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Global Solutions to Isothermal System with Source

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Abstract. In this short note, we are concerned with the global existence of solutions to the isothermal system with source, where the inhomogeneous terms $f(x,t,\rho,u) = b(x,t)\rho + \frac{a'(x)}{a(x)}\rho u^2 + \alpha(x,t)\rho u|u|$ are appeared in the momentum equation. Our work extended the results in the previous papers "Resonance for the Isothermal System of Isentropic Gas Dynamics" (Proc. A.M.S.139(2011),2821-2826), "Global Existence and Stability to the Polytropic Gas Dynamics with an Outer Force" (Appl. Math. Letters, 95(2019), 35-40) and "Existence of Global Solutions for Isentropic Gas Flow with Friction" (Nonlinearity, 33(2020), 3940-3969), where the global solution was obtained for the source $f(x,t,\rho,u) = \frac{a'(x)}{a(x)}\rho u^2, f(x,t,\rho,u) = b(x,t)\rho, f(x,t,\rho,u) = \alpha(x,t)\rho u|u|$ respectively.

Keywords: Global L^∞ solution, isothermal system, source terms, compensated compactness.

MSC2010: 35L45, 35L60, 46T99.

Soluciones globales para sistema isotérmico con fuente

Resumen. En esta nota estamos interesados en la existencia global de soluciones para el sistema isotérmico con fuente, donde los términos no homogéneos $f(x,t,\rho,u) = b(x,t)\rho + \frac{a'(x)}{a(x)}\rho u^2 + \alpha(x,t)\rho u|u|$ aparecen en la ecuación de momento. Nuestros resultados extienden los presentados en "Resonance for the Isothermal System of Isentropic Gas Dynamics" (Proc. A.M.S.139(2011),2821-2826), "Global Existence and Stability to the Polytropic Gas Dynamics with an Outer Force" (Appl. Math. Letters, 95(2019), 35-40) y "Existence of Global Solutions for Isentropic Gas Flow with Friction"

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(Nonlinearity, 33(2020), 3940-3969), en los cuales la solución global se obtuvo, respectivamente, para las fuentes $f(x,t,\rho,u)=\frac{a'(x)}{a(x)}\rho u^2$, $f(x,t,\rho,u)=b(x,t)\rho$ and $f(x,t,\rho,u)=\alpha(x,t)\rho u|u|$.

Palabras clave: Soluciones L^∞ globales, sistemas isotérmicos, términos fuente, compacidad compensada.

1. Introduction

In this paper, we studied the global entropy solutions for the Cauchy problem of isentropic gas dynamics system with source

$$\begin{cases} \rho_t+(\rho u)_x=-\frac{a'(x)}{a(x)}\rho u, \\ (\rho u)_t+(\rho u^2+P(\rho))_x+b(x,t)\rho+\alpha(x,t)\rho u|u|=-\frac{a'(x)}{a(x)}\rho u^2, \end{cases} \tag{1}$$

with bounded measurable initial data

$$(\rho(x,0),u(x,0))=(\rho_0(x),u_0(x)),\quad \rho_0(x)\geq 0, \tag{2}$$

where ρ is the density of gas, u the velocity, $P=P(\rho)$ the pressure. The function $b(x,t)$ corresponds physically to the slope of the topography, $\alpha(x,t)\rho|u|$ to a friction term, where $\alpha(x,t)$ denotes a coefficient function and $a(x)$ is a slowly variable cross section area at x in the nozzle.

The pressure-density relation is $P(\rho)=\frac{1}{\gamma}\rho^\gamma$, where $\gamma>1$ is the adiabatic exponent and for the isothermal gas, $\gamma=1$.

System (1) is of interest because it has different physical backgrounds. For the case of nozzle flow without the friction, namely $b(x,t)=0$ and $\alpha(x,t)=0$, the global solution of the Cauchy problem was well studied (cf. [1, 2, 7, 9] and the references cited therein); When $a(x)=0$ and $\alpha(x,t)=0$, the source term $b(x,t)$ in System (1) is corresponding to an outer force [3, 8], and when $b(x,t)=0, a(x)=0$, $\alpha(x,t)u|u|$ in (1) corresponds physically to a friction term [5].

In this paper we study the isothermal case $P(\rho)=\rho$ and prove the global existence of weak solutions for the Cauchy problem (1)-(2) for general bounded initial data. The main result is given in the following:

Theorem 1.1. *Let $P(\rho)=\rho, 0<a_L\leq a(x)\leq M$ for x in any compact set $x\in(-L,L), A(x)=-\frac{a'(x)}{a(x)}\in C^1(R), 0\leq\alpha(x,t)\in C^1(R\times R^+)$ and $|A(x)|+\alpha(x,t)\leq M$, where M, a_L are positive constants, but a_L could depend on L. Then the Cauchy problem (1)-(2) has a bounded weak solution (ρ,u) which satisfies system (1) in the sense of*

distributions

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \rho\phi_t + (\rho u)\phi_x - \frac{a'(x)}{a(x)}(\rho u)\phi dxdt + \int_{-\infty}^\infty \rho_0(x)\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \rho u\phi_t + (\rho u^2 + P(\rho))\phi_x - \left(\frac{a'(x)}{a(x)}\rho u^2 + b(x,t)\rho + \alpha(x,t)\rho u|u|\right)\phi dxdt \\ + \int_{-\infty}^\infty \rho_0(x)u_0(x)\phi(x,0)dx = 0, \end{cases} \tag{3}$$

for all test function $\phi \in C_0^1(R \times R^+)$, *and*

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \eta(\rho,m)\phi_t + q(\rho,m)\phi_x - \frac{a'(x)}{a(x)}(\eta(\rho,m)_\rho \rho u + \eta(\rho,m)_m \rho u^2)\phi \\ -\eta(\rho,m)_m(b(x,t)\rho + \alpha(x,t)\rho u|u|)\phi dxdt \geq 0, \end{cases} \tag{4}$$

where (η, q) *is a pair of entropy-entropy flux of system (1),* η *is convex, and* $\phi \in C_0^\infty(R \times R^+ - \{t = 0\})$ *is a positive function.*

2. Proof of Theorem 1

In this section, we shall prove Theorem 1. Let $v = \rho a(x)$, then we may rewrite (1) as

$$\begin{cases} v_t + (vu)_x = 0, \\ (vu)_t + (vu^2 + v)_x + (A(x) + b(x,t))v + \alpha(x,t)vu|u| = 0, \end{cases} \tag{5}$$

By simple calculations, the two eigenvalues of (5) are

$$\lambda_1 = u - 1, \quad \lambda_2 = u + 1 \tag{6}$$

with corresponding Riemann invariants

$$z(v,m) = \ln v - \frac{m}{v}, \quad w(v,m) = \ln v + \frac{m}{v}, \quad m = vu. \tag{7}$$

To prove Theorem 1, we consider the Cauchy problem for the following parabolic system

$$\begin{cases} v_t + (vu)_x = \varepsilon v_{xx} \\ (vu)_t + (vu^2 + v)_x + (A(x) + b(x,t))v + \alpha(x,t)vu|u| = \varepsilon(vu)_{xx}, \end{cases} \tag{8}$$

with initial data

$$(v(x,0), v(x,0)u(x,0)) = (v_0^\delta(x), v_0^\delta(x)u_0^\delta(x)), \tag{9}$$

where $\delta > 0, \varepsilon > 0$ denote a regular perturbation constant, the viscosity coefficient,

$$(v_0^\delta(x), u_0^\delta(x)) = (a(x)\rho_0(x) + \delta, u_0(x)) * G^\delta \tag{10}$$

and G^δ is a mollifier.

Then

$$(v_0^\delta(x), u_0^\delta(x)) \in C^\infty(R) \times C^\infty(R), \tag{11}$$

and

$$v_0^\delta(x) \geq \delta, \quad v_0^\delta(x) + |u_0^\delta(x))| \leq M. \tag{12}$$

We multiply (8) by (w_v, w_m) and (z_v, z_m), respectively, to obtain

$$\begin{aligned} z_t + \lambda_1 z_x - (A(x)+b(x,t)) - \alpha(x,t)u|u| &= \varepsilon z_{xx} - \varepsilon(z_{vv}v_x^2 + 2z_{vm}v_x m_x + z_{mm}m_x^2) \\ &= \varepsilon z_{xx} + \frac{2\epsilon}{v}v_x z_x - \frac{\epsilon v_x^2}{v^2} \leq \varepsilon z_{xx} + \frac{2\epsilon}{v}v_x z_x \end{aligned} \tag{13}$$

and

$$\begin{aligned} w_t + \lambda_2 w_x + (A(x)+b(x,t)) + \alpha(x,t)u|u| &= \varepsilon w_{xx} - \varepsilon(w_{vv}v_x^2 + 2w_{vm}v_x m_x + w_{mm}m_x^2) \\ &= \varepsilon w_{xx} + \frac{2\epsilon}{v}v_x w_x - \frac{\epsilon v_x^2}{v^2} \leq \varepsilon w_{xx} + \frac{2\epsilon}{v}v_x w_x. \end{aligned} \tag{14}$$

Letting $z = \bar{z} + Mt, w = \bar{w} + Mt$, where M is the bound of $|A(x)| + \alpha(x,t)$, we have from (13)-(14) that

$$\bar{z}_t + \lambda_1 \bar{z}_x + \alpha(x,t)|u|(\bar{z} - \bar{w}) \leq \varepsilon \bar{z}_{xx} + \tfrac{2\epsilon}{v} v_x \bar{z}_x \tag{15}$$

and

$$\bar{w}_t + \lambda_2 \bar{w}_x + \alpha(x,t)|u|(\bar{w} - \bar{z}) \leq \varepsilon \bar{w}_{xx} + \tfrac{2\epsilon}{v} v_x \bar{w}_x. \tag{16}$$

Since $\alpha(x,t) \geq 0$, using the maximum principle to (15)-(16) (See Theorem 8.5.1 in [6] for the details), we have the estimates on the solutions $(v^{\delta,\epsilon}, m^{\delta,\epsilon})$ of the Cauchy problem (8)-(9)

$$\bar{z}(v^{\delta,\epsilon}, m^{\delta,\epsilon}) \leq M_1, \quad \bar{w}(v^{\delta,\epsilon}, m^{\delta,\epsilon}) \leq M_1$$

or

$$z(v^{\delta,\epsilon}, m^{\delta,\epsilon}) \leq M_1 + Mt = M(t), \quad w(v^{\delta,\epsilon}, m^{\delta,\epsilon}) \leq M_1 + Mt = M(t), \tag{17}$$

where M_1 is a positive constant depending only on the bounds of the initial date.

Therefore we have the following estimates from (17)

$$v^{\delta,\epsilon} \leq e^{M(t)}, \quad \ln v^{\delta,\epsilon} - M(t) \leq u^{\delta,\epsilon} \leq M(t) - \ln v^{\delta,\epsilon}, \quad |m^{\delta,\epsilon}| \leq M_1(t), \tag{18}$$

which deduce the following positive, lower bound of $v^{\delta,\epsilon}$, by using the results in Theorem 1.0.2 in [6],

$$v^{\delta,\epsilon} \geq c(t, \delta, \epsilon) > 0, \tag{19}$$

where $c(t, \delta, \epsilon)$ could tend to zero as the time t tends to infinity or δ, ϵ tend to zero, and $M_1(t)$ is a suitable positive function of t, independent of ε, δ.

With the uniform estimates given in (18) and (19), we may apply the compactness framework in [4] to obtain the pointwise convergence of the viscosity solutions

$$(v^{\delta,\epsilon}(x,t), m^{\delta,\epsilon}(x,t)) \to (v(x,t), m(x,t)) \text{ a.e., as } \varepsilon, \delta \to 0 \tag{20}$$

or

$$(\rho^{\varepsilon,\delta}(x,t), (\rho^{\varepsilon,\delta}u^{\varepsilon,\delta})(x,t)) \to (\rho(x,t), (\rho u)(x,t)) \text{ a.e., as } \varepsilon, \delta \to 0. \tag{21}$$

We rewrite (8) as

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u + \varepsilon\rho_{xx} + 2\varepsilon\frac{a'(x)}{a(x)}\rho_x + \varepsilon\frac{a''(x)}{a(x)}\rho \\ (\rho u)_t + (\rho u^2+\rho)_x + b(x,t)\rho + \alpha(x,t)\rho u|u| \\ \qquad = -\frac{a'(x)}{a(x)}\rho u^2 + \varepsilon(\rho u)_{xx} + 2\varepsilon\frac{a'(x)}{a(x)}(\rho u)_x + \varepsilon\frac{a''(x)}{a(x)}(\rho u). \end{cases} \tag{22}$$

Multiplying a suitable test function ϕ to system (22) and letting ϵ go to zero, we can prove that the limit $(\rho(x,t), u(x,t))$ in (21) satisfies system (1) in the sense of distributions and the Lax entropy condition (3) and (4). So, we complete the proof of Theorem 1. ☑

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