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Topological Relations

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Abstract. A family of constructs is proposed that generalizes the notion of closure operator associated to a partial order. The constructs of the family (and some of its sub constructs) hold adjoint relations with **Gconv** which ensure a topological resemblance; furthermore, it is shown that the constructs are topological categories.

Keywords: Topological categories, constructs, adjunctions.

MSC2010: 18D35, 18B30, 18A40.

Relaciones Topológicas

Resumen. Se propone una familia de constructos que generaliza la noción de operador clausura asociado a un orden parcial. Los constructos de la familia (y algunos de sus subconstructos) cumplen relaciones de adjunción con **Gconv** lo que nos asegura un símil topológico; aún más, se demuestra que los constructos son categorías topológicas.

Palabras clave: Categorías topológicas, constructos, adjunciones.

1. Introduction

In [5] it is said that familiarity with categorical techniques can help those who are confronted with a new field to detect analogies and connections to familiar fields, to organize the new field properly, and to separate the general concepts, problems and results from the specials ones which deserve special investigation. Thus, categorical knowledge can help us to direct and organize our thoughts.

Closure operators have had multiple uses and applications. These applications have used finite set models [3, 4, 6], endorsing a pretopological structure to the relationships obtained from modelling the interactions between the elements (relationships that represent nearness, academic influence, domination, etc.).

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A categorical approach to the definitions of previously cited works helps to achive the goals stated at the beginning of this section. Therefore, we propose structures that generalize the closure operator used in those works.

Moreover, Blass [2] says that a useful methodological principle in modern mathematics is that, when a kind of mathematics structures are defined, the corresponding morphism between two of such structures should be defined. Therefore, not only new structures are defined but also the morphisms among them.

The constructs proposed in this work have been defined in such a way that adjunctions to classical topological structures (**Gconv** and some of its full subconstructs) are obtained. We expect that analogous topological concepts can be defined in the future.

First, we determine the nature of the spaces to work as mathematical structures in the sense of [1]. Then our mathematical structures are laid inside a construct in the sense of [1, 5, 7]. As our main concern is to work with generalized forms of pretopological spaces, we endowed our constructs with the properties needed to be topological in the sense of [7]. We proceed to verify the "correct" behavior of our structures settling adjunctions with familiar topological constructs.

2. Preliminaries

Following [7], a *construct* is a category $\mathscr{C}$ whose objects are structured sets, i.e. pairs (X, ε) where X is a set and ε is $\mathscr{C}$-structure on X, whose morphisms $f : (X, \varepsilon) \to (Y, \eta)$ are suitable maps between X and Y and whose composition law is the usual composition of maps.

We call X the *underlying set* of (X, ε) and $f : X \to Y$ the *underlying map* of $f : (X, \varepsilon) \to (Y, \eta)$. As an abuse of notation, we say *a map* $f : (X, \varepsilon) \to (Y, \eta)$ to refer to the underlying map f. An example of this is the following definition:

If (X, ε) and (X, η) are $\mathscr{C}$-constructs, we say that η is *coarser* than ε (or ε is *finer* than η) if the identity map $1_X : (X, \varepsilon) \to (X, \eta)$ is a $\mathscr{C}$-morphism.

Again, according to [7], a construct $\mathscr{C}$ is called *topological* if it satisfies the following conditions:

(1) *Existence of initial structures:* For any set X, any family $\big((X_i, \varepsilon_i)\big)_{i \in I}$ of $\mathscr{C}$-objects indexed by a class I and any family $(f_i : X \to X_i)_{i \in I}$ of maps indexed by I there exists a unique $\mathscr{C}$-structure ε on X which is *initial* with respect to $\big(X, f_i, (X_i, \varepsilon_i, I)\big)$; i.e., such that for any $\mathscr{C}$-object (Y, η) a map $g : (Y, \eta) \to (X, \varepsilon)$ is a $\mathscr{C}$-morphism iff for every $i \in I$ the composite map $f_i \circ g : (Y, \eta) \to (X_i, \varepsilon_i)$ is a $\mathscr{C}$-morphism.

(2) For any set X, the class $\{(Y, \eta) \in Ob(\mathscr{C}) : X = Y\}$ of all $\mathscr{C}$-objects with underlying set X is a set.

(3) For any set X with cardinality at most one, there exists exactly one $\mathscr{C}$-object with underlying set X.

We mainly work with the construct **Gconv** and some of its full subcategories. Recall that **Gconv** denotes the category of generalized convergence spaces (and continuous maps), that is:

- For each set X let $F(X)$ be the set of all filters on X. Then a *generalized convergence space* is a pair (X, q) where X is a set and $q \subseteq F(X) \times X$ such that the following axioms are satisfied:
  1) $(\dot{x}, x) \in q$ for each $x \in X$, where $\dot{x} = \{A \subseteq X : x \in A\}$;
  2) $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{G} \supseteq \mathcal{F}$.
- A map $f : (X, q) \to (Y, p)$ between generalized convergence spaces is *continuous* provided that $(f(\mathcal{F}), f(x)) \in p$ for each $(\mathcal{F}, x) \in q$.

Now, we recall some important subcategories of **Gconv**: A generalized convergence space (X, q) is called

a) a *Kent convergence space* provided that the following condition is satisfied:
   - $(\mathcal{F} \cap \dot{x}, x) \in q$ whenever $(\mathcal{F}, x) \in q$,

b) a *limit space* provided that the following condition is satisfied:
   - $(\mathcal{F} \cap \mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x)$ and $(\mathcal{G}, x) \in q$,

c) a *pretopological space* provided that the following condition is satisfied:
   - $(\mathscr{V}_q(x), x) \in q$ for all $x \in X$, where $\mathscr{V}_q(x) = \bigcap\{\mathcal{F} \in F(X) : (\mathcal{F}, x) \in q\}$.

A pretopological space (X, q) is called a *topological space* provided that the following condition is satisfied:

- For each $U \in \mathscr{V}_q(x)$ there is some $V \in \mathscr{V}_q(x)$ such that $U \in \mathscr{V}_q(y)$ for all $y \in V$.

The corresponding full subcategories of **Gconv** are denoted by **Kent**, **Lim**, **Prtop** and **Top**, respectively.

3. Topological Relations

In this section the concept of topological relation is defined. Also, interesting properties of these relations are proved. Constructs are built with them, we show that those constructs are topological.

Let X be a set and $R \subseteq X \times \mathcal{P}(X)$. We say that R is a R_1-*relation on* X if:

(RT_0) $\forall x \in X \left[(x, \emptyset) \notin R\right]$,

(RT_1) $\forall x \in X \left[(x, \{x\}) \in R\right]$.

If R is a R_1-relation on X, we call the pair (X, R) a R_1-space. A function $f : (X, R) \to (Y, Q)$ between R_1-spaces is R_1-continuous if:

- $\forall (x, U) \in R\ \exists V \left[(f(x), V) \in Q \wedge f[U] \subseteq V \right]$.

The class of R_1-spaces, and R_1-continuous functions forms a construct. This construct is denoted by $\mathscr{R}_1$. Observe that the identity map $1_X : (X, R) \to (X, R_1)$ is an $\mathscr{R}_1$-morphism between the $\mathscr{R}_1$-objects (X, R) and (X, R_1) if $R \subseteq R_1$. We denote by ${}_xR$ the set $\{U \subseteq X \mid (x, U) \in R\}$. The union of this set, $\bigcup {}_xR$ is denoted by $\mathscr{U}_x^R$ or just $\mathscr{U}_x$ when it is clear from context what relation we are referring to.

The following association can be made:

We associate to each $(X, R) \in \mathscr{R}_1$ an object $(X, q_R) \in$ **Gconv** as follows:

$$(\mathcal{F}, x) \in q_R \leftrightarrow \exists U \left[x\ R\ U \wedge \mathcal{F} \supseteq U\!\uparrow \right].$$

We associate each R_1-continuous $f : (X, R) \to (Y, Q)$ the **Gconv**-morphism $f : (X, q_R) \to (Y, q_Q)$, with the same underlying function.

Remark 3.1. The above association is a functor:

Given that $x\ R\ \{x\}$ for all x, and $\dot{x} = \{x\}\!\uparrow$, we have that $(\dot{x}, x) \in q_R$ for all $x \in X$.

If $\mathcal{G} \supseteq \mathcal{F}$ and $(\mathcal{F}, x) \in q_R$, then $\exists U \left[x\ R\ U \wedge \mathcal{F} \supseteq U\!\uparrow \right]$. Thus we obtain that $(\mathcal{G}, x) \in q_R$ since $\mathcal{G} \supseteq U\!\uparrow$.

Suppose $f : (X, R) \to (Y, Q)$ is R_1-continuous and $(\mathcal{F}, x) \in q_R$. It follows that $\exists U \left[x\ R\ U \wedge \mathcal{F} \supseteq U\!\uparrow \right]$. This implies that $U \in \mathcal{F}$ and $f[U] \in f(\mathcal{F})$; by the R_1 continuity we have that, for some V, $f(x) Q V$ where $f[U] \subseteq V$. All the aforementioned implies that $f(\mathcal{F}) \supseteq f[U]\!\uparrow \supseteq V\!\uparrow$, and also $\big(f(\mathcal{F}), f(x)\big)$, which proves that $f : (X, q_R) \to (Y, q_Q)$ is continuous.

The remaining properties to verify that the association is a functor are easily derived from the construct structure. We denote the previous functor by T_1.

The following association can also be made:

We associate to each $(X, q) \in$ **Gconv** an R_1-space, (X, R_q) as follows:

$$x\ R_q\ U \leftrightarrow \exists\, (\mathcal{F}, x) \in q \left[U = \bigcap \mathcal{F} \neq \emptyset \right].$$

We associate each **Gconv**-morphism $f : (X, q) \to (Y, p)$ the $\mathscr{R}_1$-morphism $f : (X, R_q) \to (Y, R_p)$, with the same underlying function.

Remark 3.2. The previous association is also a functor:

By construction we have (RT_0). Given that $(\dot{x}, x) \in q$ for all x, in any **Gconv** space, we have that (RT_1), therefore (X, R_q) is an $\mathscr{R}_1$-object.

Suppose that $f : (X, q) \to (Y, p)$ is a **Gconv**-morphism and that $x\ R_q\ U$. Then $\exists\, (\mathcal{F}, x) \in q$ such that $U = \bigcap \mathcal{F}$. By continuity of f we have that $\big(f(\mathcal{F}), f(x)\big) \in p$. Also, by construction, $f[U] \subseteq V$, for all $V \in f(\mathcal{F})$. Thus $\bigcap f(\mathcal{F}) \supseteq f[U]$. Observe that $f[U]\uparrow \supseteq f(\mathcal{F})$, this implies that $\big(f[U]\uparrow, f(x)\big)$. By construction $f[U] = \bigcap f[U]\uparrow$, which implies that $f(x)\ R_p\ f[U]$, which makes f an $\mathscr{R}_1$-morphism.

Equally, the remaining properties are easily derived from properties of the construct.

We will denote the previous functor by W_1.

The following properties will be used to define different constructs.

Definition 3.3. Let X be a set and $R \subseteq X \times \mathcal{P}(X)$:

(RT_2) $\forall x \in X \left[x\ R\ U \Rightarrow x\ R\ U \cup \{x\}\right]$.

(RT_3) $\forall x \in X \left[x\ R\ U \wedge x\ R\ V \Rightarrow x\ R\ (U \cup V)\right]$.

(RT_4) $\forall x \in X\, [x\ R\ \mathscr{U}_x]$.

(RT_5) $\forall x \in X \left[x\ R\ U \wedge y \in U \Rightarrow \mathscr{U}_y \subseteq \mathscr{U}_x\right]$.

With these properties it is possible to define $\mathscr{R}_n$ constructs as the full subconstructs of $\mathscr{R}_1$ such that their objects satisfy the RT_i properties with $i \leq 5$.

Remark 3.4. Let $(X, R) \in \mathscr{R}_1$. We define $\check{R}$ as $(x, U) \in \check{R} \Leftrightarrow \exists (x, V) \in R\, [U \subseteq V]$.

Then

1) $T_1((X, R)) = T_1((X, \check{R}))$;

2) if $R = \check{R}$ then

$$f : (X, R) \to (Y, Q) \text{ is } \mathscr{R}_1\text{-continuous} \Leftrightarrow \forall U \left[x\ R\ U \Rightarrow f(x) Q f[U]\right].$$

This means that if we are interested in studying $\mathscr{R}_n$ constructs through the functors T_1 and W_1, then we may assume $R = \check{R}$. Furthermore, we can replace (RT_0) by

$$\forall x \in X\ \exists U\, [x\ R\ U \wedge x \in U]\,.$$

We will now see that the constructs $\mathscr{R}_n$, with $1 \leq n \leq 5$, are topological. First, we will show this for $\mathscr{R}_1$ and then for the others.

and

Theorem 3.5. *Let* $\{(X_i, R_i)\}_{i\in I}$ *and* $\{f_i : X \to (X_i, R_i)\}_{i\in I}$ *be a family of* $\mathscr{R}_1$*-spaces and maps, respectively. The structure* R *over* X *defined as*

$$x\ R\ U \leftrightarrow U \neq \emptyset \wedge \forall i \in I\ \exists V_i \subseteq X_i \left[f_i(x) R_i V_i \wedge f_i[U] \subseteq V_i\right]$$

is an initial structure.

Proof. Because $f_i(x)\ R_i\ \{f_i(x)\}$ for all $x \in X$ and all $i \in I$, we obtain that $x\ R\ \{x\}$. Therefore we obtain RT_1; the verification of RT_0 is straightforward. The previous shows that (X, R) is an $\mathscr{R}_1$ structure. Next we are going to verify the initiality.

Let $g : (Y, Q) \to (X, R)$ a map such that $f_i \circ g : (Y, Q) \to (X, R_i)$ is an $\mathscr{R}_1$-morphism. We have to prove that g is an $\mathscr{R}_1$-morphism. Let $y\ Q\ V$. We will prove that $\exists U \subseteq X\left[g(y)\ R\, U \wedge g[V] \subseteq U\right]$. Because $f_i \circ g$ is an $\mathscr{R}_1$ morphism for each $i \in I$, then

$$\forall i \in I\ \exists U_i \subseteq X_i \left[f_i \circ g(y)\ R_i\ U_i \wedge f_i \circ g[V] \subseteq U_i\right].$$

By the construction of R, we have that $g(y)\ R\ g[V]$. Now we will show that R is the coarsest structure that makes each f_i R_1-continuous. Suppose that R' does it too. Let $1_X : (X, R') \to (X, R)$ be the identity map and $(x, U) \in R'$. We will prove that

$$\forall i\ \exists V_i \subseteq X_i \left[f_i(x)\ R_i\ V_i \wedge f_i[U] \subseteq V_i\right],$$

but this is equivalent to $f_i : (X, R') \to (X, R_i)$ being $\mathscr{R}_1$-continuous, which is true by hypothesis. ☑

To show that $\mathscr{R}_i$ is also topological for $1 < i \leq 5$, it is enough to prove that the structure R defined in Theorem 3.5 is an $\mathscr{R}_i$-structure when $\{f_i : X \to (X_i, R_i)\}_{i \in I}$ is an $\mathscr{R}_i$-source.

Theorem 3.6. *The constructs $\mathscr{R}_i$ when $1 < i \leq 5$ are topological.*

Proof. 1) $\mathscr{R}_4$ is topological. Indeed, let $\{f_i : X \to (X_i, R_i)\}_{i \in I}$ with $\{(X_i, R_i)\}_{i \in I} \subseteq \mathscr{R}_4$. Let (X, R) be defined as in indeed, Theorem 3.5. We shall prove that $(x, \bigcup {}_xR) \in R$. $\forall U \in {}_xR\ \exists V_U \left[f_i(x)\ R_i\ V_U \wedge f_i[U] \subseteq V_U\right]$. Because each (X_i, R_i) is an $\mathscr{R}_4$-object, then $(f_i(x), \bigcup {}_{f_i(x)}R_i) \in R_i$. It follows that

$$f_i\left[\bigcup {}_xR\right] = \bigcup_{U \in {}_xR} f_i[U] \subseteq \bigcup_{U \in {}_xR} V_U \subseteq \bigcup {}_{f_i(x)}R_i \quad \text{for all } i \in I,$$

which concludes that $x\ R \bigcup {}_xR$.

The proof that $\mathscr{R}_2$ and $\mathscr{R}_3$ are topological constructs is similar.

2) $\mathscr{R}_5$ is topological. To see this, let $\{f_i : X \to (X_i, R_i)\}_{i \in I}$ with $\{(X_i, R_i)\}_{i \in I} \subseteq \mathscr{R}_5$; (X, R) as defined in Theorem 3.5; $(x, U) \in R$ and $y \in U$. We shall prove that $\mathscr{U}_y \subseteq \mathscr{U}_x$. Since $(x, U) \in R$, we have that $\forall i \in I\ \exists V_i \left[f_i(x)\ R_i\ V_i \wedge f_i[U] \subseteq V_i\right]$. It follows that $f_i(y) \in f_i[U] \subseteq V_i$ for each i and, since each (X_i, R_i) is $\mathscr{R}_5$, we obtain $\mathscr{U}_{f_i(y)} \subseteq \mathscr{U}_{f_i(x)}$. Let $z \in \mathscr{U}_y$ then $\exists A\left[y\ R\ A \wedge z \in A\right]$. Observe that $z \in \mathscr{U}_x$ is equivalent to $\exists U'\left[x\ R\ U' \wedge z \in U'\right]$. By construction we have that $\forall i \in I$ $\exists W_i \left[f_i(y)\ R_i\ W_i \wedge f_i[A] \subseteq W_i\right]$. From this it follows that $f_i(z) \in f_i[A] \subseteq W_A \subseteq \mathscr{U}_{f_i(y)} \subseteq \mathscr{U}_{f_i(x)}$. So, we obtain that $x\ R\ \{z\}$. ☑

From an $\mathscr{R}_1$-object, (X, R), we can construct $\mathscr{R}_n$-objects as follows:

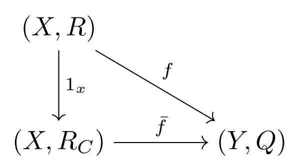
By letting $R_k = R \cup R^*$, with $R^* = \{(x, U \cup \{x\}) \mid x\ R\ U\}$, we obtain an $\mathscr{R}_2$-object. Defining $R_l = R \cup R^*$, with $R^* = \{(x, \cup A) \mid A \subseteq {}_xR \wedge |A| < \aleph_0\}$, we obtain an $\mathscr{R}_3$-object. And $R_C = R \cup R^*$, with $R^* = \{(x, \cup A) \mid A \subseteq {}_xR\}$, is an $\mathscr{R}_4$-object.

We define recursively over ω the following sets for each $x \in X$: ${}_xR^0 = {}_xR$, ${}_xR^{n+1} = \{V \in {}_yR^n \mid \exists U \in {}_xR^n\,[y \in U]\} \cup {}_xR^n$, ${}_xR^\omega = \bigcup\{{}_xR^n \mid n \in \omega\}$. Hence, $(X, (R^\omega)_C)$ is an $\mathscr{R}_5$-object.

Theorem 3.7. *$\mathscr{R}_4$ is a reflective subcategory of $\mathscr{R}_1$.*

Proof. Let (X, R) be an $\mathscr{R}_1$-object and let R_C be as previously defined. Let us see, in fact, that (X, R_C) is an $\mathscr{R}_4$-object. It is enough to show that $\bigcup {}_xR_C = \bigcup {}_xR$. One inclusion follows by definition. Let $x \in \bigcup {}_xR_C$, if $x \in U \in R$ we have finished. Suppose that $x \in U \in R^*$. This implies that $\exists V \in R\,[x \in V \in {}_xR]$, so $x \in \bigcup {}_xR$, which shows that (X, R_C) is an $\mathscr{R}_4$-object.

We will see that the identity map $1_x : (X, R) \to (X, R_C)$ is a morphism and serves as a reflector; for which we shall prove that the following diagram commutes if $\bar{f}$ has f as an underlying map.



Let $f : (X, R) \to (Y, Q)$ and $(x, U) \in R_C$. If $U \in R$, we have finished. Suppose that $U \in R^*$; then $U = \bigcup A$. By the $\mathscr{R}_1$-continuity of f, we have that $\forall W \in A \exists V_W \left[f(x)QV_W \wedge f[W] \subseteq V_W\right]$; all this implies that

$$f[U] = f[\bigcup A] = \bigcup_{W \in A} f[W] \subseteq \bigcup_{W \in A} V_W \subseteq \mathscr{U}^Q_{f(x)}.$$

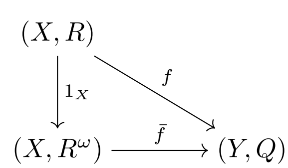
☑

The proofs that $\mathscr{R}_2$ and $\mathscr{R}_3$ are reflective are similar to that for $\mathscr{R}_4$.

Theorem 3.8. *Let $(X, R) \in \mathscr{R}_1$ and R^ω as previously defined. Then:*

a) (X, R^ω) satisfies RT_5,

b) for any $\mathscr{R}_1$-morphism $f : (X, R) \to (Y, Q)$ with (Y, Q) an $\mathscr{R}_5$-object, the diagram



commutes if $\bar{f}$ and f have the same underlying map.

Proof. a) Let $(x,U) \in R^{\omega}$ and $y \in U$. We shall prove that $\mathscr{U}_y \subseteq \mathscr{U}_x$. Let $z \in \mathscr{U}_y = \bigcup {}_y R^{\omega}$, then $\exists V \subseteq X \; \exists n \in \omega \left[z \in V \wedge V \in {}_y R^n\right]$; let $\hat{n}$ be the least $n \in \omega$ which satisfies the property. Since $U \in {}_x R^{\omega}$ we have that $\exists m \in \omega \, [U \in {}_x R^{\omega}]$; let $\hat{m}$ be the least $m \in \omega$ which satisfies the property. Let $l = \max(\hat{m}, \hat{n})$. With this we obtain that $z \in V \in {}_x R^{l+1} \subseteq {}_x R^{\omega} \subseteq \mathscr{U}_x$.

b) Let $(x,U) \in R^{\omega}$. Let $n \in \omega$ the least natural number such that $(x,U) \in R^n$. We shall prove by induction over n that, if $0 < n$, there are $N \in \omega$, $\{A_i\}_{i \in N+1} \subseteq \mathcal{P}(X)$ and $\{x_i\}_{i \in N+1} \subseteq X$ such that:

$$\begin{array}{lcl} A_0 = U, & & (x_i, A_i) \in R \text{ for each } i \in N+1, \\ x_N = x & \text{and} & x_i \in A_{i+1} \text{ for each } i \in N. \end{array}$$

Base case, $n = 1$. By definition we have that

$$\exists V \subseteq X \; \exists y \left[(y,U) \in R \wedge y \in V \wedge (x,V) \in R\right].$$

Letting $N = 0$, $A_1 = V$ and $x_0 = y$ we obtain what is required.

Inductive step. Let $(x,U) \in R^{M+1}$ (remark our hypothesis, $M+1$ is the least natural number which satisfies that membership). By definition we have that $\exists V \subseteq X \; \exists \; y \in X \left[(y,U) \in R^M \wedge y \in V \wedge (x,V) \in R^M\right]$. Using the induction hypothesis with $(y,U) \in R^M$ and $(x,V) \in R^M$ we obtain that there are $N \in \omega$, $\{A_i\}_{i \in N+1} \subseteq \mathcal{P}(X)$, $\{x_i\}_{i \in N+1} \subseteq X$ and, $N' \in \omega$, $\{A'_i\}_{i \in N'+1} \subseteq \mathcal{P}(X)$ y $\{x'_i\}_{i \in N'+1} \subseteq X$, respectively. For $j \in N + N' + 2$ we define

$$\begin{array}{lcll} B_j = A_j & \text{and} & z_j = x_j & \text{if } j \in N+1, \\ B_j = A'_k & \text{and} & z_j = x'_k & \text{if } j = k + N + 1. \end{array}$$

This construction satisfies what is required: $B_0 = A_0 = U, z_{N+N'+1} = x'_{N'} = x$ and that $z_N = y \in V = A'_0 = B_{N+1}$.

Let $M = N + N' + 1$. By construction and the $\mathscr{R}_1$-continuity of f we have that there is a family $\{V_j\}_{j \in M+1} \subseteq \mathcal{P}(Y)$ which satisfies that $f(z_j) \; Q \; V_j$ and $f[B_j] \subseteq V_j$ for each $j \in M+1$. The previous and that (Y,Q) is an $\mathscr{R}_5$-object implies that:

$$f[U] \subseteq \mathscr{U}_{f(z_0)} \subseteq \mathscr{U}_{f(z_i)} \subseteq \ldots \subseteq \mathscr{U}_{f(x)}.$$

Since $f(x)QV_M$ and from RT_4, we have that $f(x)Q\mathscr{U}_{f(x)}$, which ends the proof. ☑

Remark 3.9. To prove that $\bar{f} : (X, (R^{\omega})_C) \to (Y,Q)$ is continuous, the case when $U = \bigcup A$ with $A \subseteq {}_x R^{\omega}$ remains. In this case we proceed in the same way as Theorem 3.7, where the existence for each V_W is obtained in the same way as it done in Theorem 3.8.

To prove that $(X, (R^{\omega})_C)$ satisfies the property RT_5, observe that $\mathscr{U}_x^{R^{\omega}} = \mathscr{U}_x^{R^{\omega}_C}$.

From the previous observation we can deduce the following:

Theorem 3.10. *$\mathscr{R}_5$ is reflexive in $\mathscr{R}_1$.*

Example 3.11. *Let $X = \omega$.*

1) Let ${}_n R = \{\{n\}, \{n+1\}, \{n+2\}\}$. (X,R) is an $\mathscr{R}_1$-space but not an $\mathscr{R}_2$-space.

2) Let ${}_nR = \{\{n\}, \{n, n+1\}, \{n, n+2\}\}$. (X, R) is an $\mathscr{R}_2$-space but not an $\mathscr{R}_3$-space.

3) Let ${}_nR = \{[n, N] \subseteq \omega \mid N \in \omega \wedge n \leq N\}$. (X, R) is an $\mathscr{R}_3$-space but not an $\mathscr{R}_4$-space.

4) For $n \neq 0$, let ${}_nR = \{[n-1, \omega), \{n\}\}$. For $n = 0$, let ${}_0R = \{\{0\}, \omega\}$. (X, R) is an $\mathscr{R}_4$-space but not an $\mathscr{R}_5$-space.

5) Let ${}_nR = \{[n, \omega), \{n\}\}$. (X, R) is an $\mathscr{R}_5$-space.

Remark 3.12. We said that these constructs generalize the closure operators generated by a reflexive poset. Indeed: If $\leq$ is a reflexive partial order over X, define an $\mathscr{R}_4$-object, (X, R), as $x\ R\ U \leftrightarrow U = \{y \in X \mid x \leq z\}$. Then, the canonical closure constructed from $(X, \leq)$ coincides with the canonical closure constructed from $T_1((X, R))$.

4. Adjunction

We will show that the functors T_1 y W_1 are adjoints; since the functors assign the same underlying map to the morphism, it will be enough to show that the unity and co-unity must have the identity map as their underlying maps.

$$\begin{array}{ccccccc} (X,R) & \xhookrightarrow{\eta_R} & (X, R_{q_R}) & \qquad & (X, q_{R_q}) & \xhookrightarrow{\varepsilon_q} & (X,q) \\ \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\ (Y,Q) & \xhookrightarrow{\eta_Q} & (Y, R_{q_Q}) & & (Y, q_{R_p}) & \xhookrightarrow{\varepsilon_p} & (Y,p) \end{array}$$

Having proved that the identity maps works as unity and co-unity, we get that $T_1(\eta_{(X,R)}) : (X, q_R) \to (X, q_{R_{q_R}})$ has an identity as their underlying map. Since the identity map $\varepsilon_{(X,q_R)} : (X, q_{R_{q_R}}) \to (X, q_R)$ is a morphism and by definition of the **Gconv** continuity we can conclude that $q_R = q_{R_{q_R}}$. Those structures being equal, $\varepsilon_{(X,q_R)}$ and $T_1(\eta_{(X,R)})$ are the identity morphism over (X, q_R). This trivializes the triangular identity $\varepsilon_{T_1} \circ T_1\eta = 1_{T_1}$, since both maps from the left side are 1_{T_1}. Analogously this can be shown for $W_{1\varepsilon} \circ \eta_{W_1} = 1_{W_1}$.

Theorem 4.1. *The identity map $\varepsilon_q : (X, q_{R_q}) \to (X, q)$ is* **Gconv** *continuous.*

Proof. Let $(\mathcal{F}, x) \in q_{R_q}$. By construction we have the following

$$\exists (x, U) \in R_q \, [\mathcal{F} \supseteq U\!\uparrow]$$

and $\exists (\mathcal{G}, x) \in q \left[U = \bigcap \mathcal{G}\right]$. Then $U\!\uparrow \supseteq \mathcal{G}$, which implies that $(U\!\uparrow, x) \in q$. Since $\mathcal{F} \supseteq U\!\uparrow \supseteq \mathcal{G}$, we conclude $(\mathcal{F}, x) \in q$. ☑

Theorem 4.2. *The identity map $\eta_R : (X, R) \to (X, R_{q_R})$ is $\mathscr{R}_1$-continuous.*

Proof. Let xRU. By definition, $(U\!\uparrow, x) \in q_R$. Since $U = \bigcap U\!\uparrow$, we conclude that $x\ R_{q_R}\ U$. ☑

Example 4.3. *Let (X, R) be an $\mathscr{R}_1$-object.*

If (X, R) is an $\mathscr{R}_1$-object such that $X \in {}_xR$, then $(\mathcal{F}, x) \in q_R$ for all $\mathcal{F} \in F(X)$. In this case we get the proper inclusion $R \subsetneq R_{q_R}$.

Let $X = (0, 1] \subseteq \mathbb{R}$. Define V as the filter generated by the family $\{(0, \delta] \mid 0 < \delta \leq 1\}$ and let $q(x)$ denote the set $\{\mathcal{F} \in F(X) \mid (\mathcal{F}, x) \in q\}$. Let (X, q) be such that $q(1)$ has $\dot{1}$, V and all the superfilters of V as members. Then $q_{R_q}(1) = \{\dot{1}\}$. In this case we get the proper inclusion $q_{R_q} \subsetneq q$.

The following observation will be useful to prove Theorem 4.5 below.

Remark 4.4. Let I be a non empty set, $M = \{\mathcal{F}_i\}_{i \in I} \subseteq F(X)$ and $N = \{\bigcup\{g(i)\}_{i \in I} \mid g \in \prod_{i \in I} \mathcal{F}_i\}$. Therefore,

1) $\bigcap M = N$.

2) If $L = \{\bigcap \mathcal{F}_i \mid i \in I\}$ then $\bigcap\bigcap M = \bigcup L$.

1) For $\bigcap M \subseteq N$, if $U \in \bigcap M$, it is enough to choose g such that $g(i) = U$ for all $i \in I$. For $N \subseteq \bigcap M$ it is enough to see that for each $\hat{i} \in I$, $g(\hat{i}) \subseteq \bigcup\{g(i)\}_{i \in I}$.

2) For $\bigcap\bigcap M \supseteq \bigcup L$, observe that $\bigcap M \subseteq \mathcal{F}_i$ for each $i \in I$. For $\bigcap\bigcap M \subseteq \bigcup L$, let us see $\bigcap M$ as $N = \{\bigcup\{g(i)\}_{i \in I} \mid g \in \prod_{i \in I} \mathcal{F}_i\}$. Let $z \in \bigcap\bigcap M$. This means that for each $g \in \prod_{i \in I} \mathcal{F}_i$, there is an i such that $z \in g(i)$. Suppose that $z \notin \bigcup L$. This implies that, for each $i \in I$, we can select an $A_i \in \mathcal{F}_i$ such that $z \notin A_i$. If we define $\hat{g}(i)$ as A_i we have that $z \notin \bigcup\{\hat{g}(i)\}_{i \in I}$, a contradiction.

Restricting the functor T_1 over $\mathscr{R}_i$ and W_1 over **Kent**, **Lim**, **Prtop** and **Top** (which will be denoted by T_n and W_n, respectively) we obtain the following:

Theorem 4.5. *The following constructs are adjoints:*

(a) $\mathscr{R}_2$ and **Kent***,*

(b) $\mathscr{R}_3$ and **Lim***,*

(c) $\mathscr{R}_4$ and **Prtop***,*

(d) $\mathscr{R}_5$ and **Top***.*

Proof. (a) T_2 is well defined: Let $(X, R) \in \mathscr{R}_2$, $(\mathcal{F}, x) \in q_R$. By construction $\exists U[x\ R\ U \wedge \mathcal{F} \supseteq U\uparrow]$. By the ($\text{RT}_2$) property we have $x\ R\ U \cup \{x\}$. Since $\mathcal{F} \cap \dot{x} \supseteq (U \cup \{x\})\uparrow$, we conclude that $(\mathcal{F} \cap \dot{x}, x) \in q_R$.

W_2 is well defined: Let $(X, q) \in$ **Kent**, $x\ R_q\ U$. By construction $\exists (\mathcal{F}, x) \in q\left[U = \bigcap \mathcal{F} \neq \emptyset\right]$. By the **Kent** property, we have that $(\mathcal{F} \cap \dot{x}, x) \in q$. By Remark 4.4 (choosing $M = \{\mathcal{F}, \dot{x}\}$) we have $\bigcap(\mathcal{F} \cap \dot{x}) = U \cup \{x\}$, and, by the functor definition, we have that $x\ R_q\ U \cup \{x\}$.

(b) T_3 is well defined: Similar to (a), it can be easily proven that if $\mathcal{F}, \mathcal{G} \in q_R(x)$, we get that there are $U, V \subseteq \mathscr{P}(X)$ such that $\mathcal{F} \supseteq U\uparrow, \mathcal{G} \supseteq V\uparrow, x\ R\ U$ and $x\ R\ V$. Finally, by $\mathcal{F} \cap \mathcal{G} \supseteq (U \cup V)\uparrow$ (Remark 4.4) we conclude that $\mathcal{F} \cap \mathcal{G} \in q_R(x)$.

W_3 is well defined: Again, using the technique from (a) and by Remark 4.4, we have that if $x\ R_q\ U$ and xR_qV, then there are $\mathcal{F}, \mathcal{G} \in q(x)$ such that $\bigcap \mathcal{F} = U \neq \emptyset$ and $\bigcap \mathcal{G} = V \neq \emptyset$. But $\bigcap \mathcal{F} \cap \mathcal{G} = U \cup V$, which implies that $xR_qU \cup V$.

(c) T_4 is well defined: Let $(X, R) \in \mathscr{R}_4$. By Remark 4.4 we have $\bigcap_{\mathcal{F} \in q_R(x)} \mathcal{F} \ni \mathscr{U}_x$.

W_4 is well defined: Let $(X, q) \in$ **Prtop**. By Remark 4.4 we have that $\bigcap \mathscr{V}_x = \bigcap \bigcap q(x) = \bigcup \{\bigcap \mathcal{F}_i \mid i \in I\} = \mathscr{U}_x$.

(d) T_5 is well defined: Let $(X, R) \in \mathscr{R}_5$. Let $U \in \mathscr{V}_x$ and take V as $\mathscr{U}_x$. If $y \in V$, then

$$\exists U' \left[y \in U' \wedge x\ R\ U' \right].$$

From this and (RT_5), we have that $\mathscr{U}_y \subseteq \mathscr{U}_x$. Thus $\mathscr{U}_x \in \mathscr{V}_y$ and $U \in \mathscr{V}_y$. This shows that (X, q_R) is a topological space; observe that $\mathscr{V}_x = \mathscr{U}_x\uparrow$ for all x, then (X, q_R) is always an Alexandroff space.

W_5 is well defined: Let $(X, q) \in$ **Top**, xR_qU and $y \in U$. By (c) it is known that $\bigcap \mathscr{V}_x = \mathscr{U}_x$. $y \in U$, then there exists $(\mathcal{F}, x) \in q$ such that $y \in \bigcap \mathcal{F} = U$. Thus $\forall V \in \mathscr{V}_x[y \in V]$. Let $V \in \mathscr{V}_x$. Since $(X, q) \in$ **Top**, we have that $\exists V' \in \mathscr{V}_x \left[\forall z \in V' \left[V \in \mathscr{V}_z\right]\right]$. In particular $V \in \mathscr{V}_y$. Therefore $\mathscr{U}_y \subseteq V$. Since V was arbitrary, $\mathscr{U}_y \subseteq \bigcap \mathscr{V}_x = \mathscr{U}_x$. ☑

It remains to determine which properties the functor W_1 preserve, and how they will be stated in an $\mathscr{R}_1$ construct.

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