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Optimal system, invariant solutions and complete classification of Lie group symmetries for a generalized Kummer-Schwarz equation and its Lie algebra representation

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Abstract. We obtain the complete classification of the Lie symmetry group and the optimal system's generating operators associated with a particular case of the generalized Kummer - Schwarz equation. Using those operators we characterize all invariant solutions, alternative solutions were found for the equation studied and the Lie algebra associated with the symmetry group is classified.

Keywords: Invariant solutions, Lie symmetry group, Optimal system, Lie algebra classification, Kummer-Schwarz equation.

MSC2010: 35A30, 58J70, 76M60.

Sistema óptimo, soluciones invariantes y clasificación completa del grupo de simetrías de Lie para la ecuación de Kummer-Schwarz generalizada y su representación del álgebra de Lie

Resumen. Obtenemos la clasificación completa del grupo de simetría de Lie y los operadores generadores del sistema optimal asociados a un caso particular de la ecuación de Kummer - Schwarz generalizada. Utilizando esos operadores, caracterizamos todas las soluciones invariantes, se encontraron soluciones alternativas para la ecuación estudiada y se clasifica el álgebra de Lie asociada al grupo de simetría.

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Palabras clave: Soluciones invariantes, grupo de simetría de Lie, Sistema optimal, Clasificación del álgebra de Lie, Ecuación de Kummer - Schwarz.

1. Introduction

The Kummer-Schwarz equation appears in various mathematical contexts like theory of functions, differential geometry, complex analysis, differential equations, integrable systems, mathematical physics, Sturm - Liouville equation, study of curves in a Lorentz space, the charge of density of dark energy [12, 32, 34]. In [26], Leach suggested several generalizations of the Kummer-Schwarz Equation ($KS-3$):

$$y_{xxx} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 = S(y(x)). \tag{1}$$

The generalizations before mentioned are important due to their connection to the Schwarzian derivative, (see Milne - Pinney [7]), the Riccati equations and its algebraic properties. The equation (1) has the Lie point symmetry algebra $\mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$. The $KS-3$ equation can be useful in the interpretation of physical systems; e.g., the case of quantum non-equilibrium dynamics of many body systems. The global dynamics of the Kummer - Schwarz differential equation was studied in [29]. Following this line, in [9, 18] considered a generalization of Kummer - Schwarz differential equation for $KS-3$, namely

$$y_{xxx} - \frac{3y_{xx}^2}{2y_x} - f(x)y_x = 0, \tag{2}$$

where f is an arbitrary function, they presented the group of symmetries

$$\Pi_1 = \partial_y, \qquad \Pi_2 = y\partial_y, \qquad \Pi_3 = y^2\partial_y. \tag{3}$$

Such a symmetry group is a Non Solvable Lie Algebra and using this algebra, in [18] the following reduction is proposed for (2)

$$v_u = \frac{v^2}{2} + f(u), \quad \text{where} \quad u = x \quad \text{and} \quad v = \frac{y_{xx}}{y_x}.$$

That is, it's corresponding Riccati equation. In [9, 20], was presented the group of Lie symmetries of (1), such symmetry group is (without explaining the details in their calculations)

$$\begin{aligned} &\Pi_1 = \partial_y, && \Pi_2 = y\partial_y, && \Pi_3 = y^2\partial_y, \\ &\Pi_4 = \partial_x, && \Pi_5 = x\partial_x, && \Pi_6 = x^2\partial_x. \end{aligned} \tag{4}$$

Thus, in [20] the following solutions are proposed for (1)

$$y(x) = \frac{1}{x} \text{ and } y(x) = -x, \tag{5}$$

these solutions are calculated using the method presented by Hydon [14], which proposes how to find a complete list of all discrete symmetry groups for a differential equation and its possible solutions. In [41], Polyanin and Zaitsev presented the following differential equation

$$y_{xxx} = A\,(y_x)^{-1}\,(y_{xx})^2\,,\ \text{where } A \neq 1, 2 \text{ is a constant.} \tag{6}$$

Note that equation (2) is a particular case of equation (6) when $f(x) = 0$. For this equation they present the following solution

$$y(x) = \frac{1-A}{(2-A)C_1}\,(C_1 x + C_2)^{\frac{2-A}{1-A}} + C_3, \tag{7}$$

where C_1, C_2 and C_3 are arbitrary constants. Using this solution proposed by [41] with $A = \frac{3}{2}$, they got

$$y(x) = -\frac{(xC_1 + C_2)^{-1}}{C_1} + C_3. \tag{8}$$

The objective of our work is: *i*) to provide a complete classification of Lie symmetries group for (2), *ii*) to present the optimal system (optimal algebra) for (1), *iii*) making use of all elements of the optimal algebra, to propose invariant solutions for (1) different from (5) and (8), finally *iv*) to classify the Lie algebra associated to (1), corresponding to the Lie symmetry group.

Lie group symmetry method is an interesting instrument employed to study different types of differential equations. This theory was introduced by the prominent Norwegian mathematician Sophus Lie [28] in the latter half of nineteenth century, following the idea of Galois theory in Algebra who clarified the relationship between the solution of polynomial equations and their symmetries. This method applied to differential equations continues to be useful in the fields of mathematics and applied physics and consequently new results are published an a regular basis. Its importance lies among many things to be built, for example, conservation laws using Noether's theorem [33]. In the same way, it is possible to construct invariant solutions for the differential equation understudy or a reduction of it, which with other traditional methods is not always possible. A huge reference in Lie group method can be found in the literature, e.g., [4, 8, 15, 36, 39].

Recently, the Lie group method approach has been applied to solve and analyze different problems in many scientific fields, e.g., in [25], the authors applied the Lie symmetry method to investigate some solutions for pZK equation, which models the nonlinear propagation of dust-ion acoustic solitary waves and shocks, and in [11] the authors studied the invariance of stochastic differential equations under random diffeomorphisms and established the determining equations for random Lie point symmetries of stochastic differential equations. Some references in the latest progress using Lie group symmetries can be found in [1, 10, 13, 27, 37, 40] and references therein, some recent publications in the area of Lie symmetry groups, optimal system, and invariant solutions can be found in [2, 21, 22, 23, 24, 30, 31, 38].

2. Lie Point Symmetries

In this section, we study the Lie point symmetries of (2). Following the classical Lie technique for calculating the symmetries of differential equations [6, 19, 35, 36], we carried out the complete group classification for (2). The corresponding result can be stated as follows.

Proposition 2.1. *The Lie point symmetry group of the generalization of Kummer - Schwarz differential equation* (2) *with arbitrary f, is generated by*

$$\Pi_1 = \partial_y, \quad \Pi_2 = y\partial_y, \quad \Pi_3 = y^2\partial_y. \quad \textit{Principal algebra,} \tag{9}$$

for the special choices of f listed below (Table 1) we have,

Case	**f(x)**	**Condition**	**Infinitesimal generators of the group**
$i)$	$f(x) = 0$	$---$	$\Pi_1,\ \Pi_2,\ \Pi_3,\ \Pi_4,\ \Pi_5,\ \Pi_6.$
$ii)$	$f(x) = a$	$a > 0$	$\Pi_1,\ \Pi_2,\ \Pi_3,\ \Pi_7,\ \Pi_8,\ \Pi_9,$
$iii)$	$f(x) = a$	$a < 0$	$\Pi_1,\ \Pi_2,\ \Pi_3,\ \Pi_{10},\ \Pi_{11},\ \Pi_{12}$
$iv)$	$f(x) \in C^0$	$c_1(x)f_x + 2c_{1,x}f + c_{1,xxx} = 0$	$\Pi_1,\ \Pi_2,\ \Pi_3,\ \Pi_{13},\ \Pi_{14},\ \Pi_{15}$

Table 1. Function and infinitesimal generators.

Where

$$\begin{array}{lll}
\Pi_1 = \partial_y, & \Pi_2 = y\partial_y, & \Pi_3 = y^2\partial_y, \\
\Pi_4 = \partial_x, & \Pi_5 = x\partial_x, & \Pi_6 = x^2\partial_x. \\
\Pi_7 = \sin(x\sqrt{2a})\partial_x, & \Pi_8 = \cos(x\sqrt{2a})\partial_x & \Pi_9 = \partial_x. \\
\Pi_{10} = e^{\sqrt{2a}x}\partial_x, & \Pi_{11} = e^{-\sqrt{2a}x}\partial_x & \Pi_{12} = \partial_x. \\
\Pi_{13} = z_1^2\partial_x, & \Pi_{14} = z_1 z_2\partial_x & \Pi_{15} = z_2^2\partial_x.
\end{array}$$

with $z_1, z_2 -$ linearly independent solutions of $2z'' - f(x)z = 0$.

Proof. A general form of the one-parameter Lie group admitted by (2) is given by

$$x \to x + \epsilon\xi(x,y) + \cdots \quad \textit{and} \quad y \to y + \epsilon\eta(x,y) + \cdots,$$

where ϵ is the group parameter. The vector field associated with the group of transformations shown above can be written as $\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. Applying its third prolongation

$$\Gamma^{(3)} \quad = \quad \Gamma + \eta_{[x]}\frac{\partial}{\partial y_x} + \eta_{[xx]}\frac{\partial}{\partial y_{xx}} + \eta_{[xxx]}\frac{\partial}{\partial y_{xxx}},$$

to (2), we must find the infinitesimals $\xi(x,y)$, $\eta(x,y)$ satisfying the symmetry condition

$$-\xi(f_x(x)y_x) - f(x)\eta_{[x]} + \frac{3y_{xx}^2}{2y_x^2}\eta_{[x]} - \frac{3y_{xx}}{y_x}\eta_{[xx]} + \eta_{[xxx]} = 0, \tag{10}$$

associated with (2). Here, $\eta_{[x]}, \eta_{[xx]}$ and $\eta_{[xxx]}$ are the coefficients in $\Gamma^{(3)}$ given by

$$\begin{aligned}
\eta_{[x]} &= D_x[\eta] - (D_x[\xi])y_x = \eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2, \\
\eta_{[xx]} &= D_x[\eta_{[x]}] - (D_x[\xi])y_{xx} \\
&= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy}y_x^3 \\
&\quad +(\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx} \\
\text{and } \eta_{[xxx]} &= D_x[\eta_{[xx]}] - (D_x[\xi])y_{xxx} \\
&= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_x + 3(\eta_{xyy} - \xi_{xxy})y_x^2 \\
&\quad + (\eta_{yyy} - 3\xi_{xyy})\, y_x^3 - \xi_{yyy}y_x^4 + 3\,(\eta_{xy} - \xi_{xx})\, y_{xx} \\
&\quad +3\,(\eta_{yy} - 3\xi_{xy})\, y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2 \\
&\quad + (\eta_y - 3\xi_x)\, y_{xxx} - 4\xi_y y_x y_{xxx};
\end{aligned} \tag{11}$$

where D_x is the total derivative operator: $D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{y_x} + y_{xxx}\partial_{y_{xx}} \cdots$. Replacing (11) into (10) we obtain:

$$\begin{aligned}
&- \quad \xi(f_x(x)y_x) - f(x)\left(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2\right) + \frac{3y_{xx}^2}{2y_x^2}\left(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2\right) \\
&- \quad \frac{3y_{xx}}{y_x}\left(\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy}y_x^3 + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}\right) \\
&+ \quad \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_x + 3(\eta_{xyy} - \xi_{xxy})y_x^2 + (\eta_{yyy} - 3\xi_{xyy})\, y_x^3 - \xi_{yyy}y_x^4 \\
&+ \quad 3\,(\eta_{xy} - \xi_{xx})\, y_{xx} + 3\,(\eta_{yy} - 3\xi_{xy})\, y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2 \\
&+ \quad (\eta_y - 3\xi_x)\, y_{xxx} - 4\xi_y y_x y_{xxx} = 0.
\end{aligned}$$

If we denote $f \cong f(x)$ and substitute $y_{xxx} = \frac{3y_{xx}^2 y_x^{-1}}{2} + f y_x$ in the last expression we have

$$\begin{aligned}
&- \quad \xi(f_x(x)y_x) - f\left(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2\right) + \frac{3y_{xx}^2}{2y_x^2}\left(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2\right) \\
&- \quad \frac{3y_{xx}}{y_x}\left(\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy}y_x^3 + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}\right) \\
&+ \quad \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_x + 3(\eta_{xyy} - \xi_{xxy})y_x^2 + (\eta_{yyy} - 3\xi_{xyy})\, y_x^3 - \xi_{yyy}y_x^4 \\
&+ \quad 3\,(\eta_{xy} - \xi_{xx})\, y_{xx} + 3\,(\eta_{yy} - 3\xi_{xy})\, y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2 \\
&+ \quad (\eta_y - 3\xi_x)\left(\frac{3y_{xx}^2 y_x^{-1}}{2} + f y_x\right) - 4\xi_y y_x\left(\frac{3y_{xx}^2 y_x^{-1}}{2} + f y_x\right) = 0.
\end{aligned}$$

Thus, we can rearrange the last expression with respect to $1, y_x, y_x^2, y_x^3, y_x^4$, $y_{xx}, y_{xx}^2, y_x^{-1}y_{xx}, y_x y_{xx}, y_x^{-1}y_{xx}^2, y_x^{-2}y_{xx}$ and $y_x^{-2}y_{xx}^2$ and canceling some terms we obtain the determining equations for the symmetry group of (2). That is:

$$\xi_y = \eta_x = \eta_{yyy} = 0, \tag{12a}$$

$$-\xi f_x - 2f\xi_x - \xi_{xxx} = 0. \tag{12b}$$

Solving in (12a), we have $\xi = c_1(x)$ and $\eta = \frac{k_1}{2}y^2 + k_2 y + k_3$, with k_1, k_2, k_3 as arbitrary constants and $c_1(x)$ arbitrary function. If f is arbitrary then in (12b) we have $\xi = 0$

and $\eta = \frac{k_1}{2}y^2 + k_2 y + k_3$, where k_1, k_2 and k_3 are arbitrary constants. Therefore the group of symmetries are $\Pi_1 = \partial_y$, $\Pi_2 = y\partial_y$, $\Pi_3 = y^2\partial_y$. This is consistent with what is presented in [18, 9]. If $f = a \neq 0$ where a is an arbitrary constant. Thus, we have two cases in (12b) which are $a > 0$ and $a < 0$.

Case I: If $a > 0$ then in (12b) we have $2a\xi_x + \xi_{xxx} = 0$ and solving for ξ we obtain $\xi = \frac{k_7}{\sqrt{2a}}\sin(x\sqrt{2a}) + \frac{k_8}{\sqrt{2a}}\cos(x\sqrt{2a}) + k_9$ where k_7, k_8 and k_9 are arbitrary constants, then of (12a) we have $\eta = \frac{k_1}{2}y^2 + k_2 y + k_3$, Therefore the group of symmetries are

$$\begin{aligned} &\Pi_1 = \partial_y, && \Pi_2 = y\partial_y, && \Pi_3 = y^2\partial_y, \\ &\Pi_7 = \sin(x\sqrt{2a})\partial_x, && \Pi_8 = \cos(x\sqrt{2a})\partial_x, && \Pi_9 = \partial_x. \end{aligned}$$

Case II: If $a < 0$ then in (12b) we have $-2a\xi_x + \xi_{xxx} = 0$ and solving for ξ we obtain $\xi = \frac{k_{10}e^{\sqrt{2a}x}}{\sqrt{2a}} + \frac{k_{11}e^{-\sqrt{2a}x}}{\sqrt{2a}} + k_{12}$ where k_{10}, k_{11} and k_{12} are arbitrary constants, then of (12a) we have $\eta = \frac{k_1}{2}y^2 + k_2 y + k_3$. Therefore the group of symmetries are

$$\begin{aligned} &\Pi_1 = \partial_y, && \Pi_2 = y\partial_y, && \Pi_3 = y^2\partial_y, \\ &\Pi_{10} = e^{\sqrt{2a}x}\partial_x, && \Pi_{11} = e^{-\sqrt{2a}x}\partial_x, && \Pi_{12} = \partial_x. \end{aligned}$$

If $f = 0$ then in (12b) we have $\xi_{xxx} = 0$ and solving for ξ we obtain $\xi = \frac{k_6}{2}x^2 + k_5 x + k_4$ where k_4, k_5 and k_6 are arbitrary constants, then of (12a) we have $\eta = \frac{k_3}{2}y^2 + k_2 y + k_1$. Therefore the group of symmetries are

$$\begin{aligned} &\Pi_1 = \partial_y, && \Pi_2 = y\partial_y, && \Pi_3 = y^2\partial_y, \\ &\Pi_4 = \partial_x, && \Pi_5 = x\partial_x, && \Pi_6 = x^2\partial_x. \end{aligned}$$

This is consistent with what is presented in [9, 20]. Let's remember that $\xi = c_1(x)$, then for the other cases of f in $(12b)$ the following differential equation must be satisfied $c_1(x)f_x + 2c_{1,x}f + c_{1,xxx} = 0$, so, if the functions $c_1(x)$ and $f(x)$ satisfy Eq. $(12b)$ then we have the following operator, additional to (9): $X_{c_1} = c_1(x)\partial_x$, therefore following [3] we have the solution is given in the terms of two linearly independent solutions of the associated to $(12b)$ homogeneous second-order Sturm-Liouville equation

$$z''(x) - \frac{f(x)}{2}z(x) = 0. \tag{13}$$

By the existence and uniqueness theorem if $f(x) \in C^0$ then there exists a nonzero solution z_1 of Eq. (13). We look for another solution z_2 of (13) such that

$$W = z_1 z_2' - z_1' z_2 = 1,$$

that is z_1 and z_2 are linearly independent solutions of (13) with Wronskian = 1. Thus therefore following [3] we have the solution $c_1(x) = k_{13}z_1^2(x) + k_{14}z_1(x)z_2(x) + k_{15}z_2^2(x)$, then it follows that for any $f \in C^0$ there are three additional symmetries. Namely:

$$\Pi_{13} = z_1^2(x)\partial_x, \;\; \Pi_{14} = z_1(x)z_2(x)\partial_x, \;\; \Pi_{15} = z_2^2(x)\partial_x.$$

Thus, the demonstration of the Proposition 2.1 is finished. ☑

3. Optimal Algebra

Taking into account [15, 17, 35, 42], we present in this section the optimal algebra associated to the symmetry group of (1), that shows a systematic way to classify the invariant solutions.

To obtain the optimal algebra, we should first calculate the corresponding commutator table, which can be obtained from the operator

$$[\Pi_\alpha, \Pi_\beta] = \Pi_\alpha \Pi_\beta - \Pi_\beta \Pi_\alpha = \sum_{i=1}^{n} \left(\Pi_\alpha(\xi^i_\beta) - \Pi_\beta(\xi^i_\alpha)\right) \frac{\partial}{\partial x^i}, \tag{14}$$

where $i = 1, 2$, with $\alpha, \beta = 1, \cdots, 6$ and $\xi^i_\alpha, \xi^i_\beta$ are the corresponding coefficients of the infinitesimal operators Π_α, Π_β. After applying the operator (14) to the symmetry group of (1), we obtain the operators that are shown in the following table

	Π_1	Π_2	Π_3	Π_4	Π_5	Π_6
Π_1	0	Π_1	$2\Pi_2$	0	0	0
Π_2	$-\Pi_1$	0	Π_3	0	0	0
Π_3	$-2\Pi_2$	$-\Pi_3$	0	0	0	0
Π_4	0	0	0	0	Π_4	$2\Pi_5$
Π_5	0	0	0	$-\Pi_4$	0	Π_6
Π_6	0	0	0	$-2\Pi_5$	$-\Pi_6$	0

Table 2. Commutators table associated to the symmetry group of (1).

Now, the next thing is to calculate the adjoint action representation of the symmetries of (1) and to do that, we use Table 2 and the operator

$$\mathrm{Ad}(\exp(\lambda\Pi))H = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(\mathrm{ad}(\Pi))^n G \text{ for the symmetries } \Pi \text{ and } G.$$

Making use of this operator, we can construct the Table 3, which shows the adjoint representation for each Π_i.

adj[,]	Π_1	Π_2	Π_3	Π_4	Π_5	Π_6
Π_1	Π_1	$\Pi_2 - \lambda\Pi_1$	$\Pi_3 - 2\lambda\Pi_2 + \lambda^2\Pi_1$	Π_4	Π_5	Π_6
Π_2	$\Pi_1 e^{\lambda}$	Π_2	$\Pi_3 e^{-\lambda}$	Π_4	Π_5	Π_6
Π_3	$\Pi_1 + 2\lambda\Pi_2 + \lambda^2\Pi_3$	$\Pi_2 + \lambda\Pi_3$	Π_3	Π_4	Π_5	Π_6
Π_4	Π_1	Π_2	Π_3	Π_4	$\Pi_5 - \lambda\Pi_4$	$\Pi_6 - 2\lambda\Pi_5 + \lambda^2\Pi_4$
Π_5	Π_1	Π_2	Π_3	$e^{\lambda}\Pi_4$	Π_5	$e^{-\lambda}\Pi_6$
Π_6	Π_1	Π_2	Π_3	$\Pi_4 + 2\lambda\Pi_5 + 2\lambda^2\Pi_6$	$\Pi_5 + \lambda\Pi_6$	Π_6

Table 3. Adjoint representation of the symmetry group of (1).

Proposition 3.1. *The optimal algebra associated to the equation* (1) *is given by the vector fields*

$$\Pi_2 + b_{30}\Pi_3\,;\; a_3\Pi_3 + \Pi_6\,;$$

$$b_{27}\Pi_1 + b_{28}\Pi_2 + b_{29}^2\Pi_3\,;\; a_3\Pi_3 + \Pi_5 + b_{15}\Pi_6\,;\; a_2\Pi_2 + b_6\Pi_3 + \Pi_6\,;\; a_3\Pi_3 + b_4\Pi_4 + 2b_5\Pi_5\,;$$

$$b_1\Pi_1 - a_3\Pi_2 + \Pi_6\,;\; a_2\Pi_2 + b_6\Pi_3 + b_7\Pi_4 + b_8\Pi_5\,;\; b_1\Pi_1 - a_3\Pi_2 + b_2\Pi_4 + b_3\Pi_5\,;$$
$$a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + \Pi_6\,;\; b_{13}\Pi_1 + 2b_{13}a_4\Pi_2 + \Pi_5 + b_{14}\Pi_6\,;\; a_2\Pi_2 + b_{16}\Pi_3 + \Pi_5 + b_{17}\Pi_6\,;$$
$$a_1\Pi_1 + b_{18}\Pi_2 + b_{19}\Pi_3 + \Pi_5 + b_{20}\Pi_6\,;\; a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + b_{11}\Pi_4 + b_{12}\Pi_5\,;$$
$$a_2\Pi_2 + b_{25}\Pi_3 + \Pi_4 + 2b_{26}\Pi_5 + 2b_{26}^2\Pi_6\,;$$
$$b_{21}\Pi_1 + b_{22}\Pi_2 + b_{23}\Pi_3 + \Pi_4 + 2b_{24}\Pi_5 + 2b_{24}^2\Pi_6.$$

Proof. To calculate the optimal algebra system, we start with the generators of symmetries (1) and a generic nonzero vector. Let

$$G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + a_6\Pi_6. \tag{15}$$

The objective is to simplify as many coefficients a_i as possible, through maps adjoint to G, using Table 3.

1) Assuming $a_6 = 1$ in (15) we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$. Applying the adjoint operator to (Π_1, G), we get

$$G_1 = \mathrm{Ad}\,(\exp\,(\lambda_1\Pi_1))\,G = \tag{16}$$
$$(a_1 - a_2\lambda_1 + a_3\lambda_1^2)\Pi_1 + (a_2 - 2a_3\lambda_1)\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6.$$

1.1) **Case** $a_3 \neq 0$**.** Using $\lambda_1 = \frac{a_2}{2a_3}$ with $a_3 \neq 0$, in (16), Π_2 is eliminated, therefore $G_1 = b_1\Pi_1 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$ with $b_1 = \frac{4a_1 - a_2^2}{4a_3}$. Now, applying the adjoint operator to (Π_2, G_1) we don't have any reductions, thus applying the adjoint operator (Π_3, G_1) we get $G_2 = \mathrm{Ad}\,(\exp\,(\lambda_2\Pi_3))\,G_1 = b_1\Pi_1 + b_1\lambda_2^2\Pi_2 + (a_3 + b_1\lambda_2^2)\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$.

1.1.A) **Case** $a_2^2 - 4a_1 \neq 0$**.** Using $\lambda_2 = 2a_3\sqrt{\frac{1}{a_2^2 - 4a_1}}$ with $a_2^2 - 4a_1 \neq 0$, is eliminated Π_3, thus $G_2 = b_1\Pi_1 - a_3\Pi_2 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$. Now, applying the adjoint operator to (Π_4, G_2), we get

$$G_3 = \mathrm{Ad}\,(\exp\,(\lambda_3\Pi_4))\,G_2 = b_1\Pi_1 - a_3\Pi_2 + (a_4 - a_5\lambda_3 + \lambda_3^2)\Pi_4 + (a_5 - 2\lambda_3)\Pi_5 + \Pi_6. \tag{17}$$

Using $\lambda_3 = \frac{a_5}{2}$ is eliminated Π_5, thus $G_3 = b_1\Pi_1 - a_3\Pi_2 + b_2\Pi_4 + \Pi_6$, with $b_2 = \frac{2a_4 - a_5^2}{4}$. Now applying the adjoint operator to (Π_5, G_3), we don't have any reduction, but applying the adjoint operator to (Π_6, G_3) we get

$$G_4 = \mathrm{Ad}\,(\exp\,(\lambda_4\Pi_6))\,G_3 = b_1\Pi_1 - a_3\Pi_2 + b_2\Pi_4 + 2b_2\lambda_4\Pi_5 + (2b_2\lambda_4^2 + 1)\Pi_6.$$

1.1.A.A_1) **Case** $a_5^2 - 2a_4 \neq 0$**.** Using $\lambda_4 = \sqrt{\frac{2}{a_5^2 - 2a_4}}$ with $a_5^2 - 2a_4 \neq 0$, is eliminated Π_6. Then we have an element of the optimal algebra

$$G_4 = b_1\Pi_1 - a_3\Pi_2 + b_2\Pi_4 + b_3\Pi_5, \quad \text{with } b_3 = 2b_2\lambda_4. \tag{18}$$

This is how a reduction of the generic element (15) ends.

$1.1.A.A_2$) **Case** $a_5^2 - 2a_4 = 0$**.** Then we get $b_2 = 0$, thus $G_4 = b_1\Pi_1 - a_3\Pi_2 + \Pi_6$. Then we have an element of the optimal algebra

$$G_4 = b_1\Pi_1 - a_3\Pi_2 + \Pi_6. \tag{19}$$

This is another way how a reduction of the generic element (15) ends.

$1.1.B$) **Case** $a_2^2 - 4a_1 = 0$**.** We get $b_1 = 0$, thus $G_2 = a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$. Now, applying the adjoint operator to (Π_4, G_2), we have

$$G_5 = \mathrm{Ad}\left(\exp\left(\lambda_5\Pi_4\right)\right) G_2 = a_3\Pi_3 + (a_4 - a_5\lambda_5 + \lambda_5^2)\Pi_4 + (a_5 - 2\lambda_5)\Pi_5 + \Pi_6. \tag{20}$$

Using $\lambda_5 = \frac{a_5}{2}$, then Π_5 is eliminated, then we obtain $G_5 = a_3\Pi_3 + b_4\Pi_4 + \Pi_6$, with $b_4 = \frac{4a_4 - a_5^2}{4}$. Now applying the adjoint operator to (Π_5, G_5), we don't have any reduction, but applying the adjoint operator to (Π_6, G_5) we have $G_6 = \mathrm{Ad}\left(\exp\left(\lambda_6\Pi_6\right)\right) G_5 = a_3\Pi_3 + b_4\Pi_4 + 2b_4\lambda_6\Pi_5 + (2\lambda_6^2 b_4 + 1)\Pi_6$.

$1.1.B.B_1$) **Case** $a_5^2 - 4a_4 \neq 0$**.** Using $\lambda_6 = \sqrt{\frac{2}{a_5^2 - 4a_4}}$ with $a_5^2 - 4a_4 \neq 0$, is eliminated Π_6. Then we have an element of the optimal algebra

$$G_6 = a_3\Pi_3 + b_4\Pi_4 + 2b_5\Pi_5, \quad \text{with } b_5 = 2b_4\lambda_6. \tag{21}$$

This is how a reduction of the generic element (15) ends.

$1.1.B.B_2$) **Case** $a_5^2 - 4a_4 = 0$**.** We get $b_4 = 0$, then we have an element of the optimal algebra

$$G_6 = a_3\Pi_3 + \Pi_6. \tag{22}$$

This is how a reduction of the generic element (15) ends.

1.2) **Case** $a_3 = 0$**.** We get, $G_1 = (a_1 - a_2\lambda_1)\Pi_1 + a_2\Pi_2 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$.

$1.2.A$) **Case** $a_2 \neq 0$**.** Using $\lambda_1 = \frac{a_1}{a_2}$, with $a_2 \neq 0$, is eliminated Π_1, thus we get $G_1 = a_2\Pi_2 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$. Now, applying the adjoint operator to (Π_2, G_5) and (Π_5, G_5), we don't have any reduction, then applying the adjoint operator to (Π_3, G_1) we get $G_7 = \mathrm{Ad}\left(\exp\left(\lambda_7\Pi_3\right)\right) G_1 = a_2\Pi_2 + a_2\lambda_7\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$, it is clear that we don't have any reduction, then we have $G_7 = a_2\Pi_2 + b_6\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$, with $b_6 = a_2\lambda_7$. Now, applying the adjoint operator to (Π_4, G_7) we get $G_8 = \mathrm{Ad}\left(\exp\left(\lambda_8\Pi_4\right)\right) G_7 = a_2\Pi_2 + b_6\Pi_3 + (a_4 - a_5\lambda_8 + \lambda_8^2)\Pi_4 + (a_5 - 2\lambda_8)\Pi_5 + \Pi_6$. Using $\lambda_8 = \frac{a_5}{2}$, is eliminated Π_5, the we have $G_8 = a_2\Pi_2 + b_6\Pi_3 + b_7\Pi_4 + \Pi_6$, with $b_7 = \frac{4a_4 - a_5^2}{4}$. Now, applying the adjoint operator to (Π_6, G_8) we get $G_9 = Ad\left(\exp\left(\lambda_9\Pi_6\right)\right) G_8 = a_2\Pi_2 + b_6\Pi_3 + b_7\Pi_4 + 2b_7\lambda_9\Pi_5 + (2b_7\lambda_9^2 + 1)\Pi_6$.

$1.2.A.A_1$) **Case** $a_5^2 - 4a_4 \neq 0$**.** Using $\lambda_9 = \sqrt{\frac{2}{a_5^2 - 4a_4}}$ with $a_5^2 - 4a_4 \neq 0$, is eliminated Π_6. Then we have an element of the optimal algebra

$$G_9 = a_2\Pi_2 + b_6\Pi_3 + b_7\Pi_4 + b_8\Pi_5, \quad \text{with } b_8 = 2b_7\lambda_9. \tag{23}$$

This is how a reduction of the generic element (15) ends.

$1.2.A.A_2$) **Case** $a_5^2 - 4a_4 = 0$**.** We get $b_7 = 0$, then we have an element of the optimal algebra

$$G_9 = a_2\Pi_2 + b_6\Pi_3 + \Pi_6. \tag{24}$$

This is how a reduction of the generic element (15) ends.

1.2.B) **Case** $a_2 = 0$**.** We get $G_1 = a_1\Pi_1 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$. Now, applying the adjoint operator to (Π_2, G_5) and (Π_5, G_5), we don't have any reduction, then applying the adjoint operator to (Π_3, G_1) we get $G_{10} = \mathrm{Ad}\,(\exp(\lambda_{10}\Pi_3))\,G_1 = a_1\Pi_1 + 2a_1\lambda_{10}\Pi_2 + a_1\lambda_{10}\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$, it is clear that we don't have any reduction, then we have $G_{10} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + a_4\Pi_4 + a_5\Pi_5 + \Pi_6$, with $b_9 = 2a_1\lambda_{10}$ and $b_{10} = a_1\lambda_{10}$.

Now, applying the adjoint operator to (Π_4, G_{10}) we get $G_{11} = \mathrm{Ad}\,(\exp(\lambda_{11}\Pi_4))\,G_{11} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + (a_4 - a_5\lambda_{11} + \lambda_{11}^2)\Pi_4 + (a_5 - 2\lambda_{11})\Pi_5 + \Pi_6$. Using $\lambda_{11} = \frac{a_5}{2}$, is eliminated Π_5, then we have $G_{11} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + b_{11}\Pi_4 + \Pi_6$, with $b_{11} = \frac{4a_4 - a_5^2}{4}$.

Now, applying the adjoint operator to (Π_6, G_{11}) we get $G_{12} = \mathrm{Ad}\,(\exp(\lambda_{12}\Pi_6))\,G_{11} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + b_{11}\Pi_4 + 2b_{11}\lambda_{12}\Pi_5 + (2b_{11}\lambda_{12}^2 + 1)\Pi_6$.

1.2.$B.A_1$) **Case** $4a_4 - a_5^2 \neq 0$**.** Using $\lambda_{12} = \sqrt{\frac{2}{a_5^2 - 4a_4}}$, is eliminated Π_6, then we have an element of the optimal algebra

$$G_{12} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + b_{11}\Pi_4 + b_{12}\Pi_5, \quad \text{with}\, b_{12} = 2b_{11}\lambda_{12}. \tag{25}$$

This is how a reduction of the generic element (15) ends.

1.2.$B.A_2$) **Case** $4a_4 - a_5^2 = 0$**.** We get $b_{11} = 0$, thus we have $G_{12} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + \Pi_6$, then we have an element of the optimal algebra

$$G_{12} = a_1\Pi_1 + b_9\Pi_2 + b_{10}\Pi_3 + \Pi_6. \tag{26}$$

This is how a reduction of the generic element (15) ends.

2) Assuming $a_6 = 0$ and $a_5 = 1$ in (15) we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Applying the adjoint operator to (Π_2, G) and (Π_5, G) we don't have any reduction, thus applying the adjoint operator to (Π_1, G) we get

$$G_{13} = \mathrm{Ad}\,(\exp(\lambda_1\Pi_1))\,G = \tag{27}$$
$$(a_1 - a_2\lambda_{13} + a_3\lambda_{13}^2)\Pi_1 + (a_2 - 2a_3\lambda_{13})\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5.$$

2.1) **Case** $a_3 \neq 0$**.** Using $\lambda_{13} = \frac{a_2}{2a_3}$, with $a_3 \neq 0$, is eliminated Π_2, we get $G_{13} = \left(a_1 - \frac{a_2^2}{2a_3} + \frac{a_2^2}{4a_3}\right)\Pi_1 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Then using $b_{13} = a_1 - \frac{a_2^2}{2a_3} + \frac{a_2^2}{4a_3}$, we have $G_{13} = b_{13}\Pi_1 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Now, applying the adjoint operator (Π_3, G_{13}), we have

$$G_{14} = \mathrm{Ad}\,(\exp(\lambda_{14}\Pi_3))\,G_{13} = b_{13}\Pi_1 + 2b_{13}\lambda_{14}\Pi_2 + (a_3 + b_{13}\lambda_{14}^2)\Pi_3 + a_4\Pi_4 + \Pi_5.$$

2.1.A) **Case** $4a_1a_3 - a_2^2 \neq 0$**.** Using $\lambda_{14} = \sqrt{\frac{4a_3^2}{a_2^2 - 4a_1a_3}}$, is eliminated Π_3, the we get $G_{14} = b_{13}\Pi_1 + 2b_{13}\lambda_{14}\Pi_2 + a_4\Pi_4 + \Pi_5$. Now applying the adjoint operator to (Π_4, G_{14}), we have:

$$G_{15} = \mathrm{Ad}\,(\exp(\lambda_{15}\Pi_4))\,G_{14} = b_{13}\Pi_1 + 2b_{13}\lambda_{15}\Pi_2 + (a_4 - \lambda_{15})\Pi_4 + \Pi_5.$$

Using $\lambda_{15} = a_4$, is eliminated Π_4, then $G_{15} = b_{13}\Pi_1 + 2b_{13}a_4\Pi_2 + \Pi_5$. Now applying the adjoint operator to (Π_6, G_{15}), we have:

$$G_{16} = \mathrm{Ad}\left(\exp\left(\lambda_{16}\Pi_6\right)\right) G_{15} = b_{13}\Pi_1 + 2b_{13}a_4\Pi_2 + \Pi_5 + \lambda_{16}\Pi_6.$$

We don't have any reduction, then we have an element of the optimal algebra

$$G_{16} = b_{13}\Pi_1 + 2b_{13}a_4\Pi_2 + \Pi_5 + b_{14}\Pi_6. \tag{28}$$

With $b_{14} = \lambda_{16}$. This is how a reduction of the generic element (15) ends.

2.1.B) **Case** $4a_1a_3 - a_2^2 = 0$**.** We get $b_{13} = 0$, then $G_{14} = a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Now applying the operator to (Π_4, G_{14}) we have

$$G_{17} = \mathrm{Ad}\left(\exp\left(\lambda_{17}\Pi_4\right)\right) G_{14} = a_3\Pi_3 + (a_4 - \lambda_{17})\Pi_4 + \Pi_5.$$

Using $\lambda_{17} = a_4$, is eliminated Π_4, then we have $G_{17} = a_3\Pi_3 + \Pi_5$. Now using the operator to (Π_6, G_{17}), we get:

$$G_{18} = \mathrm{Ad}\left(\exp\left(\lambda_{18}\Pi_6\right)\right) G_{17} = a_3\Pi_3 + \Pi_5 + \lambda_{18}\Pi_6,$$

We don‘t have any reduction, then using $\lambda_{18} = b_{15}$, we have an element of the optimal algebra

$$G_{18} = a_3\Pi_3 + \Pi_5 + b_{15}\Pi_6. \tag{29}$$

This is how a reduction of the generic element (15) ends.

2.2) **Case** $a_3 = 0$**.** We get $G_{13} = (a_1 - a_2\lambda_{13})\Pi_1 + a_2\Pi_2 + a_4\Pi_4 + \Pi_5$.

2.2.A_1) **Case** $a_2 \neq 0$**.** Using $\lambda_{13} = \frac{a_1}{a_2}$, with $a_2 \neq 0$, is eliminated Π_1, then we have $G_{13} = a_2\Pi_2 + a_4\Pi_4 + \Pi_5$. Now applying the operator to (Π_3, G_{13}) we have $G_{19} = \mathrm{Ad}\left(\exp\left(\lambda_{19}\Pi_3\right)\right) G_{13} = a_2\Pi_2 + a_2\lambda_{19}\Pi_3 + a_4\Pi_4 + \Pi_5$. It is clear that we don't have any reduction, then using $b_{16} = a_2\lambda_{19}$ we get

$$G_{19} = a_2\Pi_2 + b_{16}\Pi_3 + a_4\Pi_4 + \Pi_5.$$

Now applying the operator to (Π_4, G_{19}) we get: $G_{20} = \mathrm{Ad}\left(\exp\left(\lambda_{20}\Pi_4\right)\right) G_{20} = a_2\Pi_2 + b_{16}\Pi_3 + (a_4 - \lambda_{20})\Pi_4 + \Pi_5$. Then using $\lambda_{20} = a_4$, is eliminated Π_4, thus $G_{19} = a_2\Pi_2 + b_{16}\Pi_3 + \Pi_5$. Now applying the operator (Π_6, G_{20}), we get

$$G_{21} = \mathrm{Ad}\left(\exp\left(\lambda_{21}\Pi_6\right)\right) G_{20} = a_2\Pi_2 + b_{16}\Pi_3 + \Pi_5 + \lambda_{21}\Pi_6.$$

It is clear that we don't have any reduction. The using $\lambda_{21} = b_{17}$, we have an element of the optimal algebra

$$G_{21} = a_2\Pi_2 + b_{16}\Pi_3 + \Pi_5 + b_{17}\Pi_6. \tag{30}$$

This is how a reduction of the generic element (15) ends.

2.2.A_2) **Case** $a_2 = 0$**.** We get $G_{13} = a_1\Pi_1 + a_4\Pi_4 + \Pi_5$. Now applying the operator to (Π_3, G_{13}) we have

$$G_{22} = \mathrm{Ad}\left(\exp\left(\lambda_{22}\Pi_3\right)\right) G_{13} = a_1\Pi_1 + 2a_1\lambda_{22}\Pi_2 + a_1\lambda_{22}^2\Pi_3 + a_4\Pi_4 + \Pi_5.$$

we don't have any reduction, then using $b_{18} = 2a_1\lambda_{22}$ and $b_{19} = a_1\lambda_{22}^2$, we obtain

$$G_{22} = a_1\Pi_1 + b_{18}\Pi_2 + b_{19}\Pi_3 + a_4\Pi_4 + \Pi_5.$$

Now, applying the operator to (Π_4, G_{22}) we have

$$G_{23} = \operatorname{Ad}(\exp(\lambda_{23}\Pi_4))\, G_{22} = a_1\Pi_1 + b_{18}\Pi_2 + b_{19}\Pi_3 + (a_4 - \lambda_{23})\Pi_4 + \Pi_5.$$

Using $\lambda_{23} = a_4$, is eliminated Π_4 then we get $G_{23} = a_1\Pi_1 + b_{18}\Pi_2 + b_{19}\Pi_3 + \Pi_5$. Now applying the operator to (Π_6, G_{23}) we have

$$G_{24} = \operatorname{Ad}(\exp(\lambda_{24}\Pi_6))\, G_{23} = a_1\Pi_1 + b_{18}\Pi_2 + b_{19}\Pi_3 + \Pi_5 + \lambda_{24}\Pi_6.$$

It is clear that we don't have any reduction, then using $\lambda_{24} = b_{20}$, we have an element of the optimal algebra

$$G_{24} = a_1\Pi_1 + b_{18}\Pi_2 + b_{19}\Pi_3 + \Pi_5 + b_{20}\Pi_6. \tag{31}$$

This is how a reduction of the generic element (15) ends.

3) Assuming $a_6 = a_5 = 0$ and $a_4 = 1$ in (15), we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + \Pi_4$. Applying the adjoint operator to (Π_2, G), (Π_4, G) and (Π_5, G) we don't have any reduction, on the other hand applying the adjoint operator to (Π_1, G) we get

$$G_{25} = \operatorname{Ad}(\exp(\lambda_{25}\Pi_1))\, G = (a_1 - 2\lambda_{25} + a_3\lambda_{25}^2)\Pi_1 + (a_2 - 2a_3\lambda_{25})\Pi_2 + a_3\Pi_3 + \Pi_4. \tag{32}$$

3.1) **Case** $a_3 \neq 0$. Using $\lambda_{25} = \frac{a_2}{2a_3}$, with $a_3 \neq 0$, in (32), Π_2 is eliminated, therefore $G_{25} = b_{21}\Pi_1 + a_3\Pi_3 + \Pi_4$, with $b_{21} = a_1 + 2\lambda_{25} + a_3\lambda_{25}^2$. Now, applying the adjoint operator to (Π_3, G_{25}), we get $G_{26} = \operatorname{Ad}(\exp(\lambda_{26}\Pi_3))\, G_{25} = b_{21}\Pi_1 + 2b_{21}\lambda_{26}\Pi_2 + (a_3 + b_{21}\lambda_{26}^2)\Pi_3 + \Pi_4$. It is clear that we don't have any reduction, then $G_{26} = b_{21}\Pi_1 + b_{22}\Pi_2 + b_{23}\Pi_3 + \Pi_4$, with $b_{22} = 2b_{21}\lambda_{26}$ and $b_{23} = a_3 + b_{21}\lambda_{26}^2$. Now applying the adjoint operator to (Π_6, G_{26}) we have

$$G_{27} = \operatorname{Ad}(\exp(\lambda_{27}\Pi_6))\, G_{26} = b_{21}\Pi_1 + b_{22}\Pi_2 + b_{23}\Pi_3 + \Pi_4 + 2\lambda_{27}\Pi_5 + 2\lambda_{27}^2\Pi_6.$$

It is clear that we don't have any reduction, then using $\lambda_{27} = b_{24}$, we have other element of the optimal algebra

$$G_{27} = b_{21}\Pi_1 + b_{22}\Pi_2 + b_{23}\Pi_3 + \Pi_4 + 2b_{24}\Pi_5 + 2b_{24}^2\Pi_6. \tag{33}$$

This is how other reduction of the generic element (15) ends.

3.2) **Case** $a_3 = 0$. We get $G_{25} = (a_1 - 2\lambda_{25})\Pi_1 + a_2\Pi_2 + \Pi_4$, using $\lambda_{25} = \frac{a_1}{2}$, is eliminated Π_1, thus $G_{25} = a_2\Pi_2 + \Pi_4$. Now applying the adjoint operator to (Π_3, G_{25}) we get:

$$G_{28} = \operatorname{Ad}(\exp(\lambda_{28}\Pi_3))\, G_{25} = a_2\Pi_2 + a_2\lambda_{28}\Pi_3 + \Pi_4.$$

we don't have any reduction, then using $a_2\lambda_{28} = b_{25}$, we obtain $G_{28} = a_2\Pi_2 + b_{25}\Pi_3 + \Pi_4$. Now applying the adjoint operator to (Π_6, G_{28}) we have:

$$G_{29} = \operatorname{Ad}(\exp(\lambda_{29}\Pi_6))\, G_{28} = a_2\Pi_2 + b_{25}\Pi_3 + \Pi_4 + 2\lambda_{29}\Pi_5 + 2\lambda_{29}^2\Pi_6.$$

It is clear that we don't have any reduction, then using $\lambda_{29} = b_{26}$, we have other element of the optimal algebra

$$G_{29} = a_2\Pi_2 + b_{25}\Pi_3 + \Pi_4 + 2b_{26}\Pi_5 + 2b_{26}^2\Pi_6. \tag{34}$$

This is how other reduction of the generic element (15) ends.

4) Assuming $a_6 = a_5 = a_4 = 0$ and $a_3 = 1$ in (15), we have that $G = a_1\Pi_1 + a_2\Pi_2 + \Pi_3$. Applying the adjoint operator to (Π_2, G), (Π_4, G), (Π_5, G) and (Π_6, G) we don't have any reduction, on the other hand, applying the adjoint operator to (Π_1, G) we get

$$G_{30} = \mathrm{Ad}\left(\exp\left(\lambda_{30}\Pi_1\right)\right) G = (a_1 - a_2\lambda_{30} + \lambda_{30}^2)\Pi_1 + (a_2 - 2\lambda_{30})\Pi_2 + \Pi_3. \tag{35}$$

Using $\lambda_{30} = \frac{a_2}{2}$, in (35), is eliminated Π_2, therefore $G_{30} = b_{27}\Pi_1 + \Pi_3$, with $b_{27} = \frac{4a_1 - a_2^2}{4}$. Now, applying the adjoint operator to (Π_3, G_{30}) we get:

$$G_{31} = \mathrm{Ad}\left(\exp\left(\lambda_{31}\Pi_3\right)\right) G_{30} = b_{27}\Pi_1 + 2b_{27}\lambda_{31}\Pi_2 + (1 + \lambda_{31}^2)\Pi_3.$$

Then, we don't have any reduction, thus using $b_{28} = 2b_{27}\lambda_{31}$ and $b_{29} = 1 + \lambda_{31}^2$, we have other element of the optimal algebra

$$G_{31} = b_{27}\Pi_1 + b_{28}\Pi_2 + b_{29}\Pi_3. \tag{36}$$

This is how other reduction of the generic element (15) ends.

5) Assuming $a_6 = a_5 = a_4 = a_3 = 0$ and $a_2 = 1$ in (15), we have that $G = a_1\Pi_1 + \Pi_2$. Applying the adjoint operator to (Π_2, G), (Π_4, G), (Π_5, G) and (Π_6, G) we don't have any reduction, on the other hand, applying the adjoint operator to (Π_1, G) we get

$$G_{32} = \mathrm{Ad}\left(\exp\left(\lambda_{32}\Pi_1\right)\right) G = (a_1 - \lambda_{32})\Pi_1 + \Pi_2 \tag{37}$$

Using $\lambda_{32} = a_1$, is eliminated Π_1, thus we get $G_{32} = \Pi_2$. Now applying the adjoint operator to (Π_3, G_{32}), we have

$$G_{33} = \mathrm{Ad}\left(\exp\left(\lambda_{33}\Pi_3\right)\right) G_{32} = \Pi_2 + \lambda_{32}\Pi_3.$$

We don't have any reduction, then using $\lambda_{33} = b_{30}$ we have other element of the optimal algebra

$$G_{33} = \Pi_2 + b_{30}\Pi_3. \tag{38}$$

This is how other reduction of the generic element (15) ends.

6) Assuming $a_6 = a_5 = a_4 = a_3 = a_2 = 0$ and $a_1 = 1$ in (15), we have that $G = \Pi_1$. Applying the adjoint operator to (Π_1, G), (Π_2, G), (Π_3, G), (Π_4, G), (Π_5, G) and (Π_6, G) we don't have any reduction, on the other hand, applying the adjoint operator to (Π_3, G) we get

$$G_{34} = \mathrm{Ad}\left(\exp\left(\lambda_{34}\Pi_3\right)\right) G = \Pi_1 + 2\lambda_{34}\Pi_2 + \lambda_{34}^2\Pi_3. \tag{39}$$

It is clear that we don't have any reduction, then using $\lambda_{34} = b_{31}$ we have other element of the optimal algebra

$$G_{34} = \Pi_1 + 2b_{31}\Pi_2 + b_{31}^2\Pi_3. \tag{40}$$

This is how other reduction of the generic element (15) ends.

☑

4. Invariant solutions by generators of the optimal algebra

In this section, we characterize all invariant solutions taking into account some operators that generate the optimal algebra presented in Proposition 3.1. For this purpose, we use the method of invariant curve condition [15] (presented in section 4.3), which is given by the following equation

$$Q(x,y,y_x) = \eta - y_x\xi = 0. \tag{41}$$

Using the element $\Pi_2+\Pi_3$ from Proposition 3.1, under the condition (41), we obtain that $Q = \eta_1 - y_x\xi_1 = 0$, which implies $(y+y^2) - y_x(0) = 0$, then solving this ODE we have $\left(y(x)+\frac{1}{2}\right)^2 - \frac{1}{4} = 0$, which is trivial solution for (2) with $y(x) = 0$ or $y(x) = -1$, using an analogous procedure with all of the elements of the optimal algebra (Proposition 3.1), we obtain both implicit and explicit invariant solutions that are shown in the Table 4, with c being a constant.

	Elements	$Q(x,y,y_x)=0$	Solutions	Type Solution
1	$\Pi_2+\Pi_3$	$(y+y^2)-y_x(0)=0$	$y(x)=0, y(x)=-1$	Trivial
2	$\Pi_3+\Pi_6$	$(y^2)-y_x(x^2)=0$	$y(x)=-\frac{x}{cx-1}$	Explicit
3	$\Pi_1+\Pi_2+\Pi_3$	$(1+y+y^2)-y_x(0)=0$	$y(x)^2+y(x)+1=0$	Implicit
4	$\Pi_3+\Pi_5+\Pi_6$	$(y^2)-y_x(x+x^2)=0$	$y(x)=\frac{1}{c-\log(x)+\log(x+1)}$	Explicit
5	$\Pi_2+\Pi_3+\Pi_6$	$(y^2+y)-y_x(x^2)=0$	$y(x)=-\frac{e^c}{e^c-e^{\frac{1}{x}}}$	Explicit
6	$\Pi_3+\Pi_4+2\Pi_5$	$(y^2)-y_x(1+2x)=0$	$y(x)=-\frac{2}{c+\log(2x+1)}$	Explicit
7	$\Pi_1-\Pi_2+\Pi_6$	$(1-y)-y_x(x^2)=0$	$y(x)=c\sqrt[x]{e}+1$	Explicit
8	$\Pi_2+\Pi_3+\Pi_4+\Pi_5$	$(y^2+y)-y_x(x+1)=0$	$y(x)=-\frac{e^c(x+1)}{e^cx+e^c-1}$	Explicit
9	$\Pi_1-\Pi_2+\Pi_4+\Pi_5$	$(1-y)-y_x(1+x)=0$	$y(x)=\frac{c}{x+1}+\frac{x}{x+1}$	Explicit
10	$\Pi_1+\Pi_2+\Pi_3+\Pi_6$	$(y^2+y+1)-y_x(x^2)=0$	$y(x)=\frac{1}{2}\left(\sqrt{3}\tan\left(\frac{\sqrt{3}cx-\sqrt{3}}{2x}\right)-1\right)$	Explicit
11	$\Pi_1+2\Pi_2+\Pi_5+\Pi_6$	$(2y+1)-y_x(x^2+x)=0$	$y(x)=ce^{2(\log(x)-\log(x+1))}-\frac{x}{(x+1)^2}-\frac{1}{2(x+1)^2}$	Explicit
12	$\Pi_2+\Pi_3+\Pi_5+\Pi_6$	$(y^2+y)-y_x(x^2+x)=0$	$y(x)=-\frac{e^cx}{e^cx-x-1}$	Explicit
13	$\Pi_1+\Pi_2+\Pi_3+\Pi_5+\Pi_6$	$(y^2+y+1)-y_x(x^2+x)=0$	$y(x)=\frac{1}{2}\left(\sqrt{3}\tan\left(\frac{1}{2}\left(\sqrt{3}c+\sqrt{3}\log(x)-\sqrt{3}\log(x+1)\right)-1\right)\right)$	Explicit
14	$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_5$	$(y^2+y+1)-y_x(x+1)=0$	$y(x)=\frac{1}{2}\left(\sqrt{3}\tan\left(\frac{1}{2}\left(\sqrt{3}c+\sqrt{3}\log(x+1)\right)\right)-1\right)$	Explicit
15	$\Pi_2+\Pi_3+\Pi_4+2\Pi_5+2\Pi_6$	$(y^2+y)-y_x(2x^2+2x+1)=0$	$y(x)=-\frac{e^{c+\tan^{-1}(2x+1)}}{e^{c+\tan^{-1}(2x+1)}-1}$	Explicit
16	$\Pi_1+\Pi_2+\Pi_3+\Pi_4+2\Pi_5+2\Pi_6$	$(y^2+y+1)-y_x(x^2+2x+1)=0$	$y(x)=\frac{1}{2}\left(\sqrt{3}\tan\left(\frac{1}{2}\sqrt{3}\left(c-\frac{1}{x+1}\right)\right)-1\right)$	Explicit

Table 4. Solutions for (2) using invariant curve condition.

Remark 1. The column three in Table 4 which contain the solutions for $Q(x,y,y_x)$, was calculated using by symbolic software: Mathematica.

5. Classification of Lie algebra

In the theory of finite dimensional Lie algebra it is possible to classify the Lie algebra by using the Levi's theorem, which state that any finite dimensional Lie algebra can be write as a semidirect product of a semisimple Lie algebra and a Solvable Lie algebra, that means that, the classification of Lie algebras reduces to the classification of the Solvable and semisimple Lie algebra. In the case of a semisimple Lie algebra (see [16]) we can use the Cartan's criterion to state if a Lie algebra is or not semisimple.

Let $\mathfrak{g}$ the Lie algebra related to the symmetry group of infinitesimal generators of the equation (1) as stated by the table of the commutators, it is enough to consider the no vanish brackets: $[\Pi_1,\Pi_2]=\Pi_1$, $[\Pi_1,\Pi_3]=2\Pi_2$, $[\Pi_2,\Pi_3]=2\Pi_3$, $[\Pi_4,\Pi_5]=\Pi_4$, $[\Pi_4,\Pi_6]=2\Pi_5$, $[\Pi_5,\Pi_6]=\Pi_6$, Using that we calculate Cartan-Killing form K as follows.

$$K = \begin{bmatrix} 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \end{bmatrix},$$

which the determinant is not zero, and thus by Cartan's criterion this Lie algebra is semisimple. First, We remark that in the special lineal algebra $\mathfrak{sl}(2,\mathbb{R})$ with basis X, Y and H, we have the relations $[X,Y] = H, \quad [X,H] = 2X \quad [Y,H] = -2Y$. Next, notice that if we use the transformation: $X_1 := \Pi_1, \quad Y_1 := \Pi_3 \quad H_1 = 2\Pi_2$, $X_2 := \Pi_4, \quad Y_2 := \Pi_6 \quad H_2 = 2\Pi_5$. From that we can see that X_1, Y_1, H_1 and X_2, Y_2, H_2 generate a three dimensional Lie algebra which are isomorphic each one to the special linear Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. In fact since $[X_1, H_1] = 2x_1, \quad [X_1, Y_1] = H_1, \quad [Y_1, H_1] = -2Y_1$, and $[X_2, H_2] = 2x_2, \quad [X_2, Y_2] = H_2, \quad [Y_2, H_2] = -2Y_2$, that is we have the same structure constants that the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, and since the structure constant classify the Lie algebra up isomorphism we obtain the complete classification of the Lie algebra $\mathfrak{g}$. In general terms we can write a element of the Lie algebra as $Z = X + Y$ where $X \in \mathfrak{sl}(2,\mathbb{R})$ and $Y \in \mathfrak{sl}(2,\mathbb{R})$. Consequently, we have the next proposition

Proposition 5.1. *The* 6*-dimensional Lie algebra* $\mathfrak{g}$ *related to the symmetry group of the equation (1) is isomorphich with* $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.

6. Conclusion and future works

We obtained the complete classification of the group of symmetries of (2) (see Proposition 2.1) and using the Lie symmetry group (see Proposition 2.1 item *i*)), we calculated the optimal system, as it was presented in Proposition 3.1. Using these operators it was possible to characterize all the invariant solutions (see Table 4), these solutions are different from (5) and (8), so these solutions do not appear in the literature known until today. The Lie algebra associated to the equation (1) is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. Therefore, the goal initially proposed was achieved.

For future works, by classifying the group of symmetries it is possible to think about: to calculate the conservation laws using the Ibragimov method, to calculate the equivalence group associated with the complete classification, to get the Contact symmetries, Dynamic symmetries, Hidden symmetries and Lambda-symmetries associated with each group in the respective classification. On the other hand, the solutions obtained in Table 4, can be used to test the solution and convergence of numerical methods for this equation.

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