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Properties of the Support of Solutions of a Class of 2-Dimensional Nonlinear Evolution Equations

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Abstract. In this work we consider equations of the form

$$\partial_t u + P(D)u + u^l \partial_x u = 0,$$

where $P(D)$ is a two-dimensional differential operator, and $l \in \mathbb{N}$. We prove that if u is a sufficiently smooth solution of the equation, such that $\operatorname{supp} u(0), \operatorname{supp} u(T) \subset [-B, B] \times [-B, B]$ for some $B > 0$, then there exists $R_0 > 0$ such that $\operatorname{supp} u(t) \subset [-R_0, R_0] \times [-R_0, R_0]$ for every $t \in [0, T]$.

Keywords: Nonlinear evolution equations, weighted Sobolev spaces, Carleman estimates.

MSC2010: 35Q53, 37L50,47J35.

Propiedades del soporte de soluciones de una clase de ecuaciones de evolución no lineales en dos dimensiones

Resumen. En este trabajo consideramos ecuaciones de la forma

$$\partial_t u + P(D)u + u^l \partial_x u = 0,$$

donde $P(D)$ es un operador diferencial en dos dimensiones, y $l \in \mathbb{N}$. Probamos que si u es una solución suficientemente suave de la ecuación, tal que $\operatorname{supp} u(0), \operatorname{supp} u(T) \subset [-B, B] \times [-B, B]$ para algún $B > 0$, entonces existe $R_0 > 0$ tal que $\operatorname{supp} u(t) \subset [-R_0, R_0] \times [-R_0, R_0]$ para todo $t \in [0, T]$.

Palabras clave: Ecuaciones de evolución no lineales, espacios de Sobolev con peso, estimativos Carleman.

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1. Introduction

In this note we study nonlinear evolution equations of the form

$$\partial_t u + P(D)u + u^l \partial_x u = 0, \tag{1}$$

where $P(D)u := \sum_{j=0}^{n} \sum_{j'=0}^{n-j} a_{jj'} \partial_x^j \partial_y^{j'} u$, $a_{jj'} \in \mathbb{C}$, with $a_{00} = 0$, and $n \in \{1, 2, 3, \dots\}$. Some well-known models belong to the class defined by (1) (see [1] and [17]). For instance, the Zakharov-Kuznetsov (ZK) equation, for which

$$P(D)u = \partial_x^3 u + \partial_x \partial_y^2 u,$$

and $l = 1$. The ZK equation is a bidimensional generalization of the Korteweg-de Vries (KdV) equation which is a mathematical model to describe the propagation of nonlinear ion-acoustic waves in magnetized plasma ([18]). Some aspects concerning the behavior of the solutions of the ZK equation has been studied in [3], [7], [13], [12], [14].

The class defined by (1) also includes the two dimensional Kawahara equation, for which

$$P(D)u = \alpha \partial_x u + \partial_x^3 u + \partial_x \partial_y^2 u - \partial_x^5 u,$$

where α is equal to 1 or 0 (see [11] and references therein), and the Kawahara-Burgers equation (see [10] and references therein). Both of them are perturbations of the (ZK) equation.

In 2011, Bustamante, Isaza and Mejía, in [6], proved that if the support of a sufficiently smooth solution of the ZK equation u is contained in a square at two different times, then the solution must vanish. To obtain this, they first prove that if the hypotheses mentioned are satisfied, then exists a square in which the support of u is contained for all times. Then, using a result obtained by Panthee in [16], they manage to prove that $u = 0$.

Our main result is a generalization of the one concerning the support of the solutions of the ZK equation achieved in [6]. Specifically, we extend it to the general case of $\mathbb{R}^2$, showed in equation (1), and we present it in detail in the following theorem.

Theorem 1.1. *Let $n \in \mathbb{N}$, and $P(D)$ the operator defined by*

$$P(D)u := \sum_{j=0}^{n} \sum_{j'=0}^{n-j} a_{jj'} \partial_x^j \partial_y^{j'} u, \quad \text{with } a_{jj'} \in \mathbb{C}, \text{ and } a_{00} = 0.$$

Suppose that $u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^\infty([0,T]; L^2(e^{2\beta|x|} e^{2\beta|y|} dxdy)) \cap C^1([0,T]; L^2(\mathbb{R}^2))$, $s > n$ (in any case $s > 3$) for every $\beta > 0$, and that u is a solution of (1) in $[0,T] \times \mathbb{R}^2$. If $\operatorname{supp} u(0)$ and $\operatorname{supp} u(T)$ are contained in $[-B, B] \times [-B, B]$ for some $B > 0$, then there exists $R_0 > 0$ such that $\operatorname{supp} u(t) \subset [-R_0, R_0] \times [-R_0, R_0]$ for every $t \in [0,T]$.

(See the definition of the space $L^2(e^{2\beta|x|} e^{2\beta|y|} dxdy)$ below).

In the proof of Theorem 1.1 we follow the ideas of Bustamante, Isaza and Mejía in [6] for the ZK equation, and Kenig, Ponce and Vega in [9] for the generalized Korteweg-de Vries (KdV) equation.

It is possible to extend the result of Theorem 1.1 to the general case where P is a polynomial with n spatial variables. This would allow to study dispersive equations in higher dimensions. In particular, the use of a result like this, together with the techniques developed by Bourgain in [4], would permit to obtain unique continuation principles to dispersive models in high spatial dimensions.

This paper is organized as follows: in Section 2, we present an interpolation result which allows to obtain estimates for the spatial derivatives of a function with certain regularity. It is at this point where the restriction $s > 3$ is needed. In Section 3, we prove a Carleman estimate of $L^2 - L^2$ type. Finally, in Section 4, we establish Theorem 1.1.

Throughout this article the symbol $\hat{f}$ will denote the spatial Fourier transform of a function f in $\mathbb{R}^2$. We say that a function f belongs to the weighted L^2 space, $L^2(e^{2\beta|x|}e^{2\beta|y|}dxdy)$, if it is true that $e^{\beta|\cdot x|}e^{\beta|\cdot y|}f \in L^2(\mathbb{R}^2)$; i.e. if

$$\left(\int_{\mathbb{R}^2} |f(x,y)|^2 e^{2\beta|x|}e^{2\beta|y|}dxdy\right)^{1/2} < \infty.$$

In a similar way the spaces $L^2(e^{2\beta x}dxdy)$ and $L^2(e^{2\beta y}dxdy)$ are defined.

With respect to the weighted Sobolev space $H^n(e^{2\beta|x|}e^{2\beta|y|}dxdy)$, that we use in Theorem 3.2, we say that a function f belongs to this space if $e^{\beta|\cdot x|}e^{\beta|\cdot y|}f \in H^n(\mathbb{R}^2)$. This is true if

$$\left(\int_{\mathbb{R}^2} (1+\xi^2+\tau^2)^n \left|\left(e^{\beta|\cdot x|}e^{\beta|\cdot y|}f\right)^{\wedge}(\xi,\tau)\right|^2 d\xi d\tau\right)^{1/2} < \infty.$$

Besides, the letter C will denote diverse positive constants which may change from line to line and depend on parameters which are clearly established in each case.

2. Preliminary Estimates in Weighted Sobolev Spaces

The following lemma is an interpolation result and can be proved using the Hadamard Three-lines theorem in a similar way than Lemma 4 in [15] . We omit its proof here.

Lemma 2.1. *For $s > 0$ and $\beta > 0$ let $f \in H^s(\mathbb{R}^2) \cap L^2(e^{2\beta x}dxdy)$. Then, for $\theta \in [0,1]$,*

$$\|J^{s\theta}(e^{((1-\theta)\beta x)}f)\|_{L^2} \le C\|J^s f\|_{L^2}^{\theta}\|e^{\beta x}f\|_{L^2}^{1-\theta}, \tag{2}$$

where $[J^s f]^\wedge(\xi) := (1+|\xi|^2)^{s/2}\hat{f}(\xi)$ and $C = C(s,\beta)$. Similarly, if $f \in H^s(\mathbb{R}^2) \cap L^2(e^{2\beta y}dxdy)$, then, for $\theta \in [0,1]$,

$$\|J^{s\theta}(e^{(1-\theta)\beta y}f)\|_{L^2} \le C\|J^s f\|_{L^2}^{\theta}\|e^{\beta y}f\|_{L^2}^{1-\theta}. \tag{3}$$

Remark 2.2. If $u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^\infty([0,T]; L^2(e^{2\beta|x|}e^{2\beta|y|}dxdy))$ for every $\beta > 0$, with $s > 3$, it is easy to see that there exists $C_1 > 0$ and $C_2 > 0$ independent of t, such that

$$|\partial_x u(t)(x,y)| \le C_1 e^{-x}, \tag{4}$$

and

$$|\partial_y u(t)(x,y)| \leq C_2 e^{-y}, \tag{5}$$

for every $t \in [0,T]$. In fact, using the Sobolev embedding $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ we have that there exists $C > 0$ such that

$$\begin{aligned}
\|e^x \partial_x u(t)\|_{L^\infty(\mathbb{R}^2)} &\leq C\|e^x \partial_x u(t)\|_{H^2(\mathbb{R}^2)} = C\|\partial_x(e^x u(t)) - e^x u(t)\|_{H^2(\mathbb{R}^2)} \\
&\leq C[\|e^x u(t)\|_{H^2(\mathbb{R}^2)} + \|\partial_x(e^x u(t))\|_{H^2(\mathbb{R}^2)}] \\
&= C[\|J^2(e^x u(t))\|_{L^2(\mathbb{R}^2)} + \|J^2(\partial_x(e^x u(t)))\|_{L^2(\mathbb{R}^2)}] \\
&\leq C[\|J^2(e^x u(t))\|_{L^2(\mathbb{R}^2)} + \|J^3(e^x u(t))\|_{L^2(\mathbb{R}^2)}].
\end{aligned}$$

Since $s > 3$, we can use Lemma 2.1 taking $\theta := 3/s$ and $\beta := (1 - 3/s)^{-1}$ to conclude, by inequality (2), that

$$\|J^3(e^x u(t))\|_{L^2(\mathbb{R}^2)} \leq C\|J^s u(t)\|_{L^2(\mathbb{R}^2)}^{3/s} \|e^{(1-3/s)^{-1}x} u(t)\|_{L^2(\mathbb{R}^2)}^{1-3/s} \leq C_1,$$

and

$$\|J^2(e^x) u(t)\|_{L^2(\mathbb{R}^2)} \leq \|J^3(e^x u(t))\|_{L^2(\mathbb{R}^2)} \leq C_1.$$

Thus, for a.e. $(x,y) \in \mathbb{R}^2$,

$$|e^x \partial_x u(t)(x,y)| \leq C_1, \qquad |\partial_x u(t)(x,y)| \leq C_1 e^{-x},$$

which is (4). Obviously, (5) follows in an analogous way, using (3) instead of (2).

3. Estimates of the Carleman Type

The following lemma is used in the proof of the Carleman estimates (Theorem 3.2) and it justifies the formal computation of the temporal derivative of $\widehat{e^{\lambda x}(t)}(\xi)$ and $\widehat{e^{\lambda y}(t)}(\xi)$. Its proof is taken from [6] and it is presented here for the sake of completeness.

Lemma 3.1. *Let $w \in C^1([0,T]; L^2(\mathbb{R}^2))$ be a function such that for all $\beta > 0$, w is bounded from $[0,T]$ with values in $L^2(e^{2\beta|x|}e^{2\beta|y|}dxdy)$ and $w' \in L^1([0,T]; L^2(e^{2\beta|x|}e^{2\beta|y|}dxdy))$. Then, for all $\lambda \in \mathbb{R}$ and all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the functions $t \mapsto \widehat{e^{\lambda x} w(t)}(\xi)$ and $t \mapsto \widehat{e^{\lambda y} w(t)}(\xi)$ are absolutely continuous in $[0,T]$ with derivatives $\widehat{e^{\lambda x} w'(t)}(\xi)$ and $\widehat{e^{\lambda y} w'(t)}(\xi)$ a.e. $t \in [0,T]$, respectively.*

Proof. By symmetry, it is sufficient to prove the lemma only for the weight $e^{\lambda x}$. It is easy to see that for all $t \in [0,T]$ and $\lambda \in \mathbb{R}$, $e^{\lambda x} w(t) \in L^1(\mathbb{R}^2)$, and also that $e^{\lambda x} w' \in L^1(\mathbb{R}^2 \times [0,T])$ for all $\lambda \in \mathbb{R}$. For $R > 0$, let χ_R be the characteristic function of the square $[-R,R] \times [-R,R]$. Since $w \in C^1([0,T]; L^2(\mathbb{R}^2))$,

$$t \mapsto \int_{\mathbb{R}^2} e^{-ix\xi_1} e^{-iy\xi_2} e^{\lambda x} \chi_R(x,y) w(t)(x,y) dxdy = \left\langle w(t), e^{ix\xi_1} e^{iy\xi_2} e^{\lambda x} \chi_R \right\rangle_{L^2(\mathbb{R}^2)} \tag{6}$$

defines a C^1 function of the variable t with derivative given by

$$t \mapsto \left\langle w'(t), e^{ix\xi_1} e^{iy\xi_2} e^{\lambda x} \chi_R \right\rangle_{L^2(\mathbb{R}^2)},$$

and in consequence

$$\int_{\mathbb{R}^2} e^{-ix\xi_1} e^{-iy\xi_2} e^{\lambda x} \chi_R(x,y) w(t)(x,y) dxdy =$$

$$= \int_0^t \int_{\mathbb{R}^2} e^{-ix\xi_1} e^{-iy\xi_2} e^{\lambda x} \chi_R(x,y) w'(\tau)(x,y) dxdyd\tau$$

$$+ \int_{\mathbb{R}^2} e^{-ix\xi_1} e^{-iy\xi_2} e^{\lambda x} \chi_R(x,y) w(0)(x,y) dxdy.$$

The lemma follows from the former equality by an application of the Lebesgue Dominated Convergence Theorem. ☑

The following theorem is the main result of this section. It is a Carleman estimate of $L^2 - L^2$ type and it is crucial in the proof of Theorem 1.1.

Theorem 3.2. *For $n \in \mathbb{N}$, let $w \in C([0,T]; H^n(\mathbb{R}^2)) \cap C^1([0,T]; L^2(\mathbb{R}^2))$, be a function such that for all $\beta > 0$,*

(i) w is bounded from $[0,T]$ with values in $H^n(e^{2\beta|x|} e^{2\beta|y|} dxdy)$, and

(ii) $w' \in L^1([0,T]; L^2(e^{2\beta|x|} e^{2\beta|y|} dxdy))$.

Then, for all $\lambda \neq 0$,

$$\|e^{\lambda x} w\|_{L^2(\mathbb{R}^2)} \leq \|e^{\lambda x} w(0)\|_{L^2(\mathbb{R}^2)} + \|e^{\lambda x} w(T)\|_{L^2(\mathbb{R}^2)} + \|e^{\lambda x}(w' + P(D)w)\|_{L^2(\mathbb{R}^2 \times [0,T])},$$

where $P(D)$ is the operator defined by

$$P(D)u := \sum_{j=0}^{n} \sum_{j'=0}^{n-j} a_{jj'} \partial_x^j \partial_y^{j'} u,$$

with $a_{jj'} \in \mathbb{C}$ for $j, j' = 0, \dots, n$, and $a_{00} = 0$.

A similar estimate also holds with y instead of x in the exponents.

Proof. Let us define $g(t) := e^{\lambda x} w(t)$ and $h(t) := e^{\lambda x}(w'(t) + P(D)w(t))$. Taking into account that we can write

$$P(D)w = \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \partial_x^j w \right],$$

we have that

$$\begin{aligned}
P(D)w &= \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \partial_x^{j} (g e^{-\lambda x}) \right] \\
&= \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \sum_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} \partial_x^{j-k} g \partial_x^{k} e^{-\lambda x} \right] \\
&= e^{-\lambda x} \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \sum_{k=0}^{j} (-\lambda)^k \begin{pmatrix} j \\ k \end{pmatrix} \partial_x^{j-k} g \right] \\
&= e^{-\lambda x} \sum_{j'=0}^{n} \sum_{j=0}^{n-j'} a_{jj'} \partial_y^{j'} \left[\sum_{k=0}^{j} (-\lambda)^k \begin{pmatrix} j \\ k \end{pmatrix} \partial_x^{j-k} g \right].
\end{aligned}$$

This way,

$$h(t) = e^{\lambda x} w'(t) + \sum_{j'=0}^{n} \sum_{j=0}^{n-j'} a_{jj'} \partial_y^{j'} \left[\sum_{k=0}^{j} (-\lambda)^k \begin{pmatrix} j \\ k \end{pmatrix} \partial_x^{j-k} g \right].$$

Since $w(t) \in H^n(e^{2\beta|x|} e^{2\beta|y|} dxdy)$ for all $\beta > 0$, $t \in [0, T]$, and $w' \in L^2(e^{2\beta|x|} e^{2\beta|y|} dxdy)$ for all $\beta > 0$ a.e. $t \in [0, T]$, by using the Cauchy-Schwarz inequality, it can be seen that $h(t) \in L^1(\mathbb{R}^2)$ a.e. $t \in [0, T]$. We take the spatial Fourier transform to h and apply Lemma 3.1 to obtain

$$\frac{d}{dt} \widehat{g(t)}(\xi) + \left[\sum_{j'=0}^{n} \sum_{j=0}^{n-j'} a_{jj'} (i\xi_2)^{j'} \sum_{k=0}^{j} (-\lambda)^k \begin{pmatrix} j \\ k \end{pmatrix} (i\xi_1)^{j-k} \right] \widehat{g(t)}(\xi) = \widehat{h(t)}(\xi),$$

a.e. $t \in [0, T]$, where $\xi \equiv (\xi_1, \xi_2)$. Taking into account that the expression between squared parentheses is a polynomial function of the variables ξ_1 and ξ_2, with complex coefficients, the former equality can be written in the way

$$\frac{d}{dt} \widehat{g(t)}(\xi) + [i m_\lambda(\xi) + a_\lambda(\xi)] \widehat{g(t)}(\xi) = \widehat{h(t)}(\xi) \quad \text{a.e. } t \in [0, T], \tag{7}$$

where m_λ and a_λ are polynomial functions in $\mathbb{R}^2$. We do not show interest in the precise form of $m_\lambda(\xi)$ and $a_\lambda(\xi)$ because when we estimate $|\widehat{g(t)}(\xi)|$ we only use the fact that $m_\lambda(\xi) \in \mathbb{R}$ and $a_\lambda(\xi) \in \mathbb{R}$, considering two cases: $a_\lambda(\xi) \le 0$ and $a_\lambda(\xi) > 0$, as we can see below.

(i) When $a_\lambda(\xi) \le 0$, we solve (7) integrating between 0 and t to obtain

$$\widehat{g(t)}(\xi) = e^{i m_\lambda(\xi) t} e^{a_\lambda(\xi) t} \widehat{g(0)}(\xi) + \int_0^t e^{i m_\lambda(\xi)(t-\tau)} e^{a_\lambda(\xi)(t-\tau)} \widehat{h(\tau)}(\xi) d\tau$$

for every $t \in [0, T]$. Since $m_\lambda(\xi) \in \mathbb{R}$ and $a_\lambda(\xi) \le 0$, we have that

$$|e^{i m_\lambda(\xi) t}| = 1, \quad |e^{i m_\lambda(\xi)(t-\tau)}| = 1, \quad e^{a_\lambda(\xi) t} \in (0, 1] \text{ and } e^{a_\lambda(\xi)(t-\tau)} \in (0, 1],$$

for every $t \in [0,T]$ and each $\tau \in [0,t]$. Thus, in this case,

$$|\widehat{g(t)}(\xi)| \leq |\widehat{g(0)}(\xi)| + \int_0^t |\widehat{g(\tau)}(\xi) d\tau|, \tag{8}$$

for each $t \in [0,T]$.

(ii) When $a_\lambda(\xi) > 0$, we solve (7) this time integrating between t and T to obtain

$$\widehat{g(t)}(\xi) = e^{-im_\lambda(\xi)(T-t)} e^{-a_\lambda(\xi)(T-t)} \widehat{g(T)}(\xi) + \int_t^T e^{-im_\lambda(\xi)(\tau-t)} e^{-a_\lambda(\xi)(\tau-t)} \widehat{h(\tau)}(\xi) d\tau$$

for every $t \in [0,T]$. Since $m_\lambda(\xi) \in \mathbb{R}$ and $a_\lambda(\xi) > 0$, we have that

$$|e^{-im_\lambda(\xi)(T-t)}| = 1, \qquad |e^{-im_\lambda(\xi)(\tau-t)}| = 1,$$

$$e^{-a_\lambda(\xi)(T-t)} \in (0,1] \quad \text{and} \quad e^{-a_\lambda(\xi)(\tau-t)} \in (0,1],$$

for every $t \in [0,T]$ and each $\tau \in [t,\tau]$. Thus, in this case,

$$|\widehat{g(t)}(\xi)| \leq |\widehat{g(0)}(\xi)| + \int_t^T |\widehat{g(\tau)}(\xi) d\tau|, \tag{9}$$

for each $t \in [0,T]$.

From (8) and (9) we can conclude that, in any case, for every $t \in [0,T]$,

$$|\widehat{g(t)}(\xi)| \leq |\widehat{g(0)}(\xi)| + |\widehat{g(T)}(\xi)| + \int_0^T |\widehat{h(\tau)}(\xi)| d\tau.$$

Hence, by Plancherel's Formula,

$$\|e^{\lambda x} w\| \leq \|e^{\lambda x} w(0)\|_{L^2(\mathbb{R}^2)} + \|e^{\lambda x} w(T)\|_{L^2(\mathbb{R}^2)} + \|e^{\lambda x}(w' + Pw)\|_{L^2(\mathbb{R}^2 \times [0,T])}.$$

The proof of the estimate with the weight $e^{\lambda y}$ is similar. ☑

4. Proof of Theorem 1.1

Let $\tilde{\phi} \in C^\infty(\mathbb{R})$ a non-decreasing function such that $\tilde{\phi}(x) = 0$ for $x < 0$, and $\tilde{\phi}(x) = 1$ for $x > 1$ and, for $R > B$, let $\phi(x) \equiv \phi_R(x) := \tilde{\phi}(x - R)$. We define $w \equiv w_R := \phi(x)u$, and $v \equiv v_R := \phi(y)u$. It is easy to check that w and v satisfy the hypotheses of Theorem 3.2. Taking into account that $w(0) = w(T) = 0$, from Theorem 3.2, we conclude that, for every $\lambda \neq 0$,

$$\begin{aligned} \|e^{\lambda x} w\|_{L^2(\mathbb{R}^2 \times [0,T])} \leq & \|e^{\lambda x}(w' + P(D)w)\|_{L^2(\mathbb{R}^2 \times [0,T])} \\ = & \|e^{\lambda x}(\phi u' + P(D)w)\|_{L^2(\mathbb{R}^2 \times [0,T])}. \end{aligned} \tag{10}$$

As in the proof of Theorem 3.2, we take into account that

$$P(D)w = \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \partial_x^j w \right].$$

Hence,

$$\begin{aligned}
P(D)w = P(D)(\phi u) &= \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \partial_x^j (\phi u) \right] \\
&= \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \sum_{k=0}^{j} \binom{j}{k} \partial_x^{j-k} u \phi^{(k)} \right] \\
&= \phi \sum_{j'=0}^{n} \partial_y^{j'} \left[\sum_{j=0}^{n-j'} a_{jj'} \partial_x^j u \right] + \sum_{j'=0}^{n-1} \partial_y^{j'} \left[\sum_{j=1}^{n-j'} a_{jj'} \sum_{k=1}^{j} \binom{j}{k} \partial_x^{j-k} u \phi^{(k)} \right] \\
&= \phi P u + \sum_{j'=0}^{n-1} \partial_y^{j'} \left[\sum_{j=1}^{n-j'} a_{jj'} \sum_{k=1}^{j} \binom{j}{k} \partial_x^{j-k} u \phi^{(k)} \right].
\end{aligned}$$

Therefore, from (10), and (1), we conclude that

$$\|e^{\lambda x} w\|_{L^2(\mathbb{R}^2\times[0,T])} \leq \|e^{\lambda x}\phi u^l \partial_x u\|_{L^2(\mathbb{R}^2\times[0,T])} + \|e^{\lambda x} F_{1\phi,u}\|_{L^2(\mathbb{R}^2\times[0,T])},$$

where

$$F_{1\phi,u} := \sum_{j'=0}^{n-1} \partial_y^{j'} \left[\sum_{j=1}^{n-j'} a_{jj'} \sum_{k=1}^{j} \binom{j}{k} \partial_x^{j-k} u \phi^{(k)} \right].$$

Since all the derivatives of ϕ are supported in $[R, R+1]$, let us observe that

$$\begin{aligned}
|F_{1\phi,u}| \leq& \max\{a_{jj'} : j = 1,\dots,n; j' = 0,\dots,n-1\}\cdot \\
&\cdot \max\left\{ \binom{j}{k} : j = 1,\dots,n; k = 1,\dots,n \right\} \cdot \sum_{k=1}^{n} |\phi^{(k)}| \left| \sum_{j'=0}^{n-1} \partial_y^{j'} \left[\sum_{j=1}^{n-j'} \sum_{k=1}^{j} \partial_x^{j-k} u \right] \right| \\
\leq& C \sum_{k=1}^{n} |\phi^{(k)}| \sum_{j'=0}^{n-1} \sum_{j=0}^{n-1-j'} |\partial_y^{j'} \partial_x^j u| \;\leq\; C\chi_{[R,R+1]}(\cdot_x) \sum_{j'=0}^{n-1} \sum_{j=0}^{n-1-j'} |\partial_y^{j'} \partial_x^j u|,
\end{aligned}$$

(here χ_A is the characteristic function of a set A). Then, for $\lambda > 1$,

$$\begin{aligned}
\|e^{\lambda x} F_{1\phi,u}\|^2_{L^2(\mathbb{R}^2\times[0,T])} &\leq C \int_0^T \int_{\mathbb{R}} \int_R^{R+1} e^{2\lambda x} \left[\sum_{j'=0}^{n-1} \sum_{j=0}^{n-1-j'} |\partial_y^{j'} \partial_x^j u| \right]^2 dxdydt \\
&\leq C e^{2\lambda(R+1)} \int_0^T \int_{\mathbb{R}} \int_R^{R+1} \left[\sum_{j'=0}^{n-1} \sum_{j=0}^{n-1-j'} |\partial_y^{j'} \partial_x^j u| \right]^2 dxdydt \\
&\leq C e^{2\lambda(R+1)} \|u\|^2_{C([0,T];H^{n-1}(\mathbb{R}^2))} \leq C e^{2\lambda(R+1)},
\end{aligned}$$

and $\|e^{\lambda x} F_1 \pi, u\|_{L^2(\mathbb{R}^2\times[0,T])} \leq Ce^{\lambda(R+1)}$, where $C = C(\|u\|_{C([0,T];H^{n-1}(\mathbb{R}^2))})$ is independent from λ and R. Therefore

$$\begin{aligned}
\|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])} &\leq \|e^{\lambda x}\phi u^l \partial_x u\|_{L^2(\mathbb{R}^2\times[0,T])} + \|e^{\lambda x} F_{1\phi,u}\|_{L^2(\mathbb{R}^2\times[0,T])} \\
&\leq \|e^{\lambda x}\phi u^l\|_{L^2(\mathbb{R}^2\times[0,T])} \|\partial_x u\|_{L^\infty([R,\infty)\times\mathbb{R}\times[0,T])} + Ce^{\lambda(R+1)}.
\end{aligned}$$

Using (4) we have that $\|\partial_x u\|_{L^\infty([R,\infty)\times\mathbb{R}\times[0,T])} \leq C_1 e^{-R}$. Besides, employing the Sobolev immersion $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$,

$$\begin{aligned}\|e^{\lambda x}\phi u^l\|_{L^2(\mathbb{R}^2\times[0,T])} &= \left[\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\lambda x}|\phi u|^2 |u|^{2(l-1)} dxdydt\right]^{1/2} \\ &\leq \sup_{t\in[0,T]} \|u(t)\|^{l-1}_{L^\infty_{xy}} \|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])} \leq C\|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])}.\end{aligned}$$

This way

$$\|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])} \leq C_1 e^{-R}\|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])} + Ce^{\lambda(R+1)}.$$

Since ϕ is a bounded function, from the hypotheses it is clear that $\|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])} < \infty$. Hence, taking $R > B$ such that $C_1 e^{-R} < 1/2$, we obtain

$$\|e^{\lambda x}\phi u\|_{L^2(\mathbb{R}^2\times[0,T])} \leq Ce^{\lambda(R+1)}.$$

Thus, since $\phi(x) = 1$ for $x \geq 2R$,

$$e^{2\lambda R}\left[\int_0^T \int_{\mathbb{R}} \int_{2R}^\infty |u(t)(x,y)|^2 dxdydt\right]^{1/2} \leq \|e^{\lambda x}\phi u\| \leq Ce^{\lambda(R+1)},$$

for all $\lambda > 0$, where C is independent from λ. If we choose $R > 1$ and let $\lambda \to +\infty$, it follows that

$$\left[\int_0^T \int_{\mathbb{R}} \int_{2R}^\infty |u(t)(x,y)|^2 dxdydt\right]^{1/2} = 0.$$

Therefore $u \equiv 0$ in $[2R,\infty)\times\mathbb{R}\times[0,T]$. Now, taking into account the symmetry of the operator $P(D)$, it is easy to see that

$$P(D)v = \phi P(D)u + \sum_{j=0}^{n-1} \partial_x^j \sum_{j'=1}^{n-j} a_{jj'} \sum_{k=1}^{j'} \begin{pmatrix} j' \\ k \end{pmatrix} \partial_y^{j'-k} u\phi^{(k)}.$$

Then, reasoning as above, using (5) instead of (4), we can conclude that there exists $\tilde{R} > 0$ such that $u \equiv 0$ in $\mathbb{R}\times[2\tilde{R},\infty)\times[0,T]$. Taking $R_0 := \max\{2R, 2\tilde{R}\}$, we have that $\operatorname{supp} u(t) \subset [-R_0, \times R_0] \times [-R_0, R_0]$ for every $t \in [0,T]$. ☑

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