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Zeta function of the Burnside ring for C_{p^3}

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Abstract. The main objective of this paper is to determine the local and global Zeta function of $B(C_{p^3})$ the Burnside ring for cyclic groups of order p^3 and, to study some relations that fulfill this Zeta function.

Keywords: Burnside ring, Zeta function, Fiber product.

MSC2010: 16H20, 19A22, 11S40.

Función Zeta del anillo de Burnside para C_{p^3}

Resumen. El objetivo principal de este trabajo es determinar la función Zeta local y global del anillo de Burnside $B(C_{p^3})$ para grupos cíclicos de orden p^3, así como estudiar algunas relaciones que satisface esta función Zeta.

Palabras clave: Anillo de Burnside, Función Zeta, Producto fibrado.

1. Introduction

The Burnside ring is an invariant of the group that detects solubility, as well as being a framework for the induction theorems and having different applications in topology, see [1, 2]. On the other hand, the zeta function is an invariant of the ring that detects the distribution of prime ideals, see [5].

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According to the definition given by Solomon for the zeta function of an order, it is necessary to know all its ideals of finite index, which might be complicated. In this work, we use a method used by Bushnell C. J. and Reiner I. [4], which depends only on the finite set of the isomorphism classes of the ideals of finite index. From [11] we have that:

For $B_p(C_p)$ there are 2 isomorphism classes of fractional ideals of finite index;

For $B_p(C_{p^2})$ there are 9 isomorphism classes of fractional ideals of finite index.

In both cases, we can see that this is a better alternative than the method used in [10], where the same results were obtained by computing all the ideals. However, as we will see in this paper, for $B_p(C_{p^3})$ there are

$$82 + 7p + 5(p-1) + 3(p-2)$$

isomorphism classes of fractional ideals of finite index. So for $B_p(C_{p^n})$, this method used by I. Reiner quickly becomes unmanageable. At the present, we are trying to discover a method that only depends on the conductors, and we conjecture that there are exactly $n+1$ for the general case.

Throughout this paper, let G be a finite group. Its Burnside ring $B(G)$ is the Grothendieck ring of the category of finite left G-sets. This is the free abelian group on the isomorphism classes of transitive left G-sets of the form G/H for subgroups H of G, two of which are identified if their stabilizers H are conjugate in G; addition and multiplication are given by the disjoint union and Cartesian product, respectively.

In Section 2, we recall the Burnside ring $B(G)$ of a finite group G, along with the Zeta function $\zeta_{B(G)}(s)$ of $B(G)$ and the ideals of a fiber product of rings.

In Section 3, we determine $\zeta_{B(C_{p^3})}(s)$. In [6] this zeta function was obtained via the calculation of all de ideals of finite index in $B(C_{p^3})$. In this paper, we use a method employed by Bushnell C. J. and Reiner I. [4] which only requires the family of all isomorphism classes of the fractional ideals of the finite index of the Burnside ring for this group. First, we recall the ideals of the finite index in $B_p\left(C_{p^2}\right)$ according to [11], to compute the family of all isomorphism classes of the fractional ideals of the finite index of $B_p\left(C_{p^3}\right)$ via the fiber product of rings. Next, we determine the Zeta function $\zeta_{\widetilde{B}\left(C_{p^3}\right)}(s)$ of the maximal order $\widetilde{B}\left(C_{p^3}\right)$ of $B(C_{p^3})$ and the Zeta function $\zeta_{B(C_{p^3})}(s)$ of the Burnside ring for a cyclic group C_{p^3}. Finally, we study the relations that fulfill the Zeta functions $Z_{B_p\left(C_{p^3}\right)}(M,s)$, according to [12, Theorem 2.3] where M is a representative of an isomorphism class of the fractional ideals of finite index of $B_p\left(C_{p^3}\right)$.

Remark 1.1. In Section 3, we correct a mistake made in [6, pp 17] about the calculation of the index $(B : M_{85}(a))^{-s}$, according to which $(B : M_{85}(a))^{-s} = p^s$. However, this index must be $(B : M_{85}(a))^{-s} = p^{2s}$, and therefore, we correct $\zeta_{B(C_{p^3})}(s)$. (In the present paper, $M_{85}(a)$ was reindexed by $M_{81}(a)$.)

2. Zeta Functions of Burnside Rings

Let X be a finite G-set and let $[X]$ be its G isomorphism class. We define

$$B^+(G) := \{[X] \mid X \text{ a finite } G\text{-set}\},$$

which is a commutative semiring with the unit, with the binary operations of disjoint union and Cartesian product.

Definition 2.1. We define the Burnside ring $B(G)$ of G as the Grothendieck ring of $B^+(G)$.

Now, for subgroup H of G, we write $[H]$ for its conjugacy class. We observe that as an abelian group, $B(G)$ is free, generated by elements of the form G/H, where $[H]$ belongs to the set of representatives of all conjugacy classes of subgroups of G, which we call $\mathcal{C}(G)$. That is

$$B(G) = \bigoplus_{[H]\in\mathcal{C}(G)} \mathbb{Z}(G/H).$$

For further information about the Burnside ring, see [3].

Let $H \leq G$ be a subgroup and X a G-set, we denote the set of fixed points of X under the action of H by

$$X^H = \{x \in X \mid h \cdot x = x \,\forall\, h \in H\}.$$

We define the mark of H on X as the number of elements of X^H and we call it $\varphi_H(X)$. The reader can find some of the properties satisfied by φ_H in [12, pp 3].

We define $\widetilde{B}(G) := \prod_{[H]\in\mathcal{C}(G)} \mathbb{Z}$, thus we have the following map

$$\begin{array}{rccc} \varphi: & B^+(G) & \longrightarrow & \widetilde{B}(G) \\ & [X] & \longmapsto & (\varphi_H(X))_{[H]\in\mathcal{C}(G)}, \end{array}$$

which is a morphism of semirings that extends to a unique injective morphism of rings

$$\varphi : B(G) \to \widetilde{B}(G).$$

Let R be a Dedekind domain with a quotient field K, and let $\mathcal{B}$ be a finite-dimensional $K-$algebra. For any finite-dimensional $K-$space V, a full $R-$lattice in V is a finitely generated $R-$submodule L in V such that $KL = V$, where

$$KL = \left\{\sum \alpha_i l_i \text{ (finite sum)}: \alpha_i \in K,\, l_i \in L\right\}.$$

An $R-$order in $\mathcal{B}$ is a subring Λ of $\mathcal{B}$, such that the center of Λ contains R and such that Λ is a full $R-$lattice in $\mathcal{B}$.

A fractional ideal of R is a full $R-$lattice I in K. We can see that there is a non-zero element $r \in R$, such that $rI \subseteq R$.

Let $p \in \mathbb{Z}$ be a rational prime and let $\mathbb{Z}_p$ be the ring of $p-$adic integers. We denote the following tensor products by

$$B_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} B(G) = \bigoplus_{[H]\in\mathcal{C}(G)} \mathbb{Z}_p(G/H)$$

and

$$\widetilde{B}_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} \widetilde{B}(G) = \prod_{[H]\in\mathcal{C}(G)} \mathbb{Z}_p,$$

where we have that $B_p(G)$ is a $\mathbb{Z}_p -$ order, being $\widetilde{B}_p(G)$ its maximal order. For further information about orders, see [7, chapters 2, 3].

Let A be a finite-dimensional semisimple algebra over the rational field $\mathbb{Q}$ or over a $p-$ adic field $\mathbb{Q}_p$, and let Λ be an order in A. When A is a $\mathbb{Q}-$algebra, Λ is a $\mathbb{Z}-$order in A; when A is a $\mathbb{Q}_p-$algebra, Λ is a $\mathbb{Z}_p-$order in A. Let I be a left ideal of Λ, such that the index $(\Lambda : I)$ is finite. We use this index symbol in a general sense: if, for example, Y_1 and Y_2 are $\mathbb{Z}_p -$ lattices spanning the same $\mathbb{Q}_p -$ vector space, we put

$$(Y_1 : Y_2) = \frac{(Y_1 : Y_1 \cap Y_2)}{(Y_2 : Y_1 \cap Y_2)}.$$

The symbol $(Y_1 : Y_2)$ is therefore unambiguously defined as whether or not Y_1 contains Y_2.

Definition 2.2. We define the Solomon's zeta function $\zeta_\Lambda(s)$ of an order Λ, as follows:

$$\zeta_\Lambda(s) := \sum_{\substack{I \leq \Lambda, \text{ left ideal} \\ (\Lambda : I) < \infty}} (\Lambda : I)^{-s},$$

which is a generalization of the classical Dedekind Zeta function $\zeta_{\mathcal{K}}(s)$ of an algebraic number field $\mathcal{K}$. When Λ is a $\mathbb{Z}-$order in A, $\zeta_\Lambda(s)$ is the global zeta function; when Λ is a $\mathbb{Z}_p-$order in A, $\zeta_\Lambda(s)$ is the local zeta function.

For the commutative rings $B_p(G)$ and $\widetilde{B}_p(G)$, the sum extends over all the ideals of the finite index and converges uniformly on compact subsets of

$$\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}.$$

Let Λ and Λ_i be $\mathbb{Z}$-orders, for $i = 1, ..., n$ and let $\Lambda_p := \mathbb{Z}_p \bigotimes_{\mathbb{Z}} \Lambda$, which is an order over $\mathbb{Z}_p$. We see that the function ζ satisfies the following properties:

i) If $\Lambda = \prod_{i=1}^{n} \Lambda_i$, we have $\zeta_\Lambda(s) = \prod_{i=1}^{n} \zeta_{\Lambda_i}(s)$.

ii) $\zeta_\Lambda(s) = \prod_{p-\text{prime}} \zeta_{\Lambda_p}(s)$, the Euler product.

For further information about Solomon's Zeta function, see [9].

Theorem 2.3. *Let G be a finite group and $B(G)$ its Burnside ring, if $q \in \mathbb{Z}$ is a prime, we have*

$$\zeta_{B_q(G)}(s) = f_G\left(q^{-s}\right) \zeta_{\widetilde{B}_q(G)}(s),$$

where $f_G(q^{-s})$ is a polynomial in $\mathbb{Z}[q^{-s}]$. See [9, Theorem 1].

Remark 2.4. If q does not divide $|G|$, then we have that $B_q(G) = \widetilde{B}_q(G)$, and we conclude that $f_G(q^{-s}) = 1$ when q does not divide $|G|$.

Definition 2.5. Let M be a full $\Lambda -$ lattice in A. We define the zeta function $Z_\Lambda(M; s)$, as follows:

$$Z_\Lambda(M; s) = \sum (\Lambda : N)^{-s},$$

the sum extending over all full $\Lambda -$ sublattices N in A, such that N, M are in the same isomorphism class.

So we can express

$$\zeta_\Lambda(s) = \sum_M Z_\Lambda(M; s),$$

the finite sum extending over all the representatives of the isomorphism classes of the full $\Lambda -$ lattices in A.

We define the conductor of M in Λ, as follows:

$$\{M : \Lambda\} = \{x \in A : \quad Mx \subseteq \Lambda\}.$$

Let $\Phi_{\{M:\Lambda\}}$ be the characteristic function in A of $\{M : \Lambda\}$. Now we choose a Haar measure d^*x on the unit group A^*. For measurable sets $E \subset A$, $E' \subset A^*$, it will be convenient to write

$$\mu(E) = \int_E dx, \qquad \mu^*(E') = \int_{E'} d^*x.$$

We have that:

$$Z_\Lambda(M; s) = \mu^*(\mathrm{Aut}_\Lambda M)^{-1} (\Lambda : M)^{-s} \int_{A^*} \Phi_{\{M:\Lambda\}}(x) \|x\|_A^s \, d^*x, \text{ where}$$

$\|x\|_A = (Lx : L)$ for $x \in A^*$, which is independent of the choice of full $\mathbb{Z}_p -$ lattice L in A, and we observe that it is multiplicative. Furthermore, we can see that $\|x\|_A = 1$ whenever x is a unit in some $\mathbb{Z}_p -$ order in A. For further details on this result, see [4, 2.1 The Local Case, pp 138-139].

We assume that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f_2} & \mathcal{A}_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ \mathcal{A}_1 & \xrightarrow[g_1]{} & \overline{\mathcal{A}} \end{array}$$

is a fiber product diagram of rings, where all the maps are ring surjections. By definition

$$\mathcal{A} = \{(a_1, a_2) : a_i \in \mathcal{A}_i \text{ for } i = 1, 2 \text{ and } g_1(a_1) = g_2(a_2)\}.$$

Let $I \leq \mathcal{A}$ and $I_i \leq \mathcal{A}_i$ be left ideals, such that $I_i = f_i(I)$ for $i = 1, 2$. Let $\mathcal{A}_2$ be a PID. Then $I_2 = \mathcal{A}_2\beta$ for some $\beta \in \mathcal{A}_2$. We have $\alpha \in I_1$ such that $(\alpha, \beta) \in I$. Let $J = \{c \in \mathcal{A}_1 : (c, 0) \in I\}$, which is an ideal of $\mathcal{A}_1$. We have that

$$I = \mathcal{A}(\alpha, \beta) + (J, 0)$$

and then it is determined by the following data:

1. a generator β of a principal ideal $\mathcal{A}_2\beta$ of $\mathcal{A}_2$,

2. an ideal $J \leq \mathcal{A}_1$ such that $g_1(J) = 0$, and

3. an element $\alpha \in \mathcal{A}_1$ such that $g_1(\alpha) = g_2(\beta)$. Clearly, α is uniquely determined mod J.

4. Let $D = \{a \in \mathcal{A} : f_2(a)\beta = 0\}$ which is an ideal of $\mathcal{A}$. We have that $f_1(D)\alpha \subseteq J$. For further details on this result, see [8].

3. The Zeta function of $B\left(C_{p^3}\right)$

3.1. Isomorphism classes of the fractional ideals of $B_p\left(C_{p^3}\right)$

Let p be a prime, and let $C_{p^n} = \langle a \rangle$ be a ciclyc group of order p^n for $n \in \mathbb{N}$. We have that the conjugacy classes of C_{p^n} are $\mathcal{C}\left(C_{p^n}\right) = \left\{\langle a \rangle, \langle a^p \rangle, \left\langle a^{p^2} \right\rangle, ..., \left\langle a^{p^n} \right\rangle = \langle 1 \rangle\right\}$. Therefore, a basis for $B_p\left(C_{p^n}\right)$ is

$$\left\{a_0 = C_{p^n} \diagup \langle a \rangle, \ a_1 = C_{p^n} \diagup \langle a^p \rangle, \ a_2 = C_{p^n} \diagup \left\langle a^{p^2} \right\rangle, ..., \ a_n = C_{p^n} \diagup \left\langle a^{p^n} \right\rangle\right\},$$

and so,

$$B_p\left(C_{p^n}\right) = \bigoplus_{i=0}^{n} a_i \mathbb{Z}_p.$$

Furthermore $\widetilde{B}_p\left(C_{p^n}\right) = \mathbb{Z}_p^{n+1}$ is its maximal order.

On the other hand, we know that

$$\varphi_H\left(C_{p^n}/K\right) = \begin{cases} \left|C_{p^n}/K\right|, & \text{if } \ H \subseteq K; \\ 0, & \text{if } \ H \nsubseteq K, \end{cases}$$

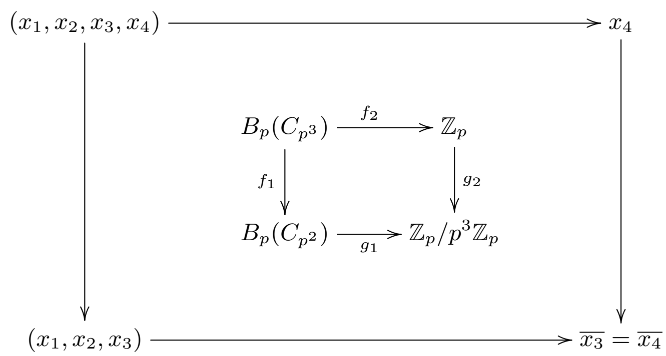
and then, we have that φ induces the following inclusion:

$$\begin{array}{ccc} B_p\left(C_{p^n}\right) & \overset{\varphi}{\hookrightarrow} & \mathbb{Z}_p^{n+1} \\ X & \longmapsto & \left(\varphi_H(X)\right)_{[H] \in \mathcal{C}\left(C_{p^n}\right)} \\ & & \\ a_0 & \longmapsto & (\underbrace{1, \ ... \ , 1}_{(n+1)-\text{times}}) \\ a_1 & \longmapsto & (0, \underbrace{p, \ ... \ , p}_{n-\text{times}}) \\ a_2 & \longmapsto & (0, 0, \underbrace{p^2, \ ... \ , p^2}_{(n-1)-\text{times}}) \\ & \vdots & \\ a_n & \longmapsto & (\underbrace{0, \ ... \ , 0}_{n-\text{times}}, p^n). \end{array}$$

Therefore, we can see $B_p\left(C_{p^n}\right)$ in $\widetilde{B}_p\left(C_{p^n}\right)$ as follows:

$$B_p\left(C_{p^n}\right) = \left\{(x_1, x_2, \ ... \ , x_{n+1}) \in \mathbb{Z}_p^{n+1}\colon \ (x_i - x_{i+1}) \in p^i \mathbb{Z}_p \ \ \text{for} \ \ i = 1, ..., n\right\}$$

and then we can give the following fiber product structure:



We observe that $\mathbb{Z}_p$ is a PID. Therefore, it has ideals of the form $p^r\mathbb{Z}_p$, for every integer $r \geq 0$, and according to the structure of the fiber product, we have that the ideals of finite index in $B_p\left(C_{p^3}\right)$ are ideals of the form:

$$I = (\alpha, p^r)\, B_p\left(C_{p^3}\right) + (J, 0)\,, \tag{1}$$

where α is an element of $B_p\left(C_{p^2}\right)$ and $J \leq B_p\left(C_{p^2}\right)$ is an ideal such that:

1. $g_1\left(J\right) = 0$,
2. $g_1\left(\alpha\right) = g_2\left(p^r\right)$, where α is uniquely determined mod J, and
3. if $D = p\mathbb{Z}_p \times p^2\mathbb{Z}_p \times p^3\mathbb{Z}_p \times \{0\}$ we have that $f_1\left(D\right)\alpha \subseteq J$.

Let $F_p = \{0, 1, ..., p-1\}$ and $F_p^* = \{1, ..., p-1\}$, from [11] we have that the following is a complete list of representatives of isomorphism classes of fractional ideals of $B_p\left(C_{p^2}\right)$:

$\boldsymbol{J_1 = \mathbb{Z}_p^3}$

$\boldsymbol{J_2 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon\ (y - x) \in p\mathbb{Z}_p\right\}}$

$\boldsymbol{J_3 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon\ (z - x) \in p\mathbb{Z}_p\right\}}$

$\boldsymbol{J_4 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon\ (z - y) \in p\mathbb{Z}_p\right\}}$

$\boldsymbol{J_5 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon\ (z - y) \in p^2\mathbb{Z}_p\right\}}$

$\boldsymbol{J_6 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon\ (y - x) \in p\mathbb{Z}_p,\ (z - y) \in p\mathbb{Z}_p\right\}}$

$\boldsymbol{J_7 = B_p\left(C_{p^2}\right)}$

$\boldsymbol{J_8 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon x - y + z \in p\mathbb{Z}_p\right\}}$

$\boldsymbol{J_9 = \left\{(x, y, z) \in \mathbb{Z}_p^3\colon px - y + z \in p^2\mathbb{Z}_p\right\}}$.

Based on the previous paragraph, we will study Eq. (1), for the nine cases above. We will denote $B_p\left(C_{p^3}\right)$ by B.

1). From Eq. (1) for J_1, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_1 = B}$

$= \left\langle(1,1,1,1),(0,p,p,p),(0,0,p^2,p^2),(0,0,0,p^3)\right\rangle$, for which: $\{M_1:B\}=B$, $\mathrm{Aut}_B M_1 = B^*$, $(B:M_1)^{-s}=1$, $\left(\left(\mathbb{Z}_p^*\right)^4:B^*\right)=p^3(p-1)^3$, $\left(\mathbb{Z}_p^4:B\right)=p^6$.

$\boldsymbol{M_2=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-v)\in p^2\mathbb{Z}_p,\ (t-w)\in p^3\mathbb{Z}_p\}}$
$= \left\langle(1,0,0,0),(0,p^2,0,0),(0,1,1,1),(0,0,0,p^3)\right\rangle$, for which: $\{M_2:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_2 = M_2^*$, $(B:M_2)^{-s}=p^s$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_2^*\right)=p^3(p-1)^2$, $\left(\mathbb{Z}_p^4:M_2\right)=p^5$.

$\boldsymbol{M_3=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-u),(w-v)\in p\mathbb{Z}_p,\ (t-w)\in p^3\mathbb{Z}_p\}}$
$= \left\langle(p,0,0,0),(0,p,0,0),(1,1,1,1),(0,0,0,p^3)\right\rangle$, for which: $\{M_3:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_3 = M_3^*$, $(B:M_3)^{-s}=p^s$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_3^*\right)=p^2(p-1)^3$, $\left(\mathbb{Z}_p^4:M_3\right)=p^5$.

$\boldsymbol{M_4=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-v)\in p\mathbb{Z}_p,\ (t-w)\in p^3\mathbb{Z}_p\}}$
$= \left\langle(1,0,0,0),(0,p,0,0),(0,1,1,1),(0,0,0,p^3)\right\rangle$, for which: $\{M_4:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_4 = M_4^*$, $(B:M_4)^{-s}=p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_4^*\right)=p^2(p-1)^2$, $\left(\mathbb{Z}_p^4:M_4\right)=p^4$.

$\boldsymbol{M_5=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-u)\in p\mathbb{Z}_p,\ (w-v),(t-w)\in p^2\mathbb{Z}_p\}}$
$= \left\langle(p,0,0,0),(0,p^2,0,0),(1,1,1,1),(0,0,0,p^2)\right\rangle$, for which: $\{M_5:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_5 = M_5^*$, $(B:M_5)^{-s}=p^s$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_5^*\right)=p^2(p-1)^3$, $\left(\mathbb{Z}_p^4:M_5\right)=p^5$.

$\boldsymbol{M_6=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-v),(t-w)\in p^2\mathbb{Z}_p\}}$
$= \left\langle(1,0,0,0),(0,p^2,0,0),(0,1,1,1),(0,0,0,p^2)\right\rangle$, for which: $\{M_6:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_6 = M_6^*$, $(B:M_6)^{-s}=p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_6^*\right)=p^2(p-1)^2$, $\left(\mathbb{Z}_p^4:M_6\right)=p^4$.

$\boldsymbol{M_7=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-u),(w-v)\in p\mathbb{Z}_p,\ (t-w)\in p^2\mathbb{Z}_p\}}$
$= \left\langle(p,0,0,0),(0,p,0,0),(1,1,1,1),(0,0,0,p^2)\right\rangle$, for which: $\{M_7:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_7 = M_7^*$, $(B:M_7)^{-s}=p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_7^*\right)=p(p-1)^3$, $\left(\mathbb{Z}_p^4:M_7\right)=p^4$.

$\boldsymbol{M_8=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-v)\in p\mathbb{Z}_p,\ (t-w)\in p^2\mathbb{Z}_p\}}$
$= \left\langle(1,0,0,0),(0,p,0,0),(0,1,1,1),(0,0,0,p^2)\right\rangle$, for which: $\{M_8:B\}=(p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_8 = M_8^*$, $(B:M_8)^{-s}=p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_8^*\right)=p(p-1)^2$, $\left(\mathbb{Z}_p^4:M_8\right)=p^3$.

$\boldsymbol{M_9=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-u)\in p\mathbb{Z}_p,\ (t-w)\in p^3\mathbb{Z}_p\}}$
$= \left\langle(p,0,0,0),(0,1,0,0),(1,0,1,1),(0,0,0,p^3)\right\rangle$, for which: $\{M_9:B\}=\left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_9 = M_9^*$, $(B:M_9)^{-s}=p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_9^*\right)=p^2(p-1)^2$, $\left(\mathbb{Z}_p^4:M_9\right)=p^4$.

$\boldsymbol{M_{10}=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (t-w)\in p^3\mathbb{Z}_p\}}$
$= \left\langle(1,0,0,0),(0,1,0,0),(0,0,1,1),(0,0,0,p^3)\right\rangle$, for which: $\{M_{10}:B\}=\left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{10} = M_{10}^*$, $(B:M_{10})^{-s}=p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_{10}^*\right)=p^2(p-1)$, $\left(\mathbb{Z}_p^4:M_{10}\right)=p^3$.

$\boldsymbol{M_{11}=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-u)\in p\mathbb{Z}_p,\ (t-w)\in p^2\mathbb{Z}_p\}}$
$= \left\langle(p,0,0,0),(0,1,0,0),(1,0,1,1),(0,0,0,p^2)\right\rangle$, for which: $\{M_{11}:B\}=\left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{11} = M_{11}^*$, $(B:M_{11})^{-s}=p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_{11}^*\right)=p(p-1)^2$, $\left(\mathbb{Z}_p^4:M_{11}\right)=p^3$.

$\boldsymbol{M_{12}=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (t-w)\in p^2\mathbb{Z}_p\}}$
$= \left\langle(1,0,0,0),(0,1,0,0),(0,0,1,1),(0,0,0,p^2)\right\rangle$, for which: $\{M_{12}:B\}=\left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{12} = M_{12}^*$, $(B:M_{12})^{-s}=p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_{12}^*\right)=p(p-1)$, $\left(\mathbb{Z}_p^4:M_{12}\right)=p^2$.

$\boldsymbol{M_{13}=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-u),(w-v),(t-w)\in p\mathbb{Z}_p\}}$
$= \langle(p,0,0,0),(0,p,0,0),(1,1,1,1),(0,0,0,p)\rangle$, for which: $\{M_{13}:B\}=\left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{13} = M_{13}^*$, $(B:M_{13})^{-s}=p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4:M_{13}^*\right)=(p-1)^3$, $\left(\mathbb{Z}_p^4:M_{13}\right)=p^3$.

$\boldsymbol{M_{14}=\{(u,v,w,t)\in\mathbb{Z}_p^4:\ (w-v),(t-w)\in p\mathbb{Z}_p\}}$

$= \langle(1,0,0,0),(0,p,0,0),(0,1,1,1),(0,0,0,p)\rangle$, for which: $\{M_{14}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{14} = M_{14}^*$, $(B:M_{14})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{14}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{14}\right) = p^2$.

$\boldsymbol{M_{15} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-u),(t-w) \in p\mathbb{Z}_p\right\}}$

$= \langle(p,0,0,0),(0,1,0,0),(1,0,1,1),(0,0,0,p)\rangle$, for which: $\{M_{15}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{15} = M_{15}^*$, $(B:M_{15})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{15}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{15}\right) = p^2$.

$\boldsymbol{M_{16} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-w) \in p\mathbb{Z}_p\right\}}$

$= \langle(1,0,0,0),(0,1,0,0),(0,0,1,1),(0,0,0,p)\rangle$, for which: $\{M_{16}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{16} = M_{16}^*$, $(B:M_{16})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{16}^*\right) = (p-1)$, $\left(\mathbb{Z}_p^4 : M_{16}\right) = p$.

$\boldsymbol{M_{17} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u),(w-v) \in p\mathbb{Z}_p,\ (t-v) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle(p,0,0,0),(1,1,1,1),(0,0,p,0),(0,0,0,p^2)\right\rangle$, for which: $\{M_{17}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{17} = M_{17}^*$, $(B:M_{17})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{17}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{17}\right) = p^4$.

$\boldsymbol{M_{18} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (w-v) \in p\mathbb{Z}_p,\ (t-v) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle(1,0,0,0),(0,1,1,1),(0,0,p,0),(0,0,0,p^2)\right\rangle$, for which: $\{M_{18}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{18} = M_{18}^*$, $(B:M_{18})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{18}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{18}\right) = p^3$.

$\boldsymbol{M_{19} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u) \in p\mathbb{Z}_p,\ (t-v) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle(p,0,0,0),(1,1,0,1),(0,0,1,0),(0,0,0,p^2)\right\rangle$, for which: $\{M_{19}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{19} = M_{19}^*$, $(B:M_{19})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{19}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{19}\right) = p^3$.

$\boldsymbol{M_{20} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-v) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle(1,0,0,0),(0,1,0,1),(0,0,1,0),(0,0,0,p^2)\right\rangle$, for which: $\{M_{20}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{20} = M_{20}^*$, $(B:M_{20})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{20}^*\right) = p(p-1)$, $\left(\mathbb{Z}_p^4 : M_{20}\right) = p^2$.

$\boldsymbol{M_{21} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u),(t-v) \in p\mathbb{Z}_p\right\}}$

$= \langle(p,0,0,0),(1,1,0,1),(0,0,1,0),(0,0,0,p)\rangle$, for which: $\{M_{21}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{21} = M_{21}^*$, $(B:M_{21})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{21}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{21}\right) = p^2$.

$\boldsymbol{M_{22} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-v) \in p\mathbb{Z}_p\right\}}$

$= \langle(1,0,0,0),(0,1,0,1),(0,0,1,0),(0,0,0,p)\rangle$, for which: $\{M_{22}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{22} = M_{22}^*$, $(B:M_{22})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{22}^*\right) = (p-1)$, $\left(\mathbb{Z}_p^4 : M_{22}\right) = p$.

$\boldsymbol{M_{23} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \langle(1,0,0,1),(0,1,0,0),(0,0,1,0),(0,0,0,p)\rangle$, for which: $\{M_{23}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{23} = M_{23}^*$, $(B:M_{23})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{23}^*\right) = (p-1)$, $\left(\mathbb{Z}_p^4 : M_{23}\right) = p$.

$\boldsymbol{M_{24} = \mathbb{Z}_p^4}$

$= \langle(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\rangle$, for which: $\{M_{24}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{24} = M_{24}^*$, $(B:M_{24})^{-s} = p^{6s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{24}^*\right) = 1$, $\left(\mathbb{Z}_p^4 : M_{24}\right) = 1$.

2). From Eq. (1) for J_2, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{25} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu-v+t) \in p^2\mathbb{Z}_p,\ (t-w) \in p^3\mathbb{Z}_p\right\}}$

$= \left\langle(1,p,0,0),(0,p^2,0,0),(0,0,p^3,0),(0,1,1,1)\right\rangle$, for which: $\{M_{25}:B\} = (p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_{25} = B^*$, $(B:M_{25})^{-s} = p^{s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : B^*\right) = p^3(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{25}\right) = p^5$.

$\boldsymbol{M_{26} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu-v+t),(t-w) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,p,0,0),(0,p^2,0,0),(0,0,p^2,0),(0,1,1,1)\right\rangle$, for which: $\{M_{26}:B\} = (p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_{26} = M_5^*$, $(B:M_{26})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_5^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{26}\right) = p^4$.

$\boldsymbol{M_{27} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-v+t)\in p\mathbb{Z}_p,\ (t-w)\in p^3\mathbb{Z}_p\right\}}$
$= \left\langle (1,1,0,0),(0,p,0,0),(0,0,p^3,0),(0,1,1,1)\right\rangle$, for which: $\{M_{27}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{27} = M_3^*$, $(B:M_{27})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_3^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{27}\right) = p^4$.

$\boldsymbol{M_{28} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (v-u)\in p\mathbb{Z}_p,\ (t-w)\in p^3\mathbb{Z}_p\right\}}$
$= \left\langle (1,1,0,0),(0,p,0,0),(0,0,1,1),(0,0,0,p^3)\right\rangle$, for which: $\{M_{28}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{28} = M_{28}^*$, $(B:M_{28})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{28}^*\right) = p^2(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{28}\right) = p^4$.

$\boldsymbol{M_{29} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-v+t)\in p\mathbb{Z}_p,\ (t-w)\in p^2\mathbb{Z}_p\right\}}$
$= \left\langle (1,1,0,0),(0,p,0,0),(0,0,p^2,0),(0,1,1,1)\right\rangle$, for which: $\{M_{29}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{29} = M_7^*$, $(B:M_{29})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_7^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{29}\right) = p^3$.

$\boldsymbol{M_{30} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (v-u)\in p\mathbb{Z}_p,\ (t-w)\in p^2\mathbb{Z}_p\right\}}$
$= \left\langle (1,1,0,0),(0,p,0,0),(0,0,p^2,0),(0,0,1,1)\right\rangle$, for which: $\{M_{30}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{30} = M_{30}^*$, $(B:M_{30})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{30}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{30}\right) = p^3$.

$\boldsymbol{M_{31} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (pu-v+t)\in p^2\mathbb{Z}_p,\ (t-w)\in p\mathbb{Z}_p\right\}}$
$= \left\langle (1,p,0,0),(0,p^2,0,0),(0,0,p,0),(0,1,1,1)\right\rangle$, for which: $\{M_{31}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{31} = M_{17}^*$, $(B:M_{31})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{17}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{31}\right) = p^3$.

$\boldsymbol{M_{32} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-v+t)\,,(t-w)\in p\mathbb{Z}_p\right\}}$
$= \langle (1,1,0,0),(0,p,0,0),(0,0,p,0),(0,1,1,1)\rangle$, for which: $\{M_{32}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{32} = M_{13}^*$, $(B:M_{32})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{13}^*\right) = (p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{32}\right) = p^2$.

$\boldsymbol{M_{33} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-v)\,,(t-w)\in p\mathbb{Z}_p\right\}}$
$= \langle (1,1,0,0),(0,p,0,0),(0,0,p,0),(0,0,1,1)\rangle$, for which: $\{M_{33}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{33} = M_{33}^*$, $(B:M_{33})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{33}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{33}\right) = p^2$.

$\boldsymbol{M_{34} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (pu-v+t)\in p^2\mathbb{Z}_p\right\}}$
$= \left\langle (1,p,0,0),(0,p^2,0,0),(0,0,1,0),(0,1,0,1)\right\rangle$, for which: $\{M_{34}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{34} = M_{19}^*$, $(B:M_{34})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{19}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{34}\right) = p^2$.

$\boldsymbol{M_{35} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-v+t)\in p\mathbb{Z}_p\right\}}$
$= \langle (1,1,0,0),(0,p,0,0),(0,0,1,0),(0,1,0,1)\rangle$, for which: $\{M_{35}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{35} = M_{21}^*$, $(B:M_{35})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{21}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{35}\right) = p$.

$\boldsymbol{M_{36} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-v)\in p\mathbb{Z}_p\right\}}$
$= \langle (1,1,0,0),(0,p,0,0),(0,0,1,0),(0,0,0,1)\rangle$, for which: $\{M_{36}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{36} = M_{36}^*$, $(B:M_{36})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{36}^*\right) = (p-1)$, $\left(\mathbb{Z}_p^4 : M_{36}\right) = p$.

3). From Eq. (1) for J_3, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{37} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (p^2u-w+t)\in p^3\mathbb{Z}_p,\ (t-v)\in p^2\mathbb{Z}_p\right\}}$
$= \left\langle (1,0,p^2,0),(0,p^2,0,0),(0,0,p^3,0),(0,1,1,1)\right\rangle$, for which: $\{M_{37}:B\} = (p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_{37} = B^*$, $(B:M_{37})^{-s} = p^{s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : B^*\right) = p^3(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{37}\right) = p^5$.

$\boldsymbol{M_{38} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (p^2u-w+t)\in p^3\mathbb{Z}_p,\ (t-v)\in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,p^2,0),(0,p,0,0),(0,0,p^3,0),(0,1,1,1)\right\rangle$, for which: $\{M_{38}:B\} = (p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_{38} = M_3^*$, $(B:M_{38})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_3^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{38}\right) = p^4$.

$\boldsymbol{M_{39} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (p^2u-w+t)\in p^3\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,p^2,0),(0,1,0,0),(0,0,p^3,0),(0,0,1,1)\right\rangle$, for which: $\{M_{39}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{39} = M_9^*$, $(B:M_{39})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_9^*\right) = p^2(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{39}\right) = p^3$.

$\boldsymbol{M_{40} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (pu-w+t),(t-v)\in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,p,0),(0,p^2,0,0),(0,0,p^2,0),(0,1,1,1)\right\rangle$, for which: $\{M_{40}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{40} = M_5^*$, $(B:M_{40})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_5^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{40}\right) = p^4$.

$\boldsymbol{M_{41} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (pu-w+t)\in p^2\mathbb{Z}_p,\ (t-v)\in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,p,0),(0,p,0,0),(0,0,p^2,0),(0,1,1,1)\right\rangle$, for which: $\{M_{41}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{41} = M_7^*$, $(B:M_{41})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_7^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{41}\right) = p^3$.

$\boldsymbol{M_{42} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (pu-w+t)\in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,p,0),(0,1,0,0),(0,0,p^2,0),(0,0,1,1)\right\rangle$, for which: $\{M_{42}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{42} = M_{11}^*$, $(B:M_{42})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{11}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{42}\right) = p^2$.

$\boldsymbol{M_{43} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-w+t)\in p\mathbb{Z}_p,\ (t-v)\in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,1,0),(0,p^2,0,0),(0,0,p,0),(0,1,1,1)\right\rangle$, for which: $\{M_{43}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{43} = M_{17}^*$, $(B:M_{43})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{17}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{43}\right) = p^3$.

$\boldsymbol{M_{44} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-w)\in p\mathbb{Z}_p,\ (t-v)\in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,1,0),(0,p^2,0,0),(0,0,p,0),(0,1,0,1)\right\rangle$, for which: $\{M_{44}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{44} = M_{44}^*$, $(B:M_{44})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{44}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{44}\right) = p^3$.

$\boldsymbol{M_{45} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-w+t),(t-v)\in p\mathbb{Z}_p\right\}}$

$= \langle (1,0,1,0),(0,p,0,0),(0,0,p,0),(0,1,1,1)\rangle$, for which: $\{M_{45}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{45} = M_{13}^*$, $(B:M_{45})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{13}^*\right) = (p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{45}\right) = p^2$.

$\boldsymbol{M_{46} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-w),(t-v)\in p\mathbb{Z}_p\right\}}$

$= \langle (1,0,1,0),(0,p,0,0),(0,0,p,0),(0,1,0,1)\rangle$, for which: $\{M_{46}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{46} = M_{46}^*$, $(B:M_{46})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{46}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{46}\right) = p^2$.

$\boldsymbol{M_{47} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-w+t)\in p\mathbb{Z}_p\right\}}$

$= \langle (1,0,1,0),(0,1,0,0),(0,0,p,0),(0,0,1,1)\rangle$, for which: $\{M_{47}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{47} = M_{15}^*$, $(B:M_{47})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{15}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{47}\right) = p$.

$\boldsymbol{M_{48} = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (u-w)\in p\mathbb{Z}_p\right\}}$

$= \langle (1,0,1,0),(0,1,0,0),(0,0,p,0),(0,0,0,1)\rangle$, for which: $\{M_{48}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{48} = M_{48}^*$, $(B:M_{48})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{48}^*\right) = (p-1)$, $\left(\mathbb{Z}_p^4 : M_{48}\right) = p$.

4). From Eq. (1) for J_4, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{49}(a) = \left\{(u,v,w,t)\in\mathbb{Z}_p^4 :\ (pav-(1+pa)w+t)\in p^3\mathbb{Z}_p,\ (u-t),(v-t)\in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,p,p^2(1+pa)^{-1}a,0),(0,0,(1+pa)^{-1}p^3,0),(1,1,1,1)\right\rangle$, for $a\in F_p^*$ and for which: $\{M_{49}(a):B\} = (p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_{49}(a) = M_{49}^*(a)$, $(B:M_{49}(a))^{-s} = p^s$,

$\left(\left(\mathbb{Z}_p^*\right)^4 : M_{49}^*(a)\right) = p^2(p-1)^3,\ \left(\mathbb{Z}_p^4 : M_{49}(a)\right) = p^5.$

$\boldsymbol{M_{50}\,(a) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pav-(1+pa)w+t) \in p^3\mathbb{Z}_p,\ (v-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,p,p^2(1+pa)^{-1}a,0),(0,0,(1+pa)^{-1}p^3,0),(0,1,1,1)\right\rangle$, for $a \in F_p^*$ and for which: $\{M_{50}\,(a) : B\} = (p,p,p,p)\,M_8$, $\mathrm{Aut}_B M_{50}\,(a) = M_{50}^*\,(a)$, $(B : M_{50}\,(a))^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{50}^*(a)\right) = p^2(p-1)^2,\ \left(\mathbb{Z}_p^4 : M_{50}(a)\right) = p^4.$

$\boldsymbol{M_{51} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (p^2v-w+t) \in p^3\mathbb{Z}_p,\ (u-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,1,p^2,0),(0,0,p^3,0),(1,0,1,1)\right\rangle$, for which: $\{M_{51} : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{51} = M_3^*$, $(B : M_{51})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_3^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{51}\right) = p^4$.

$\boldsymbol{M_{52} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (p^2v-w+t) \in p^3\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,p^2,0),(0,0,p^3,0),(0,0,1,1)\right\rangle$, for which: $\{M_{52} : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{52} = M_4^*$, $(B : M_{52})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_4^*\right) = p^2(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{52}\right) = p^3$.

$\boldsymbol{M_{53} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p^2\mathbb{Z}_p,\ (v-u)\,,(t-v) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(1,1,1,1),(0,0,p^2,0),(0,0,0,p)\right\rangle$, for which: $\{M_{53} : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{53} = M_{53}^*$, $(B : M_{53})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{53}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{53}\right) = p^4$.

$\boldsymbol{M_{54}\,(a) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (av-(1+a)w+t) \in p^2\mathbb{Z}_p,\ (t-u)\,,(v-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,p,p(1+a)^{-1}a,0),(0,0,(1+a)^{-1}p^2,0),(1,1,1,1)\right\rangle$, for $a \in \{1,...,p-2\}$ and for which: $\{M_{54}\,(a) : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{54}\,(a) = M_{54}^*\,(a)$, $(B : M_{54}\,(a))^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{54}^*(a)\right) = p(p-1)^3,\ \left(\mathbb{Z}_p^4 : M_{54}(a)\right) = p^4.$

$\boldsymbol{M_{55} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p^2\mathbb{Z}_p,\ (t-v) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,1,1),(0,0,p^2,0),(0,0,0,p)\right\rangle$, for which: $\{M_{55} : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{55} = M_{55}^*$, $(B : M_{55})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{55}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{55}\right) = p^3$.

$\boldsymbol{M_{56}\,(a) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (av-(1+a)w+t) \in p^2\mathbb{Z}_p,\ (v-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,p,p(1+a)^{-1}a,0),(0,0,(1+a)^{-1}p^2,0),(0,1,1,1)\right\rangle$, for $a \in \{1,...,p-2\}$, and for which: $\{M_{56}\,(a) : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{56}\,(a) = M_{56}^*\,(a)$, $(B : M_{56}\,(a))^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{56}^*(a)\right) = p(p-1)^2,\ \left(\mathbb{Z}_p^4 : M_{56}(a)\right) = p^3.$

$\boldsymbol{M_{57} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-w+t) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,1,p,0),(0,0,p^2,0),(1,0,1,1)\right\rangle$, for which: $\{M_{57} : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{57} = M_7^*$, $(B : M_{57})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_7^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{57}\right) = p^3$.

$\boldsymbol{M_{58} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-w+t) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,p,0),(0,0,p^2,0),(0,0,1,1)\right\rangle$, for which: $\{M_{58} : B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{58} = M_8^*$, $(B : M_{58})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_8^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{58}\right) = p^2$.

$\boldsymbol{M_{59} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-pw+t) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,p^2,0,0),(0,p,1,0),(1,-1,0,1)\right\rangle$, for which: $\{M_{59} : B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{59} = M_{17}^*$, $(B : M_{59})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{17}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{59}\right) = p^3$.

$\boldsymbol{M_{60} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-pw+t) \in p^2\mathbb{Z}_p,\right\}}$

$= \left\langle (1,0,0,0),(0,p^2,0,0),(0,p,1,0),(0,-1,0,1)\right\rangle$, for which: $\{M_{60} : B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{60} = M_{18}^*$, $(B : M_{60})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{18}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{60}\right) = p^2$.

$\boldsymbol{M_{61} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t)\,,(t-u) \in p\mathbb{Z}_p\right\}}$

$= \langle (p,0,0,0),(0,p,0,0),(0,1,1,0),(1,-1,0,1)\rangle$, for which: $\{M_{61}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{61} = M_{13}^*$, $(B:M_{61})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{13}^*\right) = (p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{61}\right) = p^2$.

$\boldsymbol{M_{62} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p\mathbb{Z}_p\right\}}$

$= \langle (1,0,0,0),(0,p,0,0),(0,1,1,0),(0,-1,0,1)\rangle$, for which: $\{M_{62}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{62} = M_{14}^*$, $(B:M_{62})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{14}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{62}\right) = p$.

$\boldsymbol{M_{63} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w),(t-u) \in p\mathbb{Z}_p\right\}}$

$= \langle (p,0,0,0),(0,p,0,0),(0,1,1,0),(1,0,0,1)\rangle$, for which: $\{M_{63}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{63} = M_{63}^*$, $(B:M_{63})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{63}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{63}\right) = p^2$.

$\boldsymbol{M_{64} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p\mathbb{Z}_p\right\}}$

$= \langle (1,0,0,0),(0,p,0,0),(0,1,1,0),(0,0,0,1)\rangle$, for which: $\{M_{64}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{64} = M_{64}^*$, $(B:M_{64})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{64}^*\right) = (p-1)$, $\left(\mathbb{Z}_p^4 : M_{64}\right) = p$.

5). From Eq. (1) for J_5, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{65} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-w+t) \in p^3\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,1,p,0),(0,0,p^3,0),(1,0,1,1)\right\rangle$, for which: $\{M_{65}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{65} = B^*$, $(B:M_{65})^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : B^*\right) = p^3(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{65}\right) = p^4$.

$\boldsymbol{M_{66} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-w+t) \in p^3\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,p,0),(0,0,p^3,0),(0,0,1,1)\right\rangle$, for which: $\{M_{66}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{66} = M_2^*$, $(B:M_{66})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_2^*\right) = p^3(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{66}\right) = p^3$.

$\boldsymbol{M_{67} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,1,1,0),(0,0,p^2,0),(1,0,1,1)\right\rangle$, for which: $\{M_{67}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{67} = M_5^*$, $(B:M_{67})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_5^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{67}\right) = p^3$.

$\boldsymbol{M_{68} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+pt) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,1,1,0),(0,0,p^2,0),(1,0,p,1)\right\rangle$, for which: $\{M_{68}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{68} = M_{53}^*$, $(B:M_{68})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{53}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{68}\right) = p^3$.

$\boldsymbol{M_{69} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p^2\mathbb{Z}_p,\ (t-u) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,1,1,0),(0,0,p^2,0),(1,0,0,1)\right\rangle$, for which: $\{M_{69}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{69} = M_{69}^*$, $(B:M_{69})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{69}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{69}\right) = p^3$.

$\boldsymbol{M_{70} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,1,0),(0,0,p^2,0),(0,0,1,1)\right\rangle$, for which: $\{M_{70}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{70} = M_6^*$, $(B:M_{70})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_6^*\right) = p^2(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{70}\right) = p^2$.

$\boldsymbol{M_{71} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+pt) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,1,0),(0,0,p^2,0),(0,0,p,1)\right\rangle$, for which: $\{M_{71}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{71} = M_{55}^*$, $(B:M_{71})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{55}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{71}\right) = p^2$.

$\boldsymbol{M_{72} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,0,0),(0,1,1,0),(0,0,p^2,0),(0,0,0,1)\right\rangle$, for which: $\{M_{72}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{72} = M_{72}^*$, $(B:M_{72})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{72}^*\right) = p(p-1)$, $\left(\mathbb{Z}_p^4 : M_{72}\right) = p^2$.

6). From Eq. (1) for J_6, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{73}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pau - v + t) \in p^2\mathbb{Z}_p,\ (p^2u - w + t) \in p^3\mathbb{Z}_p\}}$
$= \langle (1, pa, p^2, 0), (0, p^2, 0, 0), (0, 0, p^3, 0), (0, 1, 1, 1)\rangle$, for $a \in F_p^*$. $\{M_{73}(a) : B\} = (p, p, p, p)\, M_8$, $\mathrm{Aut}_B M_{73}(a) = B^*$, $(B : M_{73}(a))^{-s} = p^s$, $\left((\mathbb{Z}_p^*)^4 : B^*\right) = p^3(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{73}(a)\right) = p^5$.

$\boldsymbol{M_{74}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (p^2u - w + t) \in p^3\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$
$= \langle (1, 1, p^2, 0), (0, p, 0, 0), (0, 0, p^3, 0), (0, a, 1, 1)\rangle$, for $a \in F_p$. $\{M_{74}(a) : B\} = (p, p^2, p^2, p^2)\, M_{16}$, $\mathrm{Aut}_B M_{74}(a) = M_3^*$, $(B : M_{74}(a))^{-s} = p^{2s}$, $\left((\mathbb{Z}_p^*)^4 : M_3^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{74}(a)\right) = p^4$.

$\boldsymbol{M_{75}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pau - v + t) \in p^2\mathbb{Z}_p,\ (pu - w + t) \in p^2\mathbb{Z}_p\}}$
$= \langle (1, pa, p, 0), (0, p^2, 0, 0), (0, 0, p^2, 0), (0, 1, 1, 1)\rangle$ for $a \in F_p^*$. $\{M_{75}(a) : B\} = (p, p^2, p^2, p^2)\, M_{16}$, $\mathrm{Aut}_B M_{75}(a) = M_5^*$, $(B : M_{75}(a))^{-s} = p^{2s}$, $\left((\mathbb{Z}_p^*)^4 : M_5^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{75}(a)\right) = p^4$.

$\boldsymbol{M_{76}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pu - w + t) \in p^2\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$
$= \langle (1, 1, p, 0), (0, p, 0, 0), (0, 0, p^2, 0), (0, a, 1, 1)\rangle$, for $a \in F_p$. $\{M_{76}(a) : B\} = (p, p^2, p^2, p^2)\, M_{16}$, $\mathrm{Aut}_B M_{76}(a) = M_7^*$, $(B : M_{76}(a))^{-s} = p^{3s}$, $\left((\mathbb{Z}_p^*)^4 : M_7^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{76}(a)\right) = p^3$.

$\boldsymbol{M_{77}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pu - v + t) \in p^2\mathbb{Z}_p,\ (u - w + at) \in p\mathbb{Z}_p\}}$
$= \langle (1, p, 1, 0), (0, p^2, 0, 0), (0, 0, p, 0), (0, 1, a, 1)\rangle$, for $a \in F_p$. $\{M_{77}(a) : B\} = (p, p^2, p^3, p^3)\, M_{24}$, $\mathrm{Aut}_B M_{77}(a) = M_{17}^*$, $(B : M_{77}(a))^{-s} = p^{3s}$, $\left((\mathbb{Z}_p^*)^4 : M_{17}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{77}(a)\right) = p^3$.

$\boldsymbol{M_{78} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (u - w) \in p\mathbb{Z}_p,\ (u - v) \in p\mathbb{Z}_p\}}$
$= \langle (1, 1, 1, 0), (0, p, 0, 0), (0, 0, p, 0), (0, 0, 0, 1)\rangle$, for which: $\{M_{78} : B\} = (p, p^2, p^3, p^3)\, M_{24}$, $\mathrm{Aut}_B M_{78} = M_{78}^*$, $(B : M_{78})^{-s} = p^{4s}$, $\left((\mathbb{Z}_p^*)^4 : M_{78}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{78}\right) = p^2$.

$\boldsymbol{M_{79}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (u - w + t) \in p\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$
$= \langle (1, 1, 1, 0), (0, p, 0, 0), (0, 0, p, 0), (0, a, 1, 1)\rangle$, for $a \in F_p$. $\{M_{79}(a) : B\} = (p, p^2, p^3, p^3)\, M_{24}$, $\mathrm{Aut}_B M_{79}(a) = M_{13}^*$, $(B : M_{79}(a))^{-s} = p^{4s}$, $\left((\mathbb{Z}_p^*)^4 : M_{13}^*\right) = (p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{79}(a)\right) = p^2$.

$\boldsymbol{M_{80} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (u - w) \in p\mathbb{Z}_p,\ (u - v + t) \in p\mathbb{Z}_p\}}$
$= \langle (1, 1, 1, 0), (0, p, 0, 0), (0, 0, p, 0), (0, 1, 0, 1)\rangle$, for which: $\{M_{80} : B\} = (p, p^2, p^3, p^3)\, M_{24}$, $\mathrm{Aut}_B M_{80} = M_{13}^*$, $(B : M_{80})^{-s} = p^{4s}$, $\left((\mathbb{Z}_p^*)^4 : M_{13}^*\right) = (p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{80}\right) = p^2$.

7). From Eq. (1) for J_7, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{81}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (pv - w + t) \in p^3\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\}}$
$= \langle (p, 0, 0, 0), (1, 1, p, 0), (0, 0, p^3, 0), (-a, 0, 1, 1)\rangle$ for $a \in F_p$. $\{M_{81}(a) : B\} = (p, p^2, p^2, p^2)\, M_{16}$, $\mathrm{Aut}_B M_{81}(a) = B^*$, $(B : M_{81}(a))^{-s} = p^{2s}$, $\left((\mathbb{Z}_p^*)^4 : B^*\right) = p^3(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{81}(a)\right) = p^4$.

$\boldsymbol{M_{82} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v - w) \in p^2\mathbb{Z}_p,\ (u - w) \in p\mathbb{Z}_p\}}$
$= \langle (p, 0, 0, 0), (0, p^2, 0, 0), (1, 1, 1, 0), (0, 0, 0, 1)\rangle$, for which: $\{M_{82} : B\} = (p, p^2, p^3, p^3)\, M_{24}$, $\mathrm{Aut}_B M_{82} = M_{82}^*$, $(B : M_{82})^{-s} = p^{3s}$, $\left((\mathbb{Z}_p^*)^4 : M_{82}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{82}\right) = p^3$.

$\boldsymbol{M_{83} = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v - w) \in p^2\mathbb{Z}_p,\ (u - w + t) \in p\mathbb{Z}_p\}}$
$= \langle (p, 0, 0, 0), (0, p^2, 0, 0), (1, 1, 1, 0), (-1, 0, 0, 1)\rangle$, for which: $\{M_{83} : B\} = (p, p^2, p^3, p^3)\, M_{24}$, $\mathrm{Aut}_B M_{83} = M_{53}^*$, $(B : M_{83})^{-s} = p^{3s}$, $\left((\mathbb{Z}_p^*)^4 : M_{53}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{83}\right) = p^3$.

$\boldsymbol{M_{84}(a) = \{(u,v,w,t) \in \mathbb{Z}_p^4 : (v - w + pt) \in p^2\mathbb{Z}_p,\ (u - w + at) \in p\mathbb{Z}_p\}}$

$= \left\langle (p,0,0,0),(0,p^2,0,0),(1,1,1,0),(-a,-p,0,1)\right\rangle$ for $a \in F_p$. $\{M_{84}(a):B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{84}(a) = M_{53}^*$, $(B:M_{84}(a))^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{53}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{84}(a)\right) = p^3$.

$\boldsymbol{M_{85}(a) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w+t) \in p^2\mathbb{Z}_p,\ (u-w+at) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (p,0,0,0),(0,p^2,0,0),(1,1,1,0),(-a,-1,0,1)\right\rangle$ for $a \in F_p$. $\{M_{85}(a):B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{85}(a) = M_5^*$, $(B:M_{85}(a))^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_5^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{85}(a)\right) = p^3$.

8). From Eq. (1) for J_8, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{86}(a) =}$

$\boldsymbol{\left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pav-(1+pa)w-p^2u+t) \in p^3\mathbb{Z}_p,\ (v-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,-p^2(1+pa)^{-1},0),(0,p,p^2a(1+pa)^{-1},0),(0,0,0,p^3),(0,1,1,1)\right\rangle$, for $a \in F_p^*$. $\{M_{86}(a):B\} = (p,p,p,p)M_8$, $\mathrm{Aut}_B M_{86}(a) = M_{49}^*(a)$, $(B:M_{86}(a))^{-s} = p^{2s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{49}^*(a)\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{86}(a)\right) = p^4$.

$\boldsymbol{M_{87} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (p^2v-w-p^2u+t) \in p^3\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,-p^2,0),(0,1,p^2,0),(0,0,p^3,0),(0,0,1,1)\right\rangle$, for which: $\{M_{87}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{87} = M_3^*$, $(B:M_{87})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_3^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{87}\right) = p^3$.

$\boldsymbol{M_{88}(a) =}$

$\boldsymbol{\left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (av-(1+a)w-pu+t) \in p^2\mathbb{Z}_p,\ (v-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,-p(1+a)^{-1},0),(0,p,pa(1+a)^{-1},0),(0,0,p^2,0),(0,1,1,1)\right\rangle$, for $a \in \{1,...,p-2\}$. $\{M_{88}(a):B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{88}(a) = M_{54}^*(a)$, $(B:M_{88}(a))^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{54}^*(a)\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{88}(a)\right) = p^3$.

$\boldsymbol{M_{89} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w-pu) \in p^2\mathbb{Z}_p,\ (v-t) \in p\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,-p,0),(0,1,1,1),(0,0,p^2,0),(0,0,0,p)\right\rangle$, for which: $\{M_{89}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{89} = M_{53}^*$, $(B:M_{89})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{53}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{89}\right) = p^3$.

$\boldsymbol{M_{90} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv-pu-w+t) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,0,-p,0),(0,1,p,0),(0,0,p^2,0),(0,0,1,1)\right\rangle$, for which: $\{M_{90}:B\} = \left(p,p^2,p^2,p^2\right)M_{16}$, $\mathrm{Aut}_B M_{90} = M_7^*$, $(B:M_{90})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_7^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{90}\right) = p^2$.

$\boldsymbol{M_{91} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu-v+pw+t) \in p^2\mathbb{Z}_p\right\}}$

$= \left\langle (1,p,0,0),(0,p^2,0,0),(0,p,1,0),(0,1,0,1)\right\rangle$, for which: $\{M_{91}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{91} = M_{17}^*$, $(B:M_{91})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{17}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{91}\right) = p^2$.

$\boldsymbol{M_{92} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-u-w+t) \in p\mathbb{Z}_p\right\}}$

$= \langle (1,1,0,0),(0,p,0,0),(0,1,1,0),(0,-1,0,1)\rangle$, for which: $\{M_{92}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{92} = M_{13}^*$, $(B:M_{92})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{13}^*\right) = (p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{92}\right) = p$.

$\boldsymbol{M_{93} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (v-w-u) \in p\mathbb{Z}_p\right\}}$

$= \langle (1,1,0,0),(0,p,0,0),(0,1,1,0),(0,0,0,1)\rangle$, for which: $\{M_{93}:B\} = \left(p,p^2,p^3,p^3\right)M_{24}$, $\mathrm{Aut}_B M_{93} = M_{78}^*$, $(B:M_{93})^{-s} = p^{5s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{78}^*\right) = (p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{93}\right) = p$.

9). From Eq. (1) for J_9, we obtain the following list of representatives of isomorphism classes of fractional ideals of B :

$\boldsymbol{M_{94} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pv - w - p^2u + t) \in p^3\mathbb{Z}_p\right\}}$
$= \left\langle(1,0,-p^2,0),(0,1,p,0),(0,0,p^3,0),(0,0,1,1)\right\rangle$, for which: $\{M_{94}:B\} = \left(p,p^2,p^2,p^2\right) M_{16}$, $\mathrm{Aut}_B M_{94} = B^*$, $(B:M_{94})^{-s} = p^{3s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : B^*\right) = p^3(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{94}\right) = p^3$.

$\boldsymbol{M_{95} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu + w - v + t) \in p^2\mathbb{Z}_p\right\}}$
$= \left\langle(1,0,-p,0),(0,1,1,0),(0,0,p^2,0),(0,0,-1,1)\right\rangle$, for which: $\{M_{95}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{95} = M_5^*$, $(B:M_{95})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_5^*\right) = p^2(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{95}\right) = p^2$.

$\boldsymbol{M_{96} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu + w - v + pt) \in p^2\mathbb{Z}_p\right\}}$
$= \left\langle(1,0,-p,0),(0,1,1,0),(0,0,p^2,0),(0,0,-p,1)\right\rangle$, for which: $\{M_{96}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{96} = M_{53}^*$, $(B:M_{96})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{53}^*\right) = p(p-1)^3$, $\left(\mathbb{Z}_p^4 : M_{96}\right) = p^2$.

$\boldsymbol{M_{97} = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 :\ (pu + w - v) \in p^2\mathbb{Z}_p\right\}}$
$= \left\langle(1,0,-p,0),(0,1,1,0),(0,0,p^2,0),(0,0,0,1)\right\rangle$, for which: $\{M_{97}:B\} = \left(p,p^2,p^3,p^3\right) M_{24}$, $\mathrm{Aut}_B M_{97} = M_{82}^*$, $(B:M_{97})^{-s} = p^{4s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : M_{82}^*\right) = p(p-1)^2$, $\left(\mathbb{Z}_p^4 : M_{97}\right) = p^2$.

Remark 3.1. We have that $M_1, ..., M_{97}$ form a set of representatives of all the isomorphism classes of fractional ideals of finite index in $B_p\left(C_{p^3}\right)$. In the previous list we observe that the only conductors are $B_p\left(C_{p^3}\right)$, $(p,p,p,p)\left(\mathbb{Z}_p \times B_p\left(C_{p^2}\right)\right)$, $\left(p,p^2,p^2,p^2\right)\left(\mathbb{Z}_p^2 \times B_p\left(C_p\right)\right)$ and $\left(\left(p,p^2,p^3,p^3\right)\right)\mathbb{Z}_p^4$. Therefore, in the following subsection (The Local Zeta Function for $B_p\left(C_{p^3}\right)$) it will be sufficient to compute the integrals corresponding to these four conductors.

Remark 3.2. The way to get $\{M_i : B\}$, $\mathrm{Aut}_B M_i$, $(B:M_i)^{-s}$, $\left(\left(\mathbb{Z}_p^*\right)^4 : \mathrm{Aut}_B M_i\right)$ and $\left(\mathbb{Z}_p^4 : M_i\right)$, for $i = 1, \ldots, 97$, is very similar. As an example, we present a sketch of proof for M_{81} :

We choose a Haar measure d^*x on $\left(\mathbb{Q}_p^*\right)^4$, such as

$$1 = \int_{(\mathbb{Z}_p^*)^4} d^*x = \int_{\dot{\bigcup_j}(a_j(\mathrm{Aut}_B M_{81}))} d^*x = \sum_j \int_{a_j(\mathrm{Aut}_B M_{81})} d^*x$$

$$= \sum_j \int_{\mathrm{Aut}_B M_{81}} d^*x = \left((\mathbb{Z}_p^*)^4 : \mathrm{Aut}_B M_{81}\right) \int_{\mathrm{Aut}_B M_{81}} d^*x$$

and then, we have that $\mu^*(\mathrm{Aut}_B M_{81})^{-1} = \left((\mathbb{Z}_p^*)^4 : \mathrm{Aut}_B M_{81}\right)$.

Now, if $M_{81}\left(a\right) = \left\{(u,v,w,t) \in \mathbb{Z}_p^4 : (pv - w + t) \in p^3\mathbb{Z}_p,\ (u - v + at) \in p\mathbb{Z}_p\right\}$ for $a \in F_p$, and $(u,v,w,t) \in M_{81}(a)$, then $w = t + pv + p^3w'$ and $u = v - at + pu'$ for $w', u' \in \mathbb{Z}_p$. It follows that

$$(u,v,w,t) = u'(p,0,0,0) + v(1,1,p,0) + w'(0,0,p^3,0) + t(-a,0,1,1),$$

and it is easy to see that as $\mathbb{Z}_p$-module

$$\boldsymbol{M_{81}(a) = \left\langle(p,0,0,0),(1,1,p,0),(0,0,p^3,0),(-a,0,1,1)\right\rangle}. \tag{2}$$

First, let's see how to calculate $\{M_{81}(a):B\}$. We know that

$$\{M_{81}(a):B\} = \left\{(x,y,w,z) \in \mathbb{Q}_p^4 :\ (x,y,w,z)M_{81}(a) \subseteq B\right\}.$$

If $(x, y, w, z) \in \{M_{81}(a) : B\}$, then from Eq. (2) we have that

$$(px, 0, 0, 0) \in B \tag{3}$$

$$(x, y, pw, 0) \in B \tag{4}$$

$$(0, 0, p^3 w, 0) \in B \tag{5}$$

$$(-ax, 0, w, z) \in B \tag{6}$$

From Eq. (6) it follows that $w \in p^2\mathbb{Z}_p$ and $z - w \in p^3\mathbb{Z}_p$, then $z \in p^2\mathbb{Z}_p$. From Eq. (4) it follows that $pw - y \in p^2\mathbb{Z}_p$ and $y - x \in p\mathbb{Z}_p$, then $y \in p^2\mathbb{Z}_p$ and $x \in p\mathbb{Z}_p$. From Eq. (3) and Eq. (5) it follows that $x \in \mathbb{Z}_p$ and $w \in \mathbb{Z}_p$ respectively, then $y, z \in \mathbb{Z}_p$. So, it is easy to see that

$$\boldsymbol{\{M_{81}(a) : B\} = (p, p^2, p^2, p^2) M_{16} = (p, p^2, p^2, p^2)} \left(\mathbb{Z}_{\boldsymbol{p}}^{\boldsymbol{2}} \times \boldsymbol{B_p(C_p)}\right).$$

Next, let's see how to calculate $\mathrm{Aut}_B M_{81}(a)$. We know that

$$\mathrm{End}_B M_{81}(a) = \left\{(x, y, w, z) \in \mathbb{Q}_p^4 : \ (x, y, w, z) M_{81}(a) \subseteq M_{81}(a)\right\}.$$

If $(x, y, w, z) \in \mathrm{End}_B M_{81}(a)$, then from Eq. (2) we have that

$$(px, 0, 0, 0) \in M_{81}(a) \tag{7}$$

$$(x, y, pw, 0) \in M_{81}(a) \tag{8}$$

$$(0, 0, p^3 w, 0) \in M_{81}(a) \tag{9}$$

$$(-ax, 0, w, z) \in M_{81}(a) \tag{10}$$

From Eq. (8) and Eq. (10), it follows that $y - x \in p\mathbb{Z}_p,\ w - y \in p^2\mathbb{Z}_p$ and $z - w \in p^3\mathbb{Z}_p$. From Eq. (7) and Eq. (9), it follows that $x,\ w \in \mathbb{Z}_p$ respectively, and then $y, z \in \mathbb{Z}_p$. So, it is easy to see that $\mathrm{End}_B M_{81}(a) = M_1 = B$, and then

$$\mathrm{Aut}_{\boldsymbol{B}} \boldsymbol{M_{81}(a) = B^*}.$$

To calculate $\mu^*(\mathrm{Aut}_B M_{81}(a))^{-1} = \left((\mathbb{Z}_p^*)^4 : B^*\right)$. Let

$$\begin{array}{ccc} & \varphi & \\ (\mathbb{Z}_p^*)^4 & \longrightarrow & \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^* \times \left(\frac{\mathbb{Z}}{p^2\mathbb{Z}}\right)^* \times \left(\frac{\mathbb{Z}}{p^3\mathbb{Z}}\right)^* \\ (x, y, w, z) & \longmapsto & \left(y_0^{-1} x_0, (w_0 + pw_1)^{-1}(y_0 + py_1), \left(z_0 + pz_1 + p^2 z_2\right)^{-1}\left(w_0 + pw_1 + p^2 w_2\right)\right) \end{array}$$

where $x = x_0 + px_1 + p^2 x_2 + \cdots;\ \ y = y_0 + py_1 + p^2 y_2 + \cdots;\ \ w = w_0 + pw_1 + p^2 w_2 + \cdots;$ $z = z_0 + pz_1 + p^2 z_2 + \cdots$, for $x_0, y_0, w_0, z_0 \in F_p^*$ and $x_i, y_i, w_i, z_i \in F_p$ for $i = 1, 2, ...$; It is easy to see that φ is a surjective homomorphism of multiplicative groups, such that $ker(\varphi) = B^*$. The first isomorphism theorem gives

$$\frac{(\mathbb{Z}_p^*)^4}{B^*} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^* \times \left(\frac{\mathbb{Z}}{p^2\mathbb{Z}}\right)^* \times \left(\frac{\mathbb{Z}}{p^3\mathbb{Z}}\right)^*$$

and then

$$\boldsymbol{\left((\mathbb{Z}_p^*)^4 : B^*\right) = p^3(p-1)^3}.$$

To calculate $\left(\mathbb{Z}_p^4 : M_{81}(a)\right)$. Let

$$\begin{array}{ccc} \mathbb{Z}_p^4 & \overset{\varphi}{\longrightarrow} & \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p^3\mathbb{Z}} \\ (x,y,w,z) & \longmapsto & \left(x_0 - y_0 + az_0, -\left(w_0 + pw_1 + p^2w_2\right) + \left(z_0 + pz_1 + p^2z_2\right) + p(y_0 + py_1)\right) \end{array}$$

where $x = x_0 + px_1 + p^2x_2 + \cdots$; $y = y_0 + py_1 + p^2y_2 + \cdots$; $w = w_0 + pw_1 + p^2w_2 + \cdots$; $z = z_0 + pz_1 + p^2z_2 + \cdots$, and $x_i, y_i, w_i, z_i \in F_p$ for $i = 0, 1, 2, ...$; It is easy to see that φ is a surjective homomorphism of additive groups, such that $ker(\varphi) = M_{81}(a)$. The first isomorphism theorem gives

$$\frac{\mathbb{Z}_p^4}{M_{81}(a)} \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p^3\mathbb{Z}}$$

and then

$$\boldsymbol{\left(\mathbb{Z}_p^4 : M_{81}(a)\right) = p^4}.$$

Finally, we know that $(M_{81}(a) : B) = \left(M_{81}(a) : \mathbb{Z}_p^4\right)\left(\mathbb{Z}_p^4 : B\right) = p^6\left(\mathbb{Z}_p^4 : M_{81}(a)\right)^{-1}$ then

$$\boldsymbol{(B : M_{81}(a))^{-s} = p^{6s}\left(\mathbb{Z}_p^4 : M_{81}(a)\right)^{-s} = p^{6s}p^{-4s} = p^{2s}}.$$

3.2. The Zeta function of $B\left(C_{p^3}\right)$

Proposition 3.3. *Let G be a finite group and let $\widetilde{B}(G)$ be the maximal order of $B(G)$. Then we have that $\zeta_{\widetilde{B}_p(G)}(s) = \frac{1}{\left(1-p^{-s}\right)^{|\mathcal{C}(G)|}}$ and $\zeta_{\widetilde{B}(G)}(s) = \left[\sum_{n=1}^{\infty} n^{-s}\right]^{|\mathcal{C}(G)|}$. Moreover, $\zeta_{B_p(G)}(s) = \frac{f_G\left(p^{-s}\right)}{\left(1-p^{-s}\right)^{|\mathcal{C}(G)|}}$, where $f_G\left(p^{-s}\right) \in \mathbb{Z}\left[p^{-s}\right]$.*

Proof. We have that $\widetilde{B}_p(G) = \mathbb{Z}_p^{|\mathcal{C}(G)|}$. Since

$$\zeta_{\mathbb{Z}_p}(s) = \sum_{t=0}^{\infty}\left(\mathbb{Z}_p : p^t\mathbb{Z}_p\right)^{-s} = \sum_{t=0}^{\infty}\left(p^{-s}\right)^t = \frac{1}{(1-p^{-s})},$$

it follows that

$$\zeta_{\widetilde{B}_p(G)}(s) = \left(\zeta_{\mathbb{Z}_p}(s)\right)^{|\mathcal{C}(G)|} = \frac{1}{(1-p^{-s})^{|\mathcal{C}(G)|}}.$$

Now, by the Euler product, we have that

$$\zeta_{\widetilde{B}(G)}(s) = \prod_{p-\text{prime}} \zeta_{\widetilde{B}_p(G)}(s) = \prod_{p-\text{prime}} \frac{1}{(1-p^{-s})^{|\mathcal{C}(G)|}} = \left[\sum_{n=1}^{\infty} n^{-s}\right]^{|\mathcal{C}(G)|}.$$

Finally, from the Theorem 2.3 we obtain that

$$\zeta_{B_p(G)}(s) = \frac{f_G(p^{-s})}{(1-p^{-s})^{|\mathcal{C}(G)|}},$$

where $f_G(p^{-s}) \in \mathbb{Z}[p^{-s}]$. ☑

Corollary 3.4. *Let $n \in \mathbb{N}$ and $B(C_{p^n})$ be the Burnside ring for a cyclic group of order p^n, and let $\widetilde{B}(C_{p^n})$ be its maximal order. Then we have that $\zeta_{B_p(C_{p^n})}(s) = \frac{f_{C_{p^n}}(p^{-s})}{(1-p^{-s})^{n+1}}$, where $f_{C_{p^n}}(p^{-s}) \in \mathbb{Z}[p^{-s}]$. Moreover, $\zeta_{B(C_{p^n})}(s) = f_{C_{p^n}}(p^{-s})\left[\sum_{n=1}^{\infty} n^{-s}\right]^{n+1}$.*

Proof. We have that there are $n+1$ conjugacy classes of C_{p^n}, therefore $\widetilde{B}_p(C_{p^n}) = \mathbb{Z}_p^{(n+1)}$, then from the above proposition, it follows that

$$\zeta_{\widetilde{B}_p(C_{p^n})}(s) = \frac{1}{(1-p^{-s})^{n+1}} \quad \text{and} \quad \zeta_{B_p(C_{p^n})}(s) = \frac{f_{C_{p^n}}(p^{-s})}{(1-p^{-s})^{n+1}},$$

where $f_{C_{p^n}}(p^{-s}) \in \mathbb{Z}[p^{-s}]$.

Now, by the Euler product, it follows that

$$\zeta_{B(C_{p^n})}(s) = \prod_{q-\text{prime}} \zeta_{B_q(C_{p^n})}(s) = \zeta_{B_p(C_{p^n})}(s) \prod_{\substack{q-\text{prime}\\ q\neq p}} \zeta_{B_q(C_{p^n})}(s).$$

Now, by Remark 2.4, since $f_{C_{p^n}}(q^{-s}) = 1$, when $q \neq p$, according to Theorem 2.3 we obtain:

$$\zeta_{B(C_{p^n})}(s) = f_{C_{p^n}}(p^{-s})\,\zeta_{\widetilde{B}_p(C_{p^n})}(s) \prod_{\substack{q-\text{prime}\\ q\neq p}} \zeta_{\widetilde{B}_q(C_{p^n})}(s) = f_{C_{p^n}}(p^{-s}) \prod_{q-\text{prime}} \zeta_{\widetilde{B}_q(C_{p^n})}(s)$$

$$= f_{C_{p^n}}(p^{-s})\,\zeta_{\widetilde{B}(C_{p^n})}(s) = f_{C_{p^n}}(p^{-s})\,\zeta_{\mathbb{Z}^{n+1}}(s) = f_{C_{p^n}}(p^{-s})\left[\sum_{n=1}^{\infty} n^{-s}\right]^{n+1}.$$

☑

The Local Zeta Function for $B_p(C_{p^3})$.

Remember that:

$$\zeta_{B_p(C_{p^3})}(s) = \sum_{i=1}^{97} Z_{B_p(C_{p^3})}(M_i;s).$$

Hence, to compute the zeta function of $B_p(C_{p^3})$, it is necessary to compute $Z_{B_p(C_{p^3})}(M_i;s)$ for $i = 1,\ldots,97$. According to the previous subsection, we only need to compute the integrals that we will study in the following four Remarks.

Remark 3.5. We choose a Haar measure d^*x on $\left(\mathbb{Q}_p^*\right)^4$, such that $d^*x = \left(d^*\alpha\right)^4$, where $d^*\alpha$ is a Haar measure of $\mathbb{Q}_p^*$ such that $\int_{\mathbb{Z}_p^*} d^*\alpha = 1$. Thus

$$\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha = \int\limits_{\bigcup\limits_{t=0}^{\infty} p^t \mathbb{Z}_p^*} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha = \sum_{t=0}^{\infty} \left[\left(p^{-s}\right)^t \int\limits_{\mathbb{Z}_p^*} d^*\alpha \right] = \frac{1}{\left(1 - p^{-s}\right)}. \tag{11}$$

Besides, we have

$$\mathcal{I}_1 = \int\limits_{\left(\mathbb{Q}_p^*\right)^4 \bigcap \left(p, p^2, p^3, p^3\right) \mathbb{Z}_p^4} \|x\|_{\mathbb{Q}_p^4}^s \, d^*x = \left\| \left(p, p^2, p^3, p^3\right) \right\|_{\mathbb{Q}_p^4}^s \left[\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha \right]^4.$$

Thus, from Eq. (11), we obtain: $\boldsymbol{\mathcal{I}_1 =} \frac{\left(p^{-s}\right)^9}{\left(1-p^{-s}\right)^4}$.

Remark 3.6. We choose a Haar measure d^*z on $\left(\mathbb{Q}_p^*\right)^2$, such that $d^*z = \left(d^*\alpha\right)^2$. We know that, $B_p\left(C_p\right)$ is local, where $\operatorname{rad}\left(B_p\left(C_p\right)\right) = (p,p)\,\mathbb{Z}_p^2$. Thus $B_p\left(C_p\right) = B_p^*\left(C_p\right) \bigcup (p,p)\,\mathbb{Z}_p^2$ and then

$$\begin{aligned} \int\limits_{\left(\mathbb{Q}_p^*\right)^2 \bigcap B_p\left(C_p\right)} \|z\|_{\mathbb{Q}_p^2}^s \, d^*z &= \int\limits_{B_p^*\left(C_p\right)} d^*z + \int\limits_{\left(\mathbb{Q}_p^*\right)^2 \bigcap (p,p)\mathbb{Z}_p^2} \|z\|_{\mathbb{Q}_p^2}^s \, d^*z \\ &= \frac{1}{(p-1)} + \|(p,p)\|_{\mathbb{Q}_p^2}^s \left[\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha \right]^2 \\ &= \frac{1}{(p-1)} + \frac{\left(p^{-s}\right)^2}{\left(1-p^{-s}\right)^2} = \frac{1 - 2\left(p^{-s}\right) + p\left(p^{-s}\right)^2}{(p-1)\left(1-p^{-s}\right)^2}. \end{aligned} \tag{12}$$

Besides, we have:

$$\begin{aligned} \mathcal{I}_2 &= \int\limits_{\left(\mathbb{Q}_p^*\right)^4 \bigcap \left(p, p^2, p^2, p^2\right)\left[\mathbb{Z}_p^2 \times B_p\left(C_p\right)\right]} \|x\|_{\mathbb{Q}_p^4}^s \, d^*x \\ &= \left\| \left(p, p^2, p^2, p^2\right) \right\|_{\mathbb{Q}_p^4}^s \left[\int\limits_{\mathbb{Q}_p^* \bigcap \mathbb{Z}_p} \|\alpha\|_{\mathbb{Q}_p}^s \, d^*\alpha \right]^2 \left[\int\limits_{\left(\mathbb{Q}_p^*\right)^2 \bigcap B_p\left(C_p\right)} \|z\|_{\mathbb{Q}_p^2}^s \, d^*z \right] \end{aligned}$$

Thus, from Eq. (11) and Eq. (12), we obtain: $\boldsymbol{\mathcal{I}_2 =} \frac{\left(p^{-s}\right)^7 \left(1 - 2\left(p^{-s}\right) + p\left(p^{-s}\right)^2\right)}{(p-1)\left(1-p^{-s}\right)^4}$.

Remark 3.7. We choose a Haar measure d^*y on $\left(\mathbb{Q}_p^*\right)^3$, such that $d^*y = \left(d^*\alpha\right)^3$. We know that, $B_p\left(C_{p^2}\right)$ is local, where $\operatorname{rad}\left(B_p\left(C_{p^2}\right)\right) = (p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_p\right)\right]$. Thus $B_p\left(C_{p^2}\right) = B_p^*\left(C_{p^2}\right) \bigcup (p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_p\right)\right]$ and then

$$\int\limits_{\left(\mathbb{Q}_p^*\right)^3 \bigcap B_p\left(C_{p^2}\right)} \|y\|_{\mathbb{Q}_p^3}^s \, d^*y = \int\limits_{B_p^*\left(C_{p^2}\right)} d^*y + \int\limits_{\left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_p\right)\right]} \|y\|_{\mathbb{Q}_p^3}^s \, d^*y$$

$$= \frac{1}{p\,(p-1)^2} + \|(p,p,p)\|^s_{\mathbb{Q}_p^3}\left[\int\limits_{\mathbb{Q}_p^* \cap \mathbb{Z}_p} \|\alpha\|^s_{\mathbb{Q}_p}\, d^*\alpha\right]\left[\int\limits_{(\mathbb{Q}_p^*)^2 \cap B_p(C_p)} \|z\|^s_{\mathbb{Q}_p^2}\, d^*z\right]$$

$$= \frac{1}{p\,(p-1)^2} + \frac{(p^{-s})^3\left(1-2\,(p^{-s})+p\,(p^{-s})^2\right)}{(p-1)\,(1-p^{-s})^3}$$

$$= \frac{1-3\,(p^{-s})+3\,(p^{-s})^2+\left(p^2-p-1\right)(p^{-s})^3-2p\,(p-1)\,(p^{-s})^4+p^2\,(p-1)\,(p^{-s})^5}{p\,(p-1)^2\,(1-p^{-s})^3}. \quad (13)$$

Besides, we have:

$$\mathcal{I}_3 = \int\limits_{(\mathbb{Q}_p^*)^4 \cap (p,p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_{p^2}\right)\right]} \|x\|^s_{\mathbb{Q}_p^4}\, d^*x$$

$$= \|(p,p,p,p)\|^s_{\mathbb{Q}_p^4}\left[\int\limits_{\mathbb{Q}_p^* \cap \mathbb{Z}_p} \|\alpha\|^s_{\mathbb{Q}_p}\, d^*\alpha\right]\left[\int\limits_{(\mathbb{Q}_p^*)^3 \cap B_p\left(C_{p^2}\right)} \|y\|^s_{\mathbb{Q}_p^3}\, d^*y\right]$$

Thus, from Eq. (11) and Eq. (13), we obtain:

$$\boldsymbol{\mathcal{I}_3} = \frac{(p^{-s})^4\left(1-3(p^{-s})+3(p^{-s})^2+(p^2-p-1)(p^{-s})^3-2p(p-1)(p^{-s})^4+p^2(p-1)(p^{-s})^5\right)}{p(p-1)^2(1-p^{-s})^4}.$$

Remark 3.8. We know that, $B_p\left(C_{p^3}\right)$ is local, where $\operatorname{rad}\left(B_p\left(C_{p^3}\right)\right) = (p,p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_{p^2}\right)\right]$. Thus $B_p\left(C_{p^3}\right) = B_p^*\left(C_{p^3}\right) \bigcup (p,p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_{p^2}\right)\right]$ and then

$$\boldsymbol{\mathcal{I}_4} = \int\limits_{(\mathbb{Q}_p^*)^4 \cap B_p\left(C_{p^3}\right)} \|x\|^s_{\mathbb{Q}_p^4}\, d^*x = \int\limits_{B_p^*\left(C_{p^3}\right)} d^*x + \int\limits_{(\mathbb{Q}_p^*)^4 \cap (p,p,p,p)\left[\mathbb{Z}_p \times B_p\left(C_{p^2}\right)\right]} \|x\|^s_{\mathbb{Q}_p^4}\, d^*x$$

$$= \frac{1}{p^3\,(p-1)^3} + \mathcal{I}_3$$

Proposition 3.9. *Let p be a rational prime and let $B = B_p\left(C_{p^3}\right)$ be the Burnside ring for a cyclic group C_{p^3} of order p^3. Therefore, the zeta function will be:*

$$\zeta_{B_p\left(C_{p^3}\right)}(s) = f_{C_{p^3}}\left(p^{-s}\right)\zeta_{\mathbb{Z}_p^4}(s),$$

where $\zeta_{\mathbb{Z}_p^4}(s) = \frac{1}{(1-p^{-s})^4}$ and

$$f_{C_{p^3}}\left(p^{-s}\right) = 1-3p^{-s}+\left(3+p+p^2+p^3\right)p^{-2s}+\left(-1-2p+p^2\right)\left(1+p+p^2\right)p^{-3s}$$
$$+p\left(3-p^3+p^4\right)p^{-4s}+p\,(p-1)\left(1-p+p^2+2p^3+p^4\right)p^{-5s}+p^2\,(p-1)^3\,(p+1)\,p^{-6s}$$
$$+p^3\,(p-1)\left(-1+p^2+p^3\right)p^{-7s}+p^3\,(p-1)\left(1-2p+p^2+p^3\right)p^{-8s}+p^4\,(p-1)^2\,p^{-9s}.$$

Proof. Remember that:

$$Z_B\left(M_i;s\right)=\mu^*\left(\mathrm{Aut}_B M_i\right)^{-1}\left(B:M_i\right)^{-s}\int\limits_{\left(\mathbb{Q}_p^*\right)^4}\Phi_{\{M_i:B\}}\left(x\right)\|x\|_{\mathbb{Q}_p^4}^s\,d^*x.$$

Hence, from case 1) in subsection 3.1, along with Remark 3.8, we obtain:

$$Z_B\left(M_1;s\right)=p^3\left(p-1\right)^3\mathcal{I}_4.$$

From case 1) in subsection 3.1, along with Remark 3.7, we obtain:

$$\begin{array}{lll} Z_B\left(M_2;s\right)=p^3\left(p-1\right)^2p^s\mathcal{I}_3, & Z_B\left(M_3;s\right)=p^2\left(p-1\right)^3p^s\mathcal{I}_3, & Z_B\left(M_4;s\right)=p^2\left(p-1\right)^2p^{2s}\mathcal{I}_3,\\ Z_B\left(M_5;s\right)=p^2\left(p-1\right)^3p^s\mathcal{I}_3, & Z_B\left(M_6;s\right)=p^2\left(p-1\right)^2p^{2s}\mathcal{I}_3, & Z_B\left(M_7;s\right)=p\left(p-1\right)^3p^{2s}\mathcal{I}_3,\\ Z_B\left(M_8;s\right)=p\left(p-1\right)^2p^{3s}\mathcal{I}_3. & & \end{array}$$

From case 1) in subsection 3.1, along with Remark 3.6, we obtain:

$$\begin{array}{lll} Z_B\left(M_9;s\right)=p^2\left(p-1\right)^2p^{2s}\mathcal{I}_2, & Z_B\left(M_{10};s\right)=p^2\left(p-1\right)p^{3s}\mathcal{I}_2, & Z_B\left(M_{11};s\right)=p\left(p-1\right)^2p^{3s}\mathcal{I}_2,\\ Z_B\left(M_{12};s\right)=p\left(p-1\right)p^{4s}\mathcal{I}_2, & Z_B\left(M_{13};s\right)=\left(p-1\right)^3p^{3s}\mathcal{I}_2, & Z_B\left(M_{14};s\right)=\left(p-1\right)^2p^{4s}\mathcal{I}_2,\\ Z_B\left(M_{15};s\right)=\left(p-1\right)^2p^{4s}\mathcal{I}_2, & Z_B\left(M_{16};s\right)=\left(p-1\right)p^{5s}\mathcal{I}_2, & Z_B\left(M_{17};s\right)=p\left(p-1\right)^3p^{2s}\mathcal{I}_2,\\ Z_B\left(M_{18};s\right)=p\left(p-1\right)^2p^{3s}\mathcal{I}_2. & & \end{array}$$

From case 1) in subsection 3.1, along with Remark 3.5, we obtain:

$$\begin{array}{lll} Z_B\left(M_{19};s\right)=p\left(p-1\right)^2p^{3s}\mathcal{I}_1, & Z_B\left(M_{20};s\right)=p\left(p-1\right)p^{4s}\mathcal{I}_1, & Z_B\left(M_{21};s\right)=\left(p-1\right)^2p^{4s}\mathcal{I}_1,\\ Z_B\left(M_{22};s\right)=\left(p-1\right)p^{5s}\mathcal{I}_1, & Z_B\left(M_{23};s\right)=\left(p-1\right)p^{5s}\mathcal{I}_1, & Z_B\left(M_{24};s\right)=p^{6s}\mathcal{I}_1. \end{array}$$

From case 2) in subsection 3.1, along with Remark 3.7, we obtain:

$$Z_B\left(M_{25};s\right)=p^3\left(p-1\right)^3p^s\mathcal{I}_3,\qquad Z_B\left(M_{26};s\right)=p^2\left(p-1\right)^3p^{2s}\mathcal{I}_3.$$

From case 2) in subsection 3.1, along with Remark 3.6, we obtain:

$$\begin{array}{lll} Z_B\left(M_{27};s\right)=p^2\left(p-1\right)^3p^{2s}\mathcal{I}_2, & Z_B\left(M_{28};s\right)=p^2\left(p-1\right)^2p^{2s}\mathcal{I}_2, & Z_B\left(M_{29};s\right)=p\left(p-1\right)^3p^{3s}\mathcal{I}_2,\\ Z_B\left(M_{30};s\right)=p\left(p-1\right)^2p^{3s}\mathcal{I}_2, & Z_B\left(M_{31};s\right)=p\left(p-1\right)^3p^{3s}\mathcal{I}_2, & Z_B\left(M_{32};s\right)=\left(p-1\right)^3p^{4s}\mathcal{I}_2,\\ Z_B\left(M_{33};s\right)=\left(p-1\right)^2p^{4s}\mathcal{I}_2. & & \end{array}$$

From case 2) in subsection 3.1, along with Remark 3.5, we obtain:

$$Z_B\left(M_{34};s\right)=p\left(p-1\right)^2p^{4s}\mathcal{I}_1,\quad Z_B\left(M_{35};s\right)=\left(p-1\right)^2p^{5s}\mathcal{I}_1,\quad Z_B\left(M_{36};s\right)=\left(p-1\right)p^{5s}\mathcal{I}_1.$$

From case 3) in subsection 3.1, along with Remark 3.7, we obtain:

$$Z_B\left(M_{37};s\right)=p^3\left(p-1\right)^3p^s\mathcal{I}_3,\qquad Z_B\left(M_{38};s\right)=p^2\left(p-1\right)^3p^{2s}\mathcal{I}_3.$$

From case 3) in subsection 3.1, along with Remark 3.6, we obtain:

$$\begin{array}{ll} Z_B\left(M_{39};s\right)=p^2\left(p-1\right)^2p^{3s}\mathcal{I}_2, & Z_B\left(M_{40};s\right)=p^2\left(p-1\right)^3p^{2s}\mathcal{I}_2,\\ Z_B\left(M_{41};s\right)=p\left(p-1\right)^3p^{3s}\mathcal{I}_2, & Z_B\left(M_{42};s\right)=p\left(p-1\right)^2p^{4s}\mathcal{I}_2. \end{array}$$

From case 3) in subsection 3.1, along with Remark 3.5, we obtain:

$$\begin{array}{lll} Z_B\left(M_{43};s\right)=p\left(p-1\right)^3p^{3s}\mathcal{I}_1, & Z_B\left(M_{44};s\right)=p\left(p-1\right)^2p^{3s}\mathcal{I}_1, & Z_B\left(M_{45};s\right)=\left(p-1\right)^3p^{4s}\mathcal{I}_1,\\ Z_B\left(M_{46};s\right)=\left(p-1\right)^2p^{4s}\mathcal{I}_1, & Z_B\left(M_{47};s\right)=\left(p-1\right)^2p^{5s}\mathcal{I}_1, & Z_B\left(M_{48};s\right)=\left(p-1\right)p^{5s}\mathcal{I}_1. \end{array}$$

From case 4) in subsection 3.1, along with Remark 3.7, we obtain:

$$Z_B\left(M_{49}\left(a\right);s\right)=p^2\left(p-1\right)^3p^s\mathcal{I}_3\quad Z_B\left(M_{50}\left(a\right);s\right)=p^2\left(p-1\right)^2p^{2s}\mathcal{I}_3\quad\text{for }a\in F_p^*.$$

From case 4) in subsection 3.1, along with Remark 3.6, we obtain:

$$\begin{array}{lll} Z_B(M_{51};s)=p^2(p-1)^3p^{2s}\mathcal{I}_2, & Z_B(M_{52};s)=p^2(p-1)^2p^{3s}\mathcal{I}_2, & Z_B(M_{53};s)=p(p-1)^3p^{2s}\mathcal{I}_2, \\ Z_B(M_{54}(a);s)=p(p-1)^3p^{2s}\mathcal{I}_2 & Z_B(M_{55};s)=p(p-1)^2p^{3s}\mathcal{I}_2, & Z_B(M_{56}(a);s)=p(p-1)^2p^{3s}\mathcal{I}_2 \\ Z_B(M_{57};s)=p(p-1)^3p^{3s}\mathcal{I}_2, & Z_B(M_{58};s)=p(p-1)^2p^{4s}\mathcal{I}_2. & \text{for } a\in\{1,...,p-2\}. \end{array}$$

From case 4) in subsection 3.1, along with Remark 3.5, we obtain:

$$\begin{array}{lll} Z_B(M_{59};s)=p(p-1)^3p^{3s}\mathcal{I}_1, & Z_B(M_{60};s)=p(p-1)^2p^{4s}\mathcal{I}_1, & Z_B(M_{61};s)=(p-1)^3p^{4s}\mathcal{I}_1, \\ Z_B(M_{62};s)=(p-1)^2p^{5s}\mathcal{I}_1, & Z_B(M_{63};s)=(p-1)^2p^{4s}\mathcal{I}_1, & Z_B(M_{64};s)=(p-1)p^{5s}\mathcal{I}_1. \end{array}$$

From case 5) in subsection 3.1, along with Remark 3.6, we obtain:

$$Z_B(M_{65};s)=p^3(p-1)^3p^{2s}\mathcal{I}_2, \qquad Z_B(M_{66};s)=p^3(p-1)^2p^{3s}\mathcal{I}_2.$$

From case 5) in subsection 3.1, along with Remark 3.5, we obtain:

$$\begin{array}{lll} Z_B(M_{67};s)=p^2(p-1)^3p^{3s}\mathcal{I}_1, & Z_B(M_{68};s)=p(p-1)^3p^{3s}\mathcal{I}_1, & Z_B(M_{69};s)=p(p-1)^2p^{3s}\mathcal{I}_1, \\ Z_B(M_{70};s)=p^2(p-1)^2p^{4s}\mathcal{I}_1, & Z_B(M_{71};s)=p(p-1)^2p^{4s}\mathcal{I}_1, & Z_B(M_{72};s)=p(p-1)p^{4s}\mathcal{I}_1. \end{array}$$

From case 6) in subsection 3.1, along with Remark 3.7, we obtain:

$$Z_B(M_{73}(a);s)=p^3(p-1)^3p^{s}\mathcal{I}_3 \quad \text{for } a\in F_p^*.$$

From case 6) in subsection 3.1, along with Remark 3.6, we obtain:

$$\begin{array}{ll} Z_B(M_{74}(a);s)=p^2(p-1)^3p^{2s}\mathcal{I}_2 & \text{for } a\in F_p \\ Z_B(M_{75}(a);s)=p^2(p-1)^3p^{2s}\mathcal{I}_2 & \text{for } a\in F_p^* \\ Z_B(M_{76}(a);s)=p(p-1)^3p^{3s}\mathcal{I}_2 & \text{for } a\in F_p. \end{array}$$

From case 6) in subsection 3.1, along with Remark 3.5, we obtain:

$$\begin{array}{lll} Z_B(M_{77}(a);s)=p(p-1)^3p^{3s}\mathcal{I}_1 & Z_B(M_{78};s)=(p-1)^2p^{4s}\mathcal{I}_1, & \\ Z_B(M_{79}(a);s)=(p-1)^3p^{4s}\mathcal{I}_1 & Z_B(M_{80};s)=(p-1)^3p^{4s}\mathcal{I}_1. & \text{for } a\in F_p. \end{array}$$

From case 7) in subsection 3.1, along with Remark 3.6, we obtain:

$$Z_B(M_{81}(a);s)=p^3(p-1)^3p^{2s}\mathcal{I}_2 \quad \text{for } a\in F_p.$$

From case 7) in subsection 3.1, along with Remark 3.5, we obtain:

$$\begin{array}{lll} Z_B(M_{82};s)=p(p-1)^2p^{3s}\mathcal{I}_1, & Z_B(M_{83};s)=p(p-1)^3p^{3s}\mathcal{I}_1, & \\ Z_B(M_{84}(a);s)=p(p-1)^3p^{3s}\mathcal{I}_1 & Z_B(M_{85}(a);s)=p^2(p-1)^3p^{3s}\mathcal{I}_1 & \text{for } a\in F_p. \end{array}$$

From case 8) in subsection 3.1, along with Remark 3.7, we obtain:

$$Z_B(M_{86}(a);s)=p^2(p-1)^3p^{2s}\mathcal{I}_3 \quad \text{for } a\in F_p^*.$$

From case 8) in subsection 3.1, along with Remark 3.6, we obtain:

$$\begin{array}{lll} Z_B(M_{87};s)=p^2(p-1)^3p^{3s}\mathcal{I}_2, & Z_B(M_{88}(a);s)=p(p-1)^3p^{3s}\mathcal{I}_2 & \\ Z_B(M_{89};s)=p(p-1)^3p^{3s}\mathcal{I}_2, & Z_B(M_{90};s)=p(p-1)^3p^{4s}\mathcal{I}_2. & \text{for } a\in\{1,...,p-2\}. \end{array}$$

From case 8) in subsection 3.1, along with Remark 3.5, we obtain:

$$Z_B(M_{91};s)=p(p-1)^3p^{4s}\mathcal{I}_1, \quad Z_B(M_{92};s)=(p-1)^3p^{5s}\mathcal{I}_1, \quad Z_B(M_{93};s)=(p-1)^2p^{5s}\mathcal{I}_1.$$

From case 9) in subsection 3.1, along with Remark 3.6, we obtain:

$$Z_B(M_{94};s)=p^3(p-1)^3p^{3s}\mathcal{I}_2.$$

From case 9) in subsection 3.1, along with Remark 3.5, we obtain:

$$Z_B(M_{95};s) = p^2(p-1)^3 p^{4s}\mathcal{I}_1, \quad Z_B(M_{96};s) = p(p-1)^3 p^{4s}\mathcal{I}_1, \quad Z_B(M_{97};s) = p(p-1)^2 p^{4s}\mathcal{I}_1.$$

From the 97 partial zeta functions above, we obtain that:

$$\zeta_{B_p\left(C_{p^3}\right)}(s) = \sum_{i=1}^{97} Z_{B_p\left(C_{p^3}\right)}(M_i;s) = p^{3s}(-1+p^2+p^s)\left(-p^2+p^4+(-1+p^3)p^s+p^{2s}\right)\boldsymbol{\mathcal{I}_1}+$$

$$(-1+p)p^{2s}\left(p^2-p^3-2p^4+p^5+p^6+(1-p-2p^2+2p^4+p^5)p^s+(-2+p+p^2+p^3)p^{2s}+p^{3s}\right)\boldsymbol{\mathcal{I}_2}$$

$$+(-1+p)^2p^{1+s}(p+p^s)(-1+p^2+p^3+p^s)\boldsymbol{\mathcal{I}_3}+(-1+p)^3p^3\boldsymbol{\mathcal{I}_4}$$

and finally, from Remarks 3.5 to 3.8, we obtain that

$$\zeta_{B_p\left(C_{p^3}\right)}(s) = \frac{1-3p^{-s}+\left(3+p+p^2+p^3\right)p^{-2s}+\left(-1-2p+p^2\right)\left(1+p+p^2\right)p^{-3s}}{\left(1-p^{-s}\right)^4}$$

$$+\frac{p\left(3-p^3+p^4\right)p^{-4s}+p(p-1)\left(1-p+p^2+2p^3+p^4\right)p^{-5s}+p^2(p-1)^3(p+1)p^{-6s}}{\left(1-p^{-s}\right)^4}$$

$$+\frac{p^3(p-1)\left(-1+p^2+p^3\right)p^{-7s}+p^3(p-1)\left(1-2p+p^2+p^3\right)p^{-8s}+p^4(p-1)^2p^{-9s}}{\left(1-p^{-s}\right)^4}$$

☑

The Global Zeta Function for $B\left(C_{p^3}\right)$.

By Corollary 3.4, it follows that $\zeta_{B\left(C_{p^3}\right)}(s) = f_{C_{p^3}}\left(p^{-s}\right)\zeta_{\mathbb{Z}^4}(s)$ where we have that $\zeta_{\mathbb{Z}^4}(s) = \left[\sum_{n=1}^{\infty} n^{-s}\right]^4$ and

$$f_{C_{p^3}}\left(p^{-s}\right) = 1-3p^{-s}+\left(3+p+p^2+p^3\right)p^{-2s}+\left(-1-2p+p^2\right)\left(1+p+p^2\right)p^{-3s}$$

$$+p\left(3-p^3+p^4\right)p^{-4s}+p(p-1)\left(1-p+p^2+2p^3+p^4\right)p^{-5s}+p^2(p-1)^3(p+1)p^{-6s}$$

$$+p^3(p-1)\left(-1+p^2+p^3\right)p^{-7s}+p^3(p-1)\left(1-2p+p^2+p^3\right)p^{-8s}+p^4(p-1)^2p^{-9s}.$$

3.3. Some relations for $Z_B(M_i;s)$

Lastly, we will study a couple of relations that satisfy the zeta functions $Z_B(M_i;s)$.

Let τ be the mapping such that

$$\tau: \quad \begin{array}{ccc} \mathbb{Q}_p^4 & \longrightarrow & \mathbb{Q}_p \\ (a_1,a_2,a_3,a_4) & \longmapsto & \sum_{i=1}^{4} a_i \end{array}$$

We will denote by

$$\overline{M}_i = \left((a_1,a_2,a_3,a_4)\in\mathbb{Q}_p^4 : \ \tau\left((u,v,w,t)(a_1,a_2,a_3,a_4)\right)\in\mathbb{Z}_p \ \forall \ (u,v,w,t)\in\{M_i:B\}\right).$$

a). For each

$$i\in\{19,...,24,34,...,36,43,...,48,59,...,64,67,...,72,77,...,80,82,...,85,91,...,93,95,...,97\}$$

we have that $\{M_i : B\} = \left(p, p^2, p^3, p^3\right) \mathbb{Z}_p^4$ which satisfy $\overline{M}_i = \left(p^{-1}, p^{-2}, p^{-3}, p^{-3}\right) \mathbb{Z}_p^4$. Thus

$$\overline{M}_i = \left(p^{-2}, p^{-4}, p^{-6}, p^{-6}\right) \{M_i : B\}.$$

Hence, according to the functional equation given in [12, Theorem 2.3] the following relations are fulfilled:

$$\frac{Z_B(M_i; s)}{Z_B(M_i; 1-s)} = \left[\frac{\|\alpha\|_{\mathbb{Q}_p^4}^{1-s} (B : M_i)^{1-2s}}{\left(\mathbb{Z}_p^4 : \{M_i : B\}\right)}\right] \frac{\zeta_{\mathbb{Z}_p^4}(s)}{\zeta_{\mathbb{Z}_p^4}(1-s)}, \tag{14}$$

where $\alpha = \left(p^{-2}, p^{-4}, p^{-6}, p^{-6}\right)$, thus $\|\alpha\|_{\mathbb{Q}_p^4}^{1-s} = \left(p^{18}\right)^{1-s}$, besides, $\left(\mathbb{Z}_p^4 : \{M_i : B\}\right) = p^9$, therefore, from Eq. (14) we obtain

$$\frac{Z_B(M_i; s)}{Z_B(M_i; 1-s)} = \left[\left(p^9\right)^{1-2s} (B : M_i)^{1-2s}\right] \frac{\zeta_{\mathbb{Z}_p^4}(s)}{\zeta_{\mathbb{Z}_p^4}(1-s)}.$$

For example $(B : M_{24}) = p^{-6}$. Thus $\frac{Z_B(M_{24}; s)}{Z_B(M_{24}; 1-s)} = \left[\left(p^3\right)^{1-2s}\right] \frac{\zeta_{\mathbb{Z}_p^4}(s)}{\zeta_{\mathbb{Z}_p^4}(1-s)}$.

b). For each $i \in \{9, ..., 18, 27, ..., 33, 39, ..., 42, 51, ..., 58, 65, 66, 74, ..., 76, 81, 87, ..., 90, 94\}$ we have that $\{M_i : B\} = \left(p, p^2, p^2, p^2\right) \left(\mathbb{Z}_p^2 \times B_p(C_p)\right)$ from which we obtain $\overline{M}_i = \left(p^{-1}, p^{-2}, -p^{-3}, p^{-3}\right) \left(\mathbb{Z}_p^2 \times B_p(C_p)\right)$ thus

$$\overline{M}_i = \left(p^{-2}, p^{-4}, -p^{-5}, p^{-5}\right) \{M_i : B\}.$$

Hence, for $\alpha = \left(p^{-2}, p^{-4}, -p^{-5}, p^{-5}\right)$, we have that $\|\alpha\|_{\mathbb{Q}_p^4}^{1-s} = \left(p^{16}\right)^{1-s}$, besides, $\left(\mathbb{Z}_p^4 : \{M_i : B\}\right) = p^8$, therefore, from Eq. (14) we obtain

$$\frac{Z_B(M_i; s)}{Z_B(M_i; 1-s)} = \left[\left(p^8\right)^{1-2s} (B : M_i)^{1-2s}\right] \frac{\zeta_{\mathbb{Z}_p^4}(s)}{\zeta_{\mathbb{Z}_p^4}(1-s)}.$$

For example $(B : M_{16}) = p^{-5}$, thus $\frac{Z_B(M_{16}; s)}{Z_B(M_{16}; 1-s)} = \left[\left(p^3\right)^{1-2s}\right] \frac{\zeta_{\mathbb{Z}_p^4}(s)}{\zeta_{\mathbb{Z}_p^4}(1-s)}$.

c). For each $i \in \{2, ..., 8, 25, 26, 37, 38, 49, 50, 73, 86\}$ we have that

$$\{M_i : B\} = (p, p, p, p) \left(\mathbb{Z}_p \times B_p\left(C_{p^2}\right)\right)$$

from which we obtain $\overline{M}_i = \left(p^{-1}, p^{-2}, -p^{-3}, p^{-3}\right) M_{58}$ therefore, the condition required in functional equation given in [12, Theorem 2.3], is not fulfilled.

d). Finally, for $i = 1$ we have that $\{M_1 : B\} = B$ which satisfies $\overline{M}_1 = \left(-p^{-1}, p^{-2}, -p^{-3}, p^{-3}\right) M_{94}$ therefore, the condition required in the functional equation given in [12, Theorem 2.3], is not fulfilled.

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