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Determinant Inequalities for Positive Definite Matrices Via Additive and Multiplicative Young Inequalities

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Abstract. In this paper we prove among others that, if the positive definite matrices A, B of order n satisfy the condition

$$0 < mI_n \leq B - A \leq MI_n,$$

for some constants $0 < m < M$, where I_n is the identity matrix, then

$$\begin{aligned} 0 &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A + mI_n)\right]^{-1} - \left[\det(A + mtI_n)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(B)\right]^{-1} - \left[\det((1-t)A + tB)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A + MI_n)\right]^{-1} - \left[\det(A + MtI_n)\right]^{-1}, \end{aligned}$$

for all $t \in [0, 1]$.

Keywords: Positive definite matrices, Determinants, Inequalities.

MSC2010: 47A63, 26D15, 46C05.

Desigualdades determinantes para matrices definidas positivas a través de desigualdades young aditivas y multiplicativas

Resumen. En este trabajo demostramos entre otros que, si las matrices definidas positivas A, B de orden n satisfacen la condición

$$0 < mI_n \leq B - A \leq MI_n,$$

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para algunas constantes $0 < m < M$, donde I_n es la matriz identidad, entonces

$$\begin{aligned} 0 &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A+mI_n)\right]^{-1} - \left[\det(A+mtI_n)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(B)\right]^{-1} - \left[\det((1-t)A+tB)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A+MI_n)\right]^{-1} - \left[\det(A+MtI_n)\right]^{-1}, \end{aligned}$$

para todo $t \in [0,1]$.

Palabras clave: Matrices definidas positivas, Determinantes, Desigualdades.

1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}), \; i, j = 1, ..., k; \; k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212],

$$J_n(A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx = \frac{\pi^{n/2}}{\left[\det(A)\right]^{1/2}}, \quad (1)$$

where A is a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, pp. 63] or [11, pp. 212]), namely

$$\det((1-\lambda)A + \lambda B) \geq \left[\det(A)\right]^{1-\lambda}\left[\det(B)\right]^{\lambda} \quad (2)$$

for any positive definite matrices A, B and $\lambda \in [0,1]$.

By mathematical induction we can get a generalization of (2) which was obtained by L. Mirsky in [10], see also [11, pp. 212]

$$\det\left(\sum_{j=1}^{m} \lambda_j A_j\right) \geq \prod_{j=1}^{m} \left[\det(A_j)\right]^{\lambda_j}, \; m \geq 2, \quad (3)$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

If we write (3) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right) \geq \prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{4}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m}\lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

Using the representation (1) one can also prove the result, see [11, pp. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k})\det\left(A_{(k+1)n}\right),\ k = 1, ..., n; \tag{5}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}),\ i, j = r, ..., s.$$

In particular,

$$\det(A) \leq a_{11}a_{22}...a_{nn}. \tag{6}$$

We recall also the Minkowski's type inequality,

$$\left[\det(A+B)\right]^{1/n} \geq \left[\det(A)\right]^{1/n} + \left[\det(B)\right]^{1/n} \tag{7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [3]-[6].

Motivated by the above results, in this paper we prove among others that, if the positive definite matrices A, B of order n satisfy the condition

$$0 < mI_n \leq B - A \leq MI_n,$$

for some constants $0 < m < M$, where I_n is the identity matrix, then

$$\begin{aligned} 0 &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A+mI_n)\right]^{-1} - \left[\det(A+mtI_n)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(B)\right]^{-1} - \left[\det((1-t)A+tB)\right]^{-1} \\ &\leq (1-t)\left[\det(A)\right]^{-1} + t\left[\det(A+MI_n)\right]^{-1} - \left[\det(A+MtI_n)\right]^{-1}, \end{aligned}$$

for all $t \in [0,1]$.

2. Additive Inequalities

We consider the function $f_t : [0,\infty) \to [0,\infty)$ defined for $t \in (0,1)$ by

$$f_t(u) = 1 - t + tu - u^t. \tag{8}$$

The following lemma holds.

Lemma 2.1. *For $0 \le k < K$ we have*

$$\max_{u\in[k,K]} f_t(u) = \Delta_t(k,K) := \begin{cases} f_t(k), & \text{if } K < 1; \\ \max\{f_t(k), f_t(K)\}, & \text{if } k \le 1 \le K; \\ f_t(K), & \text{if } 1 < k. \end{cases} \tag{9}$$

and

$$\min_{u\in[k,K]} f_t(u) = \delta_t(k,K) := \begin{cases} f_t(K), & \text{if } K < 1; \\ 0, & \text{if } k \le 1 \le K; \\ f_t(k), & \text{if } 1 < k. \end{cases} \tag{10}$$

Proof. The function f_t is differentiable and

$$f_t'(u) = t\left(1 - u^{t-1}\right) = t\frac{u^{1-t} - 1}{u^{1-t}},$$

which shows that the function f_t is decreasing on $[0,1]$ and increasing on $[1,\infty)$, $f_t(0) = 1-t$, $f_t(1) = 0$ and the equation $f_t(u) = 1-t$ for $u > 0$ has the unique solution $u_t = t^{\frac{1}{t-1}} > 1$.

Therefore, by considering the 3 possible situations for the location of the interval $[k,K]$ and the number 1 we get the desired bounds (9) and (10). ☑

Lemma 2.2. *Assume that a, $b > 0$ with $0 < k \le \frac{b}{a} \le K$, then*

$$0 \le \delta_t(k,K)\, a \le (1-t)\, a + tb - b^t a^{1-t} \le \Delta_t(k,K)\, a. \tag{11}$$

Proof. If $u \in [k,K]$, then by Lemma 2.1 we have

$$\delta_t(k,K) \le 1 - t + tu - u^t \le \Delta_t(k,K). \tag{12}$$

If we take $u = \frac{b}{a}$ in (12), then we get

$$\delta_t(k,K) \le 1 - t + t\frac{b}{a} - \left(\frac{b}{a}\right)^t \le \Delta_t(k,K),$$

and by multiplying with a we obtain the desired result (11). ☑

Theorem 2.3. *Assume that the positive definite matrices A, B satisfy the condition*

$$0 < mI_n \le B - A \le MI_n, \tag{13}$$

for some constants $0 < m < M$, then

$$\begin{aligned} 0 &\le (1-t)\left[\det(A)\right]^{-1/2} + t\left[\det(A + mI_n)\right]^{-1/2} - \left[\det(A + mtI_n)\right]^{-1/2} \\ &\le (1-t)\left[\det(A)\right]^{-1/2} + t\left[\det(B)\right]^{-1/2} - \left[\det((1-t)A + tB)\right]^{-1/2} \\ &\le (1-t)\left[\det(A)\right]^{-1/2} + t\left[\det(A + MI_n)\right]^{-1/2} - \left[\det(A + MtI_n)\right]^{-1/2}, \end{aligned} \tag{14}$$

for all $t \in [0,1]$.

Also,

$$
\begin{aligned}
0 &\le \frac{[\det(A)]^{-1/2} + [\det(A + mI_n)]^{-1/2}}{2} \\
&- \frac{[\det(A + mtI_n)]^{-1/2} + [\det(A + m(1-t)I_n)]^{-1/2}}{2} \\
&\le \frac{[\det(A)]^{-1/2} + [\det(B)]^{-1/2}}{2} \\
&- \frac{[\det((1-t)A + tB)]^{-1/2} + [\det(tA + (1-t)B)]^{-1/2}}{2} \\
&\le \frac{[\det(A)]^{-1/2} + [\det(A + MI_n)]^{-1/2}}{2} \\
&- \frac{[\det(A + MtI_n)]^{-1/2} + [\det(A + M(1-t)I_n)]^{-1/2}}{2}
\end{aligned} \tag{15}
$$

for all $t \in [0,1]$.

Proof. Let $a = \exp(-\langle Ax, x\rangle)$ and $b = \exp(-\langle Bx, x\rangle)$ for $x \in \mathbb{R}^n$. Then $\frac{b}{a} = \exp(-\langle (B-A)x, x\rangle)$ and since $0 < mI_n \le B - A \le MI_n$, hence for $x \in \mathbb{R}^n$,

$$
-M\|x\|^2 \le -\langle (B-A)x, x\rangle \le -m\|x\|^2
$$

which gives that

$$
\exp\left(-M\|x\|^2\right) \le \frac{b}{a} \le \exp\left(-m\|x\|^2\right) < 1.
$$

If we apply the inequality (11) for $a = \exp(-\langle Ax, x\rangle)$, $b = \exp(-\langle Bx, x\rangle)$, $k = \exp\left(-M\|x\|^2\right)$ and $K = \exp\left(-m\|x\|^2\right) < 1$, then we get

$$
\begin{aligned}
0 &\le f_t\left(\exp\left(-m\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle) \\
&\le (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle Bx, x\rangle) \\
&- \exp(-\langle ((1-t)A + tB)x, x\rangle) \\
&\le f_t\left(\exp\left(-M\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle),
\end{aligned} \tag{16}
$$

namely

$$
\begin{aligned}
0 &\le \left(1 - t + t\exp\left(-m\|x\|^2\right) - \exp\left(-mt\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle) \\
&\le (1-t)\exp(-\langle Ax, x\rangle) + t\exp(-\langle Bx, x\rangle) - \exp(-\langle ((1-t)A + tB)x, x\rangle) \\
&\le \left(1 - t + t\exp\left(-M\|x\|^2\right) - \exp\left(-Mt\|x\|^2\right)\right)\exp(-\langle Ax, x\rangle).
\end{aligned}
$$

This inequality can be written as

$$\begin{aligned}0 &\le (1-t)\exp\left(-\langle Ax,x\rangle\right)+t\exp\left(-\langle (A+mI_n)\,x,x\rangle\right)\\&\quad-\exp\left(-\langle (A+mtI_n)\,x,x\rangle\right)\\&\le (1-t)\exp\left(-\langle Ax,x\rangle\right)+t\exp\left(-\langle Bx,x\rangle\right)\\&\quad-\exp\left(-\langle ((1-t)\,A+tB)\,x,x\rangle\right)\\&\le (1-t)\exp\left(-\langle Ax,x\rangle\right)+t\exp\left(-\langle (A+MI_n)\,x,x\rangle\right)\\&\quad-\exp\left(-\langle (A+MtI_n)\,x,x\rangle\right),\end{aligned}$$

for $x\in\mathbb{R}^n$ and $t\in[0,1]$.

If we take the integral over $x\in\mathbb{R}^n$, then we get

$$\begin{aligned}0 &\le (1-t)\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle\right)dx+t\int_{\mathbb{R}^n}\exp\left(-\langle (A+mI_n)\,x,x\rangle\right)dx\\&\quad-\int_{\mathbb{R}^n}\exp\left(-\langle (A+mtI_n)\,x,x\rangle\right)dx\\&\le (1-t)\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle\right)dx+t\int_{\mathbb{R}^n}\exp\left(-\langle Bx,x\rangle\right)dx\\&\quad-\int_{\mathbb{R}^n}\exp\left(-\langle ((1-t)\,A+tB)\,x,x\rangle\right)dx\\&\le (1-t)\int_{\mathbb{R}^n}\exp\left(-\langle Ax,x\rangle\right)dx+t\int_{\mathbb{R}^n}\exp\left(-\langle (A+MI_n)\,x,x\rangle\right)dx\\&\quad-\int_{\mathbb{R}^n}\exp\left(-\langle (A+MtI_n)\,x,x\rangle\right)dx,\end{aligned}$$

for $t\in[0,1]$.

By using the representation (1) we get

$$\begin{aligned}0 &\le (1-t)\,J_n(A)+tJ_n(A+mI_n)-J_n(A+mtI_n)\\&\le (1-t)\,J_n(A)+tJ_n(B)-J_n((1-t)\,A+tB)\\&\le (1-t)\,J_n(A)++tJ_n(A+MI_n)-J_n(A+MtI_n),\end{aligned}$$

which, by the second equality in (1) gives (14).

If we replace t with $1-t$ in (14), then we have

$$\begin{aligned}0 &\le t\left[\det(A)\right]^{-1/2}+(1-t)\left[\det(A+mI_n)\right]^{-1/2}\\&\quad-\left[\det(A+m(1-t)\,I_n)\right]^{-1/2}\\&\le t\left[\det(A)\right]^{-1/2}+(1-t)\left[\det(B)\right]^{-1/2}\\&\quad-\left[\det(tA+(1-t)\,B)\right]^{-1/2}\\&\le t\left[\det(A)\right]^{-1/2}+(1-t)\left[\det(A+MI_n)\right]^{-1/2}\\&\quad-\left[\det(A+M(1-t)\,I_n)\right]^{-1/2},\end{aligned}\tag{17}$$

for $t\in[0,1]$.

If we add (14) with (17) and divide by 2, then we get (15). ☑

Corollary 2.4. *With the assumptions of Theorem 2.3 we have*

$$\begin{aligned} 0 &\le \frac{[\det(A)]^{-1/2} + [\det(A+mI_n)]^{-1/2}}{2} - \int_0^1 [\det(A+mtI_n)]^{-1/2}\,dt \\ &\le \frac{[\det(A)]^{-1/2} + [\det(B)]^{-1/2}}{2} - \int_0^1 [\det((1-t)A+tB)]^{-1/2}\,dt \\ &\le \frac{[\det(A)]^{-1/2} + [\det(A+MI_n)]^{-1/2}}{2} - \int_0^1 [\det(A+MtI_n)]^{-1/2}\,dt. \end{aligned} \tag{18}$$

The proof follows by taking the integral over $t \in [0,1]$ in (14).

If we take the square in the representation (1), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)dxdy, \end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)dxdy = \frac{\pi^n}{\det(A)} \tag{19}$$

for A a positive definite matrix of order n and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

We have:

Theorem 2.5. *Assume that the positive definite matrices A, B satisfy the condition (13) for some constants $0 < m < M$, then*

$$\begin{aligned} 0 &\le (1-t)[\det(A)]^{-1} + t[\det(A+mI_n)]^{-1} - [\det(A+mtI_n)]^{-1} \\ &\le (1-t)[\det(A)]^{-1} + t[\det(B)]^{-1} - [\det((1-t)A+tB)]^{-1} \\ &\le (1-t)[\det(A)]^{-1} + t[\det(A+MI_n)]^{-1} - [\det(A+MtI_n)]^{-1}, \end{aligned} \tag{20}$$

for all $t \in [0,1]$.

Also,

$$\begin{aligned}0 &\le \frac{[\det(A)]^{-1}+[\det(A+mI_n)]^{-1}}{2} \\ &\quad - \frac{[\det(A+mtI_n)]^{-1}+[\det(A+m(1-t)I_n)]^{-1}}{2} \\ &\le \frac{[\det(A)]^{-1}+[\det(B)]^{-1}}{2} \\ &\quad - \frac{[\det((1-t)A+tB)]^{-1}+[\det(tA+(1-t)B)]^{-1}}{2} \\ &\le \frac{[\det(A)]^{-1}+[\det(A+MI_n)]^{-1}}{2} \\ &\quad - \frac{[\det(A+MtI_n)]^{-1}+[\det(A+M(1-t)I_n)]^{-1}}{2}\end{aligned} \tag{21}$$

for all $t\in[0,1]$.

Proof. Let $a=\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)$ and $b=\exp(-\langle Bx,x\rangle-\langle By,y\rangle)$ for $x,\, y\in\mathbb{R}^n$. Then

$$\frac{b}{a}=\exp(-\langle (B-A)x,x\rangle-\langle (B-A)y,y\rangle)$$

and since $0<mI_n\le B-A\le MI_n$, hence for $x,\, y\in\mathbb{R}^n$,

$$-M\left(\|x\|^2+\|y\|^2\right)\le -\langle (B-A)x,x\rangle-\langle (B-A)y,y\rangle\le -m\left(\|x\|^2+\|y\|^2\right),$$

which implies that

$$\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)\le\frac{b}{a}\le\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)<1.$$

If we apply the inequality (11) for

$$a=\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle),\ b=\exp(-\langle Bx,x\rangle-\langle By,y\rangle),$$

$k=\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)$ and $K=\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)<1$, then we get

$$\begin{aligned}0&\le f_t\left(\exp\left(-m\left(\|x\|^2+\|y\|^2\right)\right)\right)\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)\\ &\le (1-t)\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)+t\exp(-\langle Bx,x\rangle-\langle By,y\rangle)\\ &\quad-\exp(-\langle((1-t)A+tB)x,x\rangle-\langle((1-t)A+tB)y,y\rangle)\\ &\le f_t\left(\exp\left(-M\left(\|x\|^2+\|y\|^2\right)\right)\right)\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle),\end{aligned}$$

namely

$$\begin{aligned}
0 &\leq (1-t)\exp\left(-\langle Ax,x\rangle - \langle Ay,y\rangle\right) \\
&+ t\exp\left(-\langle (A+mI_n)\,x,x\rangle - \langle (A+mI_n)\,y,y\rangle\right) \\
&- \exp\left(-\langle (A+mtI_n)\,x,x\rangle - \langle (A+mtI_n)\,y,y\rangle\right) \\
&\leq (1-t)\exp\left(-\langle Ax,x\rangle - \langle Ay,y\rangle\right) + t\exp\left(-\langle Bx,x\rangle - \langle By,y\rangle\right) \\
&- \exp\left(-\langle ((1-t)\,A+tB)\,x,x\rangle - \langle ((1-t)\,A+tB)\,y,xy\rangle\right) \\
&\leq (1-t)\exp\left(-\langle Ax,x\rangle - \langle Ay,y\rangle\right) \\
&+ t\exp\left(-\langle (A+MI_n)\,x,x\rangle - \langle (A+MI_n)\,y,y\rangle\right) \\
&- \exp\left(-\langle (A+MtI_n)\,x,x\rangle - \langle (A+MtI_n)\,y,y\rangle\right),
\end{aligned}$$

for $x,\ y \in \mathbb{R}^n$ and $t \in [0,1]$.

If we take the double integral over $x, y \in \mathbb{R}^n$, then we get

$$\begin{aligned}
0 &\leq (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle Ax,x\rangle - \langle Ay,y\rangle\right) dxdy \\
&+ t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A+mI_n)\,x,x\rangle - \langle (A+mI_n)\,y,y\rangle\right) dxdy \\
&- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A+mtI_n)\,x,x\rangle - \langle (A+mtI_n)\,y,y\rangle\right) dxdy \\
&\leq (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle Ax,x\rangle - \langle Ay,y\rangle\right) dxdy \\
&+ t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle Bx,x\rangle - \langle By,y\rangle\right) dxdy \\
&- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle ((1-t)\,A+tB)\,x,x\rangle - \langle ((1-t)\,A+tB)\,y,xy\rangle\right) dxdy \\
&\leq (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle Ax,x\rangle - \langle Ay,y\rangle\right) dxdy \\
&+ t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A+MI_n)\,x,x\rangle - \langle (A+MI_n)\,y,y\rangle\right) dxdy \\
&- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle (A+MtI_n)\,x,x\rangle - \langle (A+MtI_n)\,y,y\rangle\right) dxdy,
\end{aligned}$$

and by making use of the representation (19). ☑

Corollary 2.6. *With the assumptions of Theorem 2.5 we have*

$$\begin{aligned}
0 &\leq \frac{[\det(A)]^{-1} + [\det(A+mI_n)]^{-1}}{2} - \int_0^1 [\det(A+mtI_n)]^{-1}\,dt \qquad (22) \\
&\leq \frac{[\det(A)]^{-1} + [\det(B)]^{-1}}{2} - \int_0^1 [\det((1-t)\,A+tB)]^{-1}\,dt \\
&\leq \frac{[\det(A)]^{-1} + [\det(A+MI_n)]^{-1}}{2} - \int_0^1 [\det(A+MtI_n)]^{-1}\,dt.
\end{aligned}$$

The proof follows by taking the integral over $t \in [0,1]$ in (20).

3. Multiplicative Inequalities

We consider the function $g_t : (0, \infty) \to (0, \infty)$ defined for $t \in (0, 1)$ by

$$1 \le g_t(u) = \frac{1 - t + tu}{u^t} = (1 - t) u^{-t} + tu^{1-t}. \tag{23}$$

For $[k, K] \subset (0, \infty)$ define the quantities

$$\Gamma_t(k, K) = \begin{cases} g_t(k), & \text{if } K < 1; \\ \max\{g_t(k), g_t(K)\}, & \text{if } k \le 1 \le K; \\ g_t(K), & \text{if } 1 < k. \end{cases} \tag{24}$$

$$= \begin{cases} (1 - t) k^{-t} + tk^{1-t}, & \text{if } K < 1; \\ \max\left\{(1 - t) k^{-t} + tk^{1-t}, (1 - t) K^{-t} + tK^{1-t}\right\}, & \text{if } k \le 1 \le K; \\ (1 - t) K^{-t} + tK^{1-t}, & \text{if } 1 < k. \end{cases}$$

and

$$\gamma_t(k, K) = \begin{cases} g_t(K), & \text{if } K < 1; \\ 1, & \text{if } k \le 1 \le K; \\ g_t(k), & \text{if } 1 < k. \end{cases} \tag{25}$$

$$= \begin{cases} (1 - t) K^{-t} + tK^{1-t}, & \text{if } K < 1; \\ 1, & \text{if } k \le 1 \le K; \\ (1 - t) k^{-t} + tk^{1-t}, & \text{if } 1 < k. \end{cases}$$

The following lemma holds.

Lemma 3.1. *For $0 \le k < K$ we have*

$$\max_{u \in [k,K]} g_t(u) = \Gamma_t(k, K)$$

and

$$\min_{u \in [k,K]} g_t(u) = \gamma_t(k, K).$$

Proof. The function g_t is differentiable and

$$g_t'(u) = (1 - t) tu^{-t-1}(u - 1),$$

which shows that the function g_t is decreasing on $(0, 1)$ and increasing on $[1, \infty)$. We have $g_t(1) = 1$, $\lim_{u \to 0+} g_t(u) = +\infty$, $\lim_{u \to \infty} g_t(u) = +\infty$ and $g_t\left(\frac{1}{u}\right) = g_{1-t}(u)$ for any $u > 0$ and $t \in (0, 1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[k, K]$ and the number 1 we get the desired bounds (11) and (12). ☑

Lemma 3.2. *Assume that* a, $b > 0$ *with* $0 < k \leq \frac{b}{a} \leq K$, *then*

$$\gamma_t (k, K)\, a^{1-t} b^t \leq (1-t)\, a + tb \leq \Gamma_t (k, K)\, a^{1-t} b^t. \tag{26}$$

Proof. From Lemma 3.1 we have

$$\gamma_t (k, K) \leq \frac{1 - t + t\frac{b}{a}}{\left(\frac{b}{a}\right)^t} \leq \Gamma_t (k, K),$$

namely

$$\gamma_t (k, K) \left(\frac{b}{a}\right)^t \leq 1 - t + t\frac{b}{a} \leq \Gamma_t (k, K) \left(\frac{b}{a}\right)^t.$$

If we multiply these inequalities by a, then we get (26). ☑

Theorem 3.3. *Assume that the positive definite matrices* A, B *satisfy the condition*

$$0 < mI_n \leq B - A,$$

for some constant $0 < m$, *then*

$$\begin{aligned} &(1-t) \left[\det \left((1-t) A + tB - tmI_n\right)\right]^{-1/2} \\ &+ t \left[\det \left((1-t) A + tB + (1-t) mI_n\right)\right]^{-1/2} \\ &\leq (1-t) \left[\det (A)\right]^{-1/2} + t \left[\det (B)\right]^{-1/2} \end{aligned} \tag{27}$$

for all $t \in [0, 1]$.

In particular, for $t = 1/2$,

$$\begin{aligned} &\left[\det \left(\frac{A+B}{2} - \frac{m}{2} I_n\right)\right]^{-1/2} + \left[\det \left(\frac{A+B}{2} + \frac{m}{2} I_n\right)\right]^{-1/2} \\ &\leq \left[\det (A)\right]^{-1/2} + \left[\det (B)\right]^{-1/2}. \end{aligned} \tag{28}$$

Proof. If $0 < k \leq \frac{b}{a} \leq K < 1$, then by (26) we get

$$g_t (K)\, a^{1-t} b^t \leq (1-t)\, a + tb$$

namely

$$\left[(1-t) K^{-t} + tK^{1-t}\right] a^{1-t} b^t \leq (1-t)\, a + tb \tag{29}$$

If we apply the inequality (29) for $a = \exp(-\langle Ax, x\rangle)$, $b = \exp(-\langle Bx, x\rangle)$ and $K = \exp\left(-m \|x\|^2\right) < 1$, then we get

$$\begin{aligned} &\left[(1-t) \exp\left(tm \|x\|^2\right) + t \exp\left(-m (1-t) \|x\|^2\right)\right] \\ &\times \exp\left(-\langle (1-t) Ax, x\rangle - \langle tBx, x\rangle\right) \\ &\leq (1-t) \exp(-\langle Ax, x\rangle) + t \exp(-\langle Bx, x\rangle) \end{aligned}$$

This is equivalent to

$$\begin{aligned}&(1-t)\exp\left(tm\|x\|^2-\langle(1-t)Ax,x\rangle-\langle tBx,x\rangle\right)\\&+t\exp\left(-m(1-t)\|x\|^2-\langle(1-t)Ax,x\rangle-\langle tBx,x\rangle\right)\\&\le(1-t)\exp(-\langle Ax,x\rangle)+t\exp(-\langle Bx,x\rangle)\end{aligned}$$

namely

$$\begin{aligned}&(1-t)\exp(-\langle((1-t)A+tB-tmI_n)x,x\rangle)\\&+t\exp(-\langle((1-t)A+tB+(1-t)mI_n)x,x\rangle)\\&\le(1-t)\exp(-\langle Ax,x\rangle)+t\exp(-\langle Bx,x\rangle)\end{aligned}$$

for all $x\in\mathbb{R}^n$ and $t\in[0,1]$.

Observe that

$$(1-t)A+tB-tmI_n=A+t(B-A)-tmI_n\ge A+tmI_n-tmI_n=A>0$$

and

$$(1-t)A+tB+(1-t)mI_n>0.$$

By taking the integral on $\mathbb{R}^n$, we get

$$\begin{aligned}&(1-t)\int_{\mathbb{R}^n}\exp(-\langle((1-t)A+tB-tmI_n)x,x\rangle)\,dx\\&+t\int_{\mathbb{R}^n}\exp(-\langle((1-t)A+tB+(1-t)mI_n)x,x\rangle)\,dx\\&\le(1-t)\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)\,dx+t\int_{\mathbb{R}^n}\exp(-\langle Bx,x\rangle)\,dx,\end{aligned}$$

namely, by (1)

$$\begin{aligned}&(1-t)J_n((1-t)A+tB-tmI_n)+tJ_n((1-t)A+tB+(1-t)mI_n)\\&\le(1-t)J_n(A)+tJ_n(B),\end{aligned}$$

which gives (27). ☑

By utilizing a similar argument to the one in the proof of Theorem 2.5 we can finally state:

Theorem 3.4. *Assume that the positive definite matrices A, B satisfy the condition*

$$0<mI_n\le B-A,$$

for some constant $0<m$, then

$$\begin{aligned}&(1-t)\left[\det((1-t)A+tB-tmI_n)\right]^{-1}\\&+t\left[\det((1-t)A+tB+(1-t)mI_n)\right]^{-1}\\&\le(1-t)\left[\det(A)\right]^{-1}+t\left[\det(B)\right]^{-1}\end{aligned}\tag{30}$$

for all $t \in [0,1]$.

In particular, for $t = 1/2$,

$$\left[\det\left(\frac{A+B}{2} - \frac{m}{2}I_n\right)\right]^{-1} + \left[\det\left(\frac{A+B}{2} + \frac{m}{2}I_n\right)\right]^{-1} \le [\det(A)]^{-1} + [\det(B)]^{-1}. \tag{31}$$

A complex square matrix $H = (h_{ij})$, $i,j = 1,...,n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1,...,z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [11, pp. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{32}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x,y\rangle$ is understood in the real sense, i.e. $\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 2.5 and Theorem 3.4 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

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