each $A, B \in \mathcal{A}$. It is easy to deduce from the maximality of $\mathcal{A}$ that $|\mathcal{A}| \geq \omega_2$. Consider the space $\Psi_{\omega_1}(\mathcal{A}) = \omega_1 \cup \mathcal{A}$, where each point in ω_1 is isolated and every $A \in \mathcal{A}$ has as a basis of open neighborhoods the family $\{A \setminus C : C \in [\omega_1]^{<\omega_1}\}$. It follows immediately from the definition that $\Psi_{\omega_1}(\mathcal{A})$ is locally Lindelöf and not locally compact.*

We will prove that $\Psi_{\omega_1}(\mathcal{A})$ is not normal. Suppose on the contrary, that $\Psi_{\omega_1}(\mathcal{A})$ is normal. Let $\{\mathcal{A}_\alpha\}_{\alpha<\omega_1}$ be a partition of $\mathcal{A}$ in nonempty subsets and fix $A_\alpha \in \mathcal{A}_\alpha$ for each $\alpha < \omega_1$. Given $\alpha < \omega_1$, because of the normality of $\Psi_{\omega_1}(\mathcal{A})$ we can choose an uncountable subset B_α of ω_1 such that the subsets $\mathcal{A}_\alpha \cup B_\alpha$ and $(\mathcal{A} \setminus \mathcal{A}_\alpha) \cup (\omega_1 \setminus B_\alpha)$ form a partition of $\Psi_{\omega_1}(\mathcal{A})$ in open sets. We will construct a subset $\{x_\alpha\}_{\alpha<\omega_1}$ of ω_1 recursively as follows. Fix $x_0 \in A_0$, and if $\{x_\alpha\}_{\alpha<\beta}$ is defined for some $\beta < \omega_1$, fix $x_\beta \in A_\beta \setminus \bigcup_{\alpha<\beta}(\{x_\alpha\} \cup B_\alpha)$. Consider the uncountable set $B = \{x_\alpha\}_{\alpha<\omega_1}$; then the maximality of $\mathcal{A}$ implies that $A \cap B$ is uncountable for some $A \in \mathcal{A}$. We know that $A \in \mathcal{A}_\gamma$ for some $\gamma < \omega_1$. Since $\{A\} \cup B_\gamma$ is open, we must have that $(A \cap B) \setminus B_\gamma \subset A \setminus B_\gamma$ is countable, and hence $(A \cap B) \setminus B_\gamma \subset \{x_\alpha\}_{\alpha<\beta}$ for some $\gamma < \beta < \omega_1$. Since $A \cap B$ is uncountable, we can suppose that β is such that $x_\beta \in A \cap B$. However, the construction implies that $x_\beta \notin B_\gamma$, which is not possible. Thus, the space $\Psi_{\omega_1}(\mathcal{A})$ is not normal.

Question 4.6. Is there a locally normal regular space X which is not C-normal?

Proposition 4.7. *If any countable subspace of X is discrete, then X is C-normal.*

Proof. By [1, Corollary 1.4] it is sufficient to verify that all compact subsets of X are finite; namely, under such conditions any bijection onto a discrete space witnesses the C-normality of X. Let $A \subset X$ infinite. Take $B \subset A$ infinite and countable. Note that B is closed in X because $B \cup \{x\}$ is discrete for each $x \in X$. As B is closed, discrete and infinite, it follows that A is not compact. ☑

We now describe an example of a space in which all countable subsets are discrete, and hence a C-normal space, but which is not normal. This example was obtained by Shakhmatov (see [3, Example 1.2.5]).

Example 4.8. *Let $I^{\mathfrak{c}}$ be the Tychonoff cube of weight $\mathfrak{c}$. Let*

$$\Sigma I^{\mathfrak{c}} = \{x \in I^{\mathfrak{c}} : |\{\alpha < \mathfrak{c} : \pi_\alpha(x) \neq 0\}| \leq \omega\} \subset I^{\mathfrak{c}}.$$

Observe that $|\Sigma I^{\mathfrak{c}}| = \mathfrak{c}$, and take an enumeration $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ of the elements of $\Sigma I^{\mathfrak{c}}$ where each element appears $\mathfrak{c}$-many times. Moreover, take an enumeration $\{A_\alpha\}_{\alpha<\mathfrak{c}}$ of the elements from $[\mathfrak{c}]^{\leq\omega}$ where each element appears $\mathfrak{c}$-many times. For each $\alpha < \mathfrak{c}$ define a point $y_\alpha \in I^{\mathfrak{c}}$ by:

$$\pi_\beta(y_\alpha) = \begin{cases} \pi_\beta(x_\alpha), & \text{if } \beta \leq \alpha; \\ 1, & \text{if } \beta > \alpha, \beta \in A_\alpha; \\ 0, & \text{if } \beta > \alpha, \beta \notin A_\alpha. \end{cases}$$

As it is proved in [3, Example 1.2.5], the space $Y = \{y_\alpha\}_{\alpha<\mathfrak{c}} \subset I^{\mathfrak{c}}$ is dense in $I^{\mathfrak{c}}$, pseudocompact, and every countable subset of Y is discrete. As Y is pseudocompact but not countably compact, we conclude from [5, Theorem 3.10.21] that Y is not normal.

5. Operations with C-normal spaces

In [6] Saeed showed the existence of a Tychonoff space which is not C-normal. Such example is constructed as follows: Let 2^{ω_1} be the Cantor cube of size ω_1; the product of ω_1-many copies of the discrete two points space. Now consider the subspace

$$\Sigma 2^{\omega_1} = \{x \in 2^{\omega_1} : |x^{-1}(1)| \leq \omega\} \subset 2^{\omega_1}.$$

Then the product $2^{\omega_1} \times \Sigma 2^{\omega_1}$ is Tychonoff but not C-normal (see [6, Example 8]). This example provides us a compact space and a normal space whose product is not C-normal, so C-normality is not a productive property. However, we still do not know the answer to the following question.

Question 5.1. Is there a C-normal space X such that its square is not C-normal?

We know that normality is preserved under closed subspaces and closed continuous images. In the following examples we will show that C-normality is not necessary preserved in these cases.

Example 5.2. *There exists an epi-compact space containing a closed subspace which is not C-normal.*

Proof. Consider the cartesian product $Y = 2^{\omega_1} \times 2^{\omega_1}$ endowed with the product topology, and the cartesian product $X = 2^{\omega_1} \times 2^{\omega_1}$ endowed with the topology obtained from the product topology by adding $2^{\omega_1} \times \Sigma 2^{\omega_1}$ and its complement as open sets. Notice that X is epi-normal; indeed, the identity function $id : X \to Y$ is continuous and Y is compact. It is clear that $2^{\omega_1} \times \Sigma 2^{\omega_1}$ is closed in X. Moreover, the topology on $2^{\omega_1} \times \Sigma 2^{\omega_1}$ inherited from X coincides with the topology inherited from Y. Thus, $2^{\omega_1} \times \Sigma 2^{\omega_1}$ is a closed subspace of X which is not C-normal. ☑

Example 5.3. *There exists an epi-compact space admitting a closed continuous image which is not C-normal.*

Proof. Take the spaces X and Y as in Example 5.2. Now consider the function $f : X \to Y$ given by:

$$f(x) = \begin{cases} id(x), & \text{if } x \in 2^{\omega_1} \times \Sigma 2^{\omega_1}; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that f is continuous and $f(X) = 2^{\omega_1} \times \Sigma 2^{\omega_1}$. Besides, if F is closed in X, then either $f(F) = F \cap (2^{\omega_1} \times \Sigma 2^{\omega_1})$ or $f(F) = (F \cap (2^{\omega_1} \times \Sigma 2^{\omega_1})) \cup \{0\}$. It follows that f is a closed function. Finally, we know that X is epi-compact and $f(X)$ is not C-normal. ☑

Any closed continuous function is quotient; from the previous result we conclude that C-normality is not preserved under quotient functions. It happens that C-normality is also not preserved under open perfect preimages. Indeed, take the proyection $\pi : 2^{\omega_1} \times \Sigma 2^{\omega_1} \to \Sigma 2^{\omega_1}$ on the second factor. As 2^{ω_1} is compact, it follows from [5, Theorem 3.7.1.] that the function π is perfect. However, the space $\Sigma 2^{\omega_1}$ is C-normal while $2^{\omega_1} \times \Sigma 2^{\omega_1} = \pi^{-1}(\Sigma 2^{\omega_1})$ is not C-normal.

Question 5.4. Suppose $X \times K$ is C-normal for some compact K. Is it true that X is C-normal?

We will prove now that C-normality is not preserved under the union of two arbitrary subspaces; we will use an example obtained in [4].

Example 5.5. *There exists a non-C-normal space which is the union of a compact subspace and a locally compact subspace.*

Proof. Consider the topological product $(\omega_1 + 1) \times [0, 1]$, the subspace $R = \{\omega_1\} \times (0, 1)$, the space $X = \big((\omega_1 + 1) \times [0, 1]\big) \setminus R$, and the space $Y = X \times (\omega_1 + 1)$. Then the space Y is not C-normal (see [4]). Now, take $A = (\omega_1 + 1) \times \{0, 1\} \times (\omega_1 + 1)$ and $B = (\omega_1 + 1) \times (0, 1) \times (\omega_1 + 1)$. Clearly A is compact, B is locally compact, and $Y = A \cup B$. ☑

Question 5.6. Is there a non-C-normal Tychonoff space X which is the union of two C-normal closed subspaces?

Now we will answer in the positive the following question which is attributed to Arhangel'skii in [7]; Is there a normal space which is not C-paracompact?

Example 5.7. *There exists a normal space which is not C-paracompact.*

Proof. Consider the normal space $\Sigma 2^{\omega_1}$. We claim that $\Sigma 2^{\omega_1}$ is not C-paracompact. Suppose that the space $\Sigma 2^{\omega_1}$ is C-paracompact. Since 2^{ω_1} is C-compact, we can apply Proposition 3.7 and the fact that the product of a compact space and a paracompact space is paracompact (see [5, Theorem 5.1.36]), to conclude that $2^{\omega_1} \times \Sigma 2^{\omega_1}$ is C-paracompact and thus C-normal; which we know is not true. Thus, the space $\Sigma 2^{\omega_1}$ is not C-paracompact. ☑

Note that using the space described in examples 5.2 and 5.7 we can conclude that C-paracompactness is not inherited by closed subspaces. It is worth to mention that Example 5.7 also provides an epi-normal space which is not C-paracompact. This answers another question from [7].

6. Epi-normal spaces

It follows from Examples 5.2, 5.3 and 5.5 that epi-normality is not necessarily preserved under closed subspaces, unions, products, continuous and closed images, and inverse images of perfect functions. Now we will analyze other properties of epi-normal spaces.

Proposition 6.1. *Let X be an epi-normal space. If $g : C \to \mathbb{R}$ is a continuous function, where $C \subset X$ is compact, then there exists a continuous function $\hat{g} : X \to \mathbb{R}$ such that $\hat{g} \restriction_C = g$.*

Proof. Let $f : X \to Y$ be bijective and continuous, for some normal space Y. Notice that $f \restriction_C : C \to f(C)$ is a homeomorphism. As Y is normal, the function $h = g \circ (f \restriction_C)^{-1} : f(C) \to \mathbb{R}$ admits a continuous extension $\hat{h} : Y \to \mathbb{R}$. We consider the continuous function $\hat{g} = \hat{h} \circ f : X \to \mathbb{R}$. Notice that

$$\hat{g} \restriction_C = (\hat{h} \circ f) \restriction_C = (\hat{h} \restriction_{f(C)}) \circ (f \restriction_C) = g \circ (f \restriction_C)^{-1} \circ f \restriction_C = g$$

is the required extension of g. ☑

Corollary 6.2. *If X is epi-normal, then X is Urysohn.*

Proof. Given two distinct points $x, y \in X$, by Proposition 6.1 we can take a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 2$ and $f(y) = 5$. Then the open subsets $U = f^{-1}((1,3))$ and $V = f^{-1}((4,6))$ of X have disjoint closures and contain x and y, respectively. ☑

The following example shows that, in general, epi-normal spaces are not necessary regular.

Example 6.3. *Let $X = \mathbb{R}$ and consider the sequence $A = \{1/(n+1)\}_{n \in \mathbb{N}}$. Define a topology in X as the family of all sets of the form $U \setminus B$ where $B \subset A$ and U is open in the usual topology of $\mathbb{R}$. Clearly X is epi-normal, because its topology is finer than the usual topology. However, the space X is not regular, because $\{0\}$ and A cannot be separated by disjoint open subsets.*

Example 6.4. *There exists a space X which is neither C-normal nor Urysohn, but which is the union of two epi-normal closed subspaces.*

Proof. Let $X = (\mathbb{R}^2 \setminus \{0\}) \cup \{x_1, x_2\}$, where x_1 and x_2 do not belong to $\mathbb{R}^2$. Define a topology in X as follows. The space $\mathbb{R}^2 \setminus \{0\}$ endowed with its usual topology is an open subspace of X. Besides, for each $i = 1, 2$ the point x_i has as a basis of open neighborhoods the family of all sets of the form

$$U_{n,i} = \{x_i\} \cup \{(x, y) : x^2 + y^2 < 1/(n+1)^2 \text{ and } (-1)^i y > 0\},$$

where $n \in \mathbb{N}$. Note that x_1 and x_2 cannot be separated using neighborhoods with disjoint closures, thus X is not Urysohn. As an application of Corollary 6.2 we obtain that X is not epi-normal. Observe that X is Fréchet-Urysohn, so we can apply Proposition 3.2 to conclude that X is not C-normal. Choose $i \in \{1, 2\}$. Let $A_i = \{(x, y) : (-1)^i y \geq 0\} \setminus \{0\}$. Note that $A_i \cup \{x_i\}$ admits a bijective continuous function onto the subspace $A_i \cup \{0\}$ of $\mathbb{R}^2$ and hence is epi-normal. Therefore, the space $X = (A_1 \cup \{x_1\}) \cup (A_2 \cup \{x_2\})$ is the union of two closed epi-normal subspaces. ☑

It happens that C-normal spaces are not necessarily Urysohn, as the following example shows.

Example 6.5. *There exists a C-normal space which is not Urysohn.*

Proof. Consider the space ω_1 with the discrete topology. Let $L = \omega_1 + 1$ endowed with the following topology. The space ω_1 is open in L and ω_1 has as a basis of open neighborhoods the family of all sets $(\alpha, \omega_1]$, where $\alpha < \omega_1$. Consider the open subspace $O = L \times \omega_1$ of $L \times L$. Let $\{A_1, A_2\}$ be a partition of ω_1 into uncountable sets. Consider the space $X = O \cup \{x_1, x_2\}$, where $x_1, x_2 \notin L \times L$, endowed with the following topology. The space O is open in X and, for $i \in \{1, 2\}$ the point x_i has as a basis of open neighborhoods the family of all sets of the form $(U \cap (A_i \times \omega_1)) \cup \{x_i\}$ where U is a neigborhood of (ω_1, ω_1) in $L \times L$. Note that X is Hausdorff. Besides, the space X is not Urysohn because the points x_1 and x_2 cannot be separated by open sets in X with disjoint closures. However, if we take $Y = O \oplus \{x_1\} \oplus \{x_2\}$, then Y is normal and the restriction of the identity function from X onto Y to each compact subspace is a homeomorphism, that is, the space X is C-normal. ☑

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