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About Lie algebra classification, conservation laws, and invariant solutions for the relativistic fluid sphere equation

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Abstract. The optimal generating operators for the relativistic fluid sphere equation have been derived. We have characterized all invariant solutions of this equation using these operators. Furthermore, we have introduced variational symmetries and their corresponding conservation laws, employing both Noether's theorem and Ibragimov's method. Finally, we have classified the Lie algebra associated with the given equation.

Keywords: Optimal algebra, Invariant solutions, Lie algebra classification, Lie symmetry group, Ibragimov's method, Noether's theorem, Conservation laws, Variational symmetries.

MSC2010: 35A30, 58J70, 76M60.

Acerca de la clasificación del álgebra de Lie, leyes de conservación y soluciones invariantes para la ecuación de la esfera de fluidos relativista

Resumen. Se han derivado los operadores generadores óptimos para la ecuación de la esfera de fluido relativista. Hemos caracterizado todas las soluciones invariantes de esta ecuación utilizando dichos operadores. Además, hemos introducido simetrías variacionales y sus correspondientes leyes de conservación, empleando tanto el teorema de Noether como el método de Ibragimov. Finalmente, hemos clasificado el álgebra de Lie asociada a la ecuación dada.

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Palabras clave: Álgebra óptima, Soluciones invariantes, Clasificación de las álgebras de Lie, Grupo de simetría de Lie, Método de Ibragimov, Teorema de Noether, Leyes de conservación, Simetrías variacionales.

1. Introduction

The Lie group symmetry method emerges as a potent tool employed for the analysis of differential equations (DEs), encompassing both ordinary differential equations (ODEs) and partial differential equations (PDEs), as well as fractional differential equations (FDEs) and various other equation types. This theory, originating in the 19th century by the mathematician Sophus Lie [1], follows in the footsteps of Galois theory within the realm of algebra. The application of the Lie group method to differential equations has engendered considerable interest across a spectrum of scientific disciplines, including pure mathematics and both theoretical and applied physics. This stems from the invaluable physical interpretations it affords to the scrutinized equations. Consequently, this approach facilitates the construction of conservation laws, employing, for instance, the renowned Noether's Theorem [2], and even symmetry solutions, a feat unattainable through conventional methods.

Moreover, this method contributes to the formulation of frameworks and the assessment of numerical methods, among other applications [3, 4, 5]. Present-day Lie symmetries have undergone extensive scrutiny, as evidenced by the comprehensive body of work available in the literature [6, 7, 8, 9, 10].

Within the study of the diffusions, specifically in the study of a relativistic fluid sphere by considering the so-called isotropic metric, Buchdahl [11] obtained the equation

$$y_{xx} = 3y^{-1}y_x^2 + x^{-1}y_x, \tag{1}$$

the solution of which is,

$$y(x) = \frac{a}{2c^2\sqrt{(1+kx^2)}}, \text{ where } c, a, k \text{ are constants.} \tag{2}$$

In [12], the first three integrals of (1) are presented using the Prelle-Singer method. In [13], first integrals of (1) are also derived by employing the relation between $\lambda-$symmetries and extending the Prelle-Singer method. In [14], the first integrals of (1) are obtained through an extension of the Prelle-Singer method. All of the aforementioned authors have arrived at the same solution for (1)

$$y(x) = \sqrt{\frac{1}{I_1\left(I_2 - x^2\right)}} \tag{3}$$

where I_1 and I_2 are first integrals. It is worth noting that (2) and (3) are equivalent when the constants are arranged accordingly. In [15], Zaitsev and Polyanin present a solution to (1)

$$y(x) = \pm\left(C_1 \ln|x| + C_2\right)^{-4}, \text{ where } C_1, C_2 \text{ are constants.} \tag{4}$$

The primary objectives of this article are as follows: We will begin by presenting the 5−dimensional Lie symmetry group for (1), offering a comprehensive description of its computation. Next, we will utilize this Lie symmetry group to introduce an optimal system, also known as an optimal algebra, for (1). Using the elements of the optimal system, we will then proceed to derive invariant solutions for (1). Following this, we will construct the Lagrangian associated with equation (1), based on the calculated group of symmetries. This will enable us to determine variational symmetries through the application of Noether's theorem, ultimately leading to the presentation of associated conservation laws. Furthermore, we will employ Ibragimov's method to establish non-trivial conservation laws. Finally, leveraging the group of symmetries we have identified, we will undertake the classification of the Lie algebra associated with (1).

2. About the Lie group symmetries for relativistic fluid sphere equation.

The purpose of this section is to determine for (1) the group of Lie symmetries. This objective is explained in the following proposition

Proposition 2.1. *The Lie group of symmetries for the equation (1) consists of the following elements:*

$$\Gamma_1 = x\frac{\partial}{\partial x}, \qquad \Gamma_2 = x^3\frac{\partial}{\partial x} + (-x^2 y)\frac{\partial}{\partial y}, \qquad \Gamma_3 = y\frac{\partial}{\partial y}, \qquad \Gamma_4 = y^3\frac{\partial}{\partial y}, \tag{5}$$

and $\Gamma_5 = x^2 y^3 \frac{\partial}{\partial y}$.

Proof. The general form for the generator operators of a Lie group with an admissible parameter for (1) is as follows:

$$x \to x + \epsilon\xi(x,y) + O\left(\epsilon^2\right), \qquad \text{and} \qquad y \to y + \epsilon\eta(x,y) + O\left(\epsilon^2\right),$$

where ϵ is the group parameter. The vector field associated with this group of transformations is given by:

$$\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{6}$$

with ξ and η being differentiable functions in $\mathbb{R}^2$. To calculate the infinitesimals η and ξ in (6), we employ the second extension operator

$$\Gamma^{(2)} \quad = \quad \Gamma + \eta_{[x]}\frac{\partial}{\partial y_x} + \eta_{[xx]}\frac{\partial}{\partial y_{xx}}, \tag{7}$$

to the equation (1), resulting in the following symmetry condition:

$$\xi\left(x^{-2}y_x\right) + \eta\left(3y^{-2}y_x^2\right) + \eta_{[x]}\left(-6y^{-1}y_x - x^{-1}\right) + \eta_{[xx]} = 0, \tag{8}$$

where $\eta_{[x]}$ and $\eta_{[xx]}$ are the coefficients in $\Gamma^{(2)}$ given by:

$$\begin{aligned} \eta_{[x]} &= D_x[\eta] - (D_x[\xi])y_x = \eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2. \\ \eta_{[xx]} &= D_x[\eta_{[x]}] - (D_x[\xi])y_{xx}, \\ &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy}y_x^3 \\ &\quad +(\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}. \end{aligned} \tag{9}$$

where D_x is the total derivative operator: $D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{y_x} + \cdots$.

After applying (9) to (8) and substituting the resulting expression for y_{xx} with (1), we obtain the following:

$$(-3y^{-1}\xi_y - \xi_{yy})y_x^3 + (3y^{-2}\eta - 3y^{-1}\eta_y - 2x^{-1}\xi_y + \eta_{yy} - 2\xi_{xy})y_x^2 + \left(x^{-2}\xi - 6y^{-1}\eta_x - x^{-1}\xi_x + 2\eta_{xy} - \xi_{xx}\right) y_x + \eta_{xx} - x^{-1}\eta_x = 0.$$

after analyzing the coefficients in regard to the independent variables $y_x^3, y_x^2, y_x, 1$ we get the following system of determining equations:

$$3\xi_y + y\xi_{yy} = 0, \tag{10a}$$

$$3x\eta - 3xy\eta_y - 2y^2\xi_y + xy^2\eta_{yy} - 2xy^2\xi_{xy} = 0, \tag{10b}$$

$$y\xi - 6x^2\eta_x - xy\xi_x + 2x^2y\eta_{xy} - x^2y\xi_{xx} = 0, \tag{10c}$$

$$x\eta_{xx} - \eta_x = 0. \tag{10d}$$

Solving in (10a) and (10d) we get

$$\xi = c_1(x) - \frac{c_2(x)}{2y^2} \ ; \ \eta = \frac{1}{2}x^2c_3(y) + c_4(y).$$

Using these equations in (10b) and (10c) to solve for $c_1(x), c_2(x), c_3(y)$, and $c_4(y)$, we obtain the following:

$$\xi = k_1x + k_2x^3,$$
$$\eta = -k_2x^2y + k_3y + k_4y^3 + k_5x^2y^3,$$

where k_1, k_2, k_3, k_4, and k_5 are arbitrary constants. Thus, by using η and ξ in the operator (6) and grouping the constants, we obtain Γ_1 through Γ_5, which constitute the generators of the symmetry group for the equation (1), as proposed in the statement of Proposition 2.1. ☑

3. About optimal system

Considering [16, 18, 19, 20, 21, 17], we will now present the optimal system for (5). To determine the optimal algebra, it is essential to first obtain the corresponding commutator table. This can be calculated using the following operator:

$$[\Gamma_\alpha, \Gamma_\beta] = \Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha = \sum_{i=1}^{n} \left(\Gamma_\alpha(\xi_\beta^i) - \Gamma_\beta(\xi_\alpha^i)\right) \frac{\partial}{\partial x^i}, \tag{11}$$

where $i = 1, 2$, with $\alpha, \beta = 1, \cdots, 5$ and $\xi_\alpha^i, \xi_\beta^i$ are the corresponding coefficients of the $\Gamma_\alpha, \Gamma_\beta$.

In Table 1, we present the results obtained by applying the operator (11) to the symmetry group (1).

[;]	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	0	$2\Gamma_2$	0	0	$2\Gamma_5$
Γ_2	$-2\Gamma_2$	0	0	$-2\Gamma_5$	0
Γ_3	0	0	0	$2\Gamma_4$	$2\Gamma_5$
Γ_4	0	$2\Gamma_5$	$-2\Gamma_4$	0	0
Γ_5	$-2\Gamma_5$	0	$-2\Gamma_5$	0	0

Table 1. Commutators table for (5).

Following the objective for determining the optimal algebra, we must now obtain the 'Adjoint Representation' using Table 1 and the next operator (Ad):

$$Ad(exp(\lambda\Gamma))G = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(ad(\Gamma))^n G \text{ for the symmetries } \Gamma \text{ and } G.$$

In Table 2, we display the adjoint representation for each Γ_i, with each entry in this table calculated using the operator mentioned above.

Ad[,]	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	$e^{-2\lambda}\Gamma_2$	Γ_3	Γ_4	$e^{-2\lambda}\Gamma_5$
Γ_2	$\Gamma_1 + 2\lambda\Gamma_2$	Γ_2	Γ_3	$\Gamma_4 + 2\lambda\Gamma_5$	Γ_5
Γ_3	Γ_1	Γ_2	Γ_3	$e^{-2\lambda}\Gamma_4$	$e^{-2\lambda}\Gamma_5$
Γ_4	Γ_1	$\Gamma_2 - 2\lambda\Gamma_5$	$\Gamma_3 + 2\lambda\Gamma_4$	Γ_4	Γ_5
Γ_5	$\Gamma_1 + 2\lambda\Gamma_5$	Γ_2	$\Gamma_3 + 2\lambda\Gamma_5$	Γ_4	Γ_5

Table 2. Adjoint representation for 5.

Proposition 3.1. *The vector fields that represent the optimal algebra associated with the equation* (1) *are as follows:*

$$\Gamma_4, b_{13}\Gamma_5,$$

$$a_1\Gamma_1 + a_3\Gamma_3,\ a_1\Gamma_1 + a_4\Gamma_4,\ a_2\Gamma_2 + a_3\Gamma_3,\ \Gamma_4 + \Gamma_5,$$

$$a_2\Gamma_2 + b_{14}\Gamma_4,\ a_3\Gamma_3 + b_{15}\Gamma_4,\ a_1\Gamma_1 + b_5\Gamma_5,\ \Gamma_2 + b_8\Gamma_5,$$

$$\Gamma_1 + b_6\Gamma_2 + b_7\Gamma_5,\ a_2\Gamma_2 + \Gamma_4 + b_{10}\Gamma_5, \Gamma_3 + b_{17}\Gamma_4 + b_{18}\Gamma_5,$$

$$a_1\Gamma_1 - a_1\Gamma_3 + b_1\Gamma_5,\ -\Gamma_1 + \Gamma_3 + b_3\Gamma_4,\ a_2\Gamma_2 + \Gamma_3 - \frac{b_{16}}{a_2}\Gamma_4,$$

$$a_2\Gamma_2 + a_3\Gamma_3 + b_{14}\Gamma_4 + b_4\Gamma_5, a_1\Gamma_1 + \Gamma_3 + b_3\Gamma_4 + b_4\Gamma_5.$$

Proof. Considering the generic operator G, which is a linear combination of the symmetry group (5). Let

$$G = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5. \tag{12}$$

Using the adjoint operator (Ad) in G and the elements from Table 2, we can simplify the coefficients a_i as much as possible, which will be our goal at every step of this proof.

1) Assuming $a_5 = 1$ in (12) we have that $G = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + \Gamma_5$. Using the adjoint operator to (Γ_1, G) and (Γ_3, G) no reductions are available, but applying the adjoint operator to (Γ_2, G) we obtain

$$G_1 = Ad\,(\exp(\lambda_1\Gamma_2))\,G = a_1\Gamma_1 + (a_2 + 2a_1\lambda_1)\Gamma_2 + a_3\Gamma_3 + \Gamma_4 + (1 + 2a_4\lambda_1)\Gamma_5. \quad (13)$$

1.1) **Case** $a_1 \neq 0$**.** applying $\lambda_1 = \frac{-a_2}{2a_1}$ with $a_1 \neq 0$, in (13), Γ_2 is eliminated, therefore $G_1 = a_1\Gamma_1 + a_3\Gamma_3 + a_4\Gamma_4 + b_1\Gamma_5$, where $b_1 = 1 - \frac{a_2 a_4}{a_1}$. Now, using the adjoint operator to (Γ_4, G_1), we obtain $G_2 = Ad\,(\exp(\lambda_2\Gamma_4))\,G_1 = a_1\Gamma_1 + a_3\Gamma_3 + (a_4 + 2a_3\lambda_2)\Gamma_4 + b_1\Gamma_5$.

1.1.A) **Case** $a_3 \neq 0$**.** Using $\lambda_2 = \frac{-a_4}{2a_3}$, with $a_3 \neq 0$, is eliminated Γ_4, then $G_2 = a_1\Gamma_1 + a_3\Gamma_3 + b_1\Gamma_5$. Using the adjoint operator to (Γ_5, G_2), we obtain

$$G_3 = Ad\,(\exp(\lambda_3\Gamma_5))\,G_2 = a_1\Gamma_1 + a_3\Gamma_3 + (b_1 + 2\lambda_3(a_1 + a_3))\Gamma_5. \quad (14)$$

1.1.$A.A_1$) **Case** $a_1 + a_3 \neq 0$**.** Using $\lambda_3 = -\frac{b_1}{2(a_1+a_3)}$, with $a_1 + a_3 \neq 0$, in (14), Γ_5 is eliminated, therefore $G_3 = a_1\Gamma_1 + a_3\Gamma_3$. This is how the first optimal element appears

$$G_3 = a_1\Gamma_1 + a_3\Gamma_3, \text{ with } a_1, a_3 \neq 0. \quad (15)$$

Then, we obtain the first reduction of the generic element (12).

1.1.$A.A_2$) **Case** $a_1 + a_3 = 0$**.** We get in (14), $G_3 = a_1\Gamma_1 - a_1\Gamma_3 + b_1\Gamma_5$. This is how the other optimal element appears

$$G_3 = a_1\Gamma_1 - a_1\Gamma_3 + b_1\Gamma_5, \text{ with } a_1 \neq 0. \quad (16)$$

Then, we obtain one more reduction of the generic element (12).

1.1.B) **Case** $a_3 = 0$**.** We get, $G_2 = a_1\Gamma_1 + a_4\Gamma_4 + b_1\Gamma_5$. Now, applying the adjoint operator to (Γ_5, G_2), we have $G_{15} = Ad\,(\exp(\lambda_{15}\Gamma_5))\,G_2 = a_1\Gamma_1 + a_4\Gamma_4 + (b_1 + 2a_1\lambda_{15})\Gamma_5$. Then, as $a_1 \neq 0$, we can substitute $\lambda_{15} = \frac{-b_1}{2a_1}$, then Γ_5 is removed, then an another item of the optimal algebra is

$$G_{15} = a_1\Gamma_1 + a_4\Gamma_4, \text{ with } a_1 \neq 0. \quad (17)$$

1.2) **Case** $a_1 = 0$**.** Now, we have, $G_1 = a_2\Gamma_2 + a_3\Gamma_3 + \Gamma_4 + (1 + 2a_4\lambda_1)\Gamma_5$. 1.2.$A$) **Case** $a_4 \neq 0$**.** Using $\lambda_1 = \frac{-1}{2a_4}$ with $a_4 \neq 0$, Γ_5 is removed, therefore $G_1 = a_2\Gamma_2 + a_3\Gamma_3 + \Gamma_4$. Now, applying the adjoint operator to (Γ_4, G_1), we get

$$G_{16} = Ad\,(\exp(\lambda_{16}\Gamma_4))\,G_1 = a_2\Gamma_2 + a_3\Gamma_3 + (1 + 2a_3\lambda_{16})\Gamma_4 - 2a_2\lambda_{16}\Gamma_5. \quad (18)$$

1.2.$A.A_1$) **Case** $a_3 \neq 0$**.** Using $\lambda_{16} = \frac{-1}{2a_3}$ with $a_3 \neq 0$, Γ_4 is eliminated, therefore $G_{16} = a_2\Gamma_2 + a_3\Gamma_3 + b_9\Gamma_5$, where $b_9 = \frac{a_2}{a_3}$. Now, using the adjoint operator to (Γ_5, G_{16}), we obtain

$$G_{17} = Ad\,(\exp(\lambda_{17}\Gamma_5))\,G_{16} = a_2\Gamma_2 + a_3\Gamma_3 + (b_9 + 2a_3\lambda_{17})\Gamma_5.$$

Then, as $a_3 \neq 0$, we can substitute $\lambda_{17} = \frac{-b_9}{2a_3}$, then Γ_5 is removed, after we get other item

$$G_{17} = a_2\Gamma_2 + a_3\Gamma_3, \text{ with } a_3 \neq 0. \quad (19)$$

1.2.A.A_2) **Case** $a_3 = 0$**.** We get in (18), $G_{16} = a_2\Gamma_2 + \Gamma_4 - 2a_2\lambda_{16}\Gamma_5$.

1.2.A.A_2.1) **Case** $a_2 \neq 0$**.** Applying $\lambda_{16} = \frac{-b_{10}}{2a_2}$ with $a_2 \neq 0$, we get $G_{16} = a_2\Gamma_2 + \Gamma_4 + b_{10}\Gamma_5$. Now, applying the adjoint operator to (Γ_5, G_{16}), it is not possible to further reduce, this is how the other optimal element appears:

$$G_{16} = a_2\Gamma_2 + \Gamma_4 + b_{10}\Gamma_5. \tag{20}$$

1.2.A.A_2.2) **Case** $a_2 = 0$**.** We have $G_{16} = \Gamma_4$. Now, using the adjoint operator to (Γ_5, G_{16}), Therefore, there are no reductions, this is how the other optimal element appears:

$$G_{16} = \Gamma_4. \tag{21}$$

1.2.B) **Case** $a_4 = 0$**.** We get, $G_1 = a_2\Gamma_2 + a_3\Gamma_3 + \Gamma_4 + \Gamma_5$. Now, applying the adjoint operator to (Γ_4, G_1), we have

$$G_{19} = Ad\left(\exp\left(\lambda_{19}\Gamma_4\right)\right)G_1 = a_2\Gamma_2 + a_3\Gamma_3 + (1 + 2a_3\lambda_{19})\Gamma_4 + (1 - 2a_2\lambda_{19})\Gamma_5. \tag{22}$$

1.2.B.1) **Case** $a_3 \neq 0$**.** Using $\lambda_{19} = \frac{-1}{2a_3}$, with $a_3 \neq 0$, Γ_4 is eliminated, therefore $G_{19} = a_2\Gamma_2 + a_3\Gamma_3 + b_{11}\Gamma_5$, with $b_{11} = 1 + \frac{a_2}{a_3}$. Now, using the adjoint operator to (Γ_5, G_{19}), we obtain

$$G_{20} = Ad\left(\exp\left(\lambda_{20}\Gamma_5\right)\right)G_{19} = a_2\Gamma_2 + a_3\Gamma_3 + (b_{11} + 2a_3\lambda_{20})\Gamma_5. \tag{23}$$

Then, as $a_3 \neq 0$, we can substitute $\lambda_{20} = \frac{-b_{11}}{2a_3}$, then Γ_5 is removed, after we get other item of the optimal algebra

$$G_{20} = a_2\Gamma_2 + a_3\Gamma_3, \text{ with } a_3 \neq 0. \tag{24}$$

Then, we obtain one more reduction of the generic element (12).

1.2.B.2) **Case** $a_3 = 0$**.** We have in (22), $G_{19} = a_2\Gamma_2 + \Gamma_4 + (1 - 2a_2\lambda_{19})\Gamma_5$.

1.2.B.2.A_1) **Case** $a_2 \neq 0$**.** Using $\lambda_{19} = \frac{1}{2a_2}$, with $a_2 \neq 0$, Γ_5 is eliminated, therefore $G_{19} = a_2\Gamma_2 + \Gamma_4$. Now, applying the adjoint operator to (Γ_5, G_{19}), it is not possible to further reduce, this is how the other optimal element appears

$$G_{19} = a_2\Gamma_2 + \Gamma_4. \tag{25}$$

1.2.B.2.A_2) **Case** $a_2 = 0$**.** We get $G_{19} = \Gamma_4 + \Gamma_5$. Now, applying the adjoint operator to (Γ_5, G_{19}), it is not possible to further reduce, this is how the other optimal element appears:

$$G_{19} = \Gamma_4 + \Gamma_5. \tag{26}$$

2) Assuming $a_5 = 0$ and $a_4 = 1$ in (12), we have that $G = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + \Gamma_4$. Using the adjoint operator to (Γ_1, G) and (Γ_3, G) and there is no reduction, but applying the adjoint operator to (Γ_2, G) we obtain

$$G_4 = Ad\left(\exp\left(\lambda_4\Gamma_2\right)\right)G = a_1\Gamma_1 + (a_2 + 2a_1\lambda_4)\Gamma_2 + a_3\Gamma_3 + \Gamma_4 + 2\lambda_4\Gamma_5. \tag{27}$$

2.1) **Case** $a_1 \neq 0$**.** Using $\lambda_4 = \frac{-a_2}{2a_1}$ with $a_1 \neq 0$, in (27), Γ_2 is eliminated, therefore $G_4 = a_1\Gamma_1 + a_3\Gamma_3 + \Gamma_4 + b_2\Gamma_5$, where $b_2 = \lambda_4 = \frac{-a_2}{2a_1}$. Now, using the adjoint operator to (Γ_4, G_4), we obtain $G_5 = Ad\,(\exp(\lambda_5\Gamma_4))\,G_4 = a_1\Gamma_1 + a_3\Gamma_3 + (1 + 2a_3\lambda_5)\Gamma_4 + b_2\Gamma_5$.

2.1.A) **Case** $a_3 \neq 0$**.** Using $\lambda_5 = \frac{-1}{2a_3}$ with $a_3 \neq 0$, Γ_4 is eliminated, therefore $G_5 = a_1\Gamma_1 + a_3\Gamma_3 + b_2\Gamma_5$. Now, using the adjoint operator to (Γ_5, G_5), we obtain $G_6 = Ad\,(\exp(\lambda_6\Gamma_5))\,G_5 = a_1\Gamma_1 + a_3\Gamma_3 + (b_2 + 2\lambda_6(a_1 + a_3))\Gamma_5$.

2.1.$A.A_1$) **Case** $a_1 + a_3 \neq 0$**.** Using $\lambda_6 = -\frac{b_2}{2(a_1+a_3)}$, with $a_1 + a_3 \neq 0$, Γ_5 is eliminated, therefore $G_6 = a_1\Gamma_1 + a_3\Gamma_3$. This is how the other optimal element appears:

$$G_6 = a_1\Gamma_1 + a_3\Gamma_3, \text{ with } a_1, a_3 \neq 0. \tag{28}$$

2.1.$A.A_2$) **Case** $a_1 + a_3 = 0$**.** we get $G_6 = a_1\Gamma_1 - a_1\Gamma_3 + b_2\Gamma_5$. This is how the other optimal element appears:

$$G_6 = a_1\Gamma_1 - a_1\Gamma_3 + b_2\Gamma_5, \text{ with } a_1 \neq 0. \tag{29}$$

2.1.B) **Case** $a_3 = 0$**.** We have $G_5 = a_1\Gamma_1 + \Gamma_4 + b_2\Gamma_5$. Now, applying the adjoint operator to (Γ_5, G_5), we have $G_{21} = Ad\,(\exp(\lambda_{21}\Gamma_5))\,G_5 = a_1\Gamma_1 + \Gamma_4 + (b_2 + 2a_1\lambda_{21})\Gamma_5$. Then, as $a_1 \neq 0$, we can substitute $\lambda_{21} = \frac{-b_2}{2a_1}$, then Γ_5 is removed, after we get another item of the optimal algebra

$$G_{21} = a_1\Gamma_1 + \Gamma_4, \text{ with } a_1 \neq 0. \tag{30}$$

2.2) **Case** $a_1 = 0$**.** We have in (27), $G_4 = a_2\Gamma_2 + a_3\Gamma_3 + \Gamma_4 + 2\lambda_4\Gamma_5$, using $\lambda_4 = \frac{b_{13}}{2}$ get $G_4 = a_2\Gamma_2 + a_3\Gamma_3 + \Gamma_4 + b_{13}\Gamma_5$. Now, using the adjoint operator to (Γ_4, G_4), we obtain $G_{21} = Ad\,(\exp(\lambda_{21}\Gamma_4))\,G_4 = a_2\Gamma_2 + a_3\Gamma_3 + 2a_3\lambda_{21}\Gamma_4 + (b_{13} - 2a_2\lambda_{21})\Gamma_5$.

2.2.A) **Case** $a_2 \neq 0$**.** Using $\lambda_{21} = \frac{b_{13}}{2a_2}$, with $a_2 \neq 0$, Γ_5 is eliminated, therefore $G_{21} = a_2\Gamma_2 + a_3\Gamma_3 + b_{14}$, where $b_{14} = \frac{a_3 b_{13}}{a_2}$. Now, using the adjoint operator to (Γ_5, G_{21}), we obtain

$$G_{22} = Ad\,(\exp(\lambda_{22}\Gamma_5))\,G_{21} = a_2\Gamma_2 + a_3\Gamma_3 + b_{14}\Gamma_4 + 2a_3\lambda_{22}\Gamma_5.$$

2.2.A.1) **Case** $a_3 \neq 0$**.** It is clear that is not possible to further reduce, then using $\lambda_{22} = \frac{b_{15}}{2a_3}$, with $a_3 \neq 0$, this is how the other optimal element appears:

$$G_{22} = a_2\Gamma_2 + a_3\Gamma_3 + b_{14}\Gamma_4 + b_{15}\Gamma_5, \text{ with } a_2 \neq 0. \tag{31}$$

Then, we obtain one more reduction of the generic element (12).

2.2.A.2) **Case** $a_3 = 0$**.** We obtain $G_{22} = a_2\Gamma_2 + b_{14}\Gamma_4$. This is how the other optimal element appears:

$$G_{22} = a_2\Gamma_2 + b_{14}\Gamma_4, \text{ with } a_2 \neq 0. \tag{32}$$

2.2.B) **Case** $a_2 = 0$**.** We have $G_{21} = a_3\Gamma_3 + 2a_3\lambda_{21}\Gamma_4 + b_{13}\Gamma_5$.

2.2.B.1) **Case** $a_3 \neq 0$**.** Using $\lambda_{21} = \frac{b_{15}}{2a_3}$, we get $G_{21} = a_3\Gamma_3 + b_{15}\Gamma_4 + b_{13}\Gamma_5$. Now, applying the adjoint operator to (Γ_5, G_{21}), we have

$$G_{23} = Ad\,(\exp(\lambda_{23}\Gamma_5))\,G_{21} = a_3\Gamma_3 + b_{15}\Gamma_4 + (b_{13} + 2a_3\lambda_{23})\Gamma_5.$$

Then, as $a_3 \neq 0$, we can substitute $\lambda_{23} = \frac{-b_{13}}{2a_3}$, then Γ_5 is removed, after we get another item of the optimal algebra

$$G_{23} = a_3\Gamma_3 + b_{15}\Gamma_4, \text{ with } a_3 \neq 0. \tag{33}$$

2.2.B.2) **Case** $a_3 = 0$**.** We have $G_{21} = b_{13}\Gamma_5$. Now, using the adjoint operator to (Γ_5, G_{21}), therefore, there are not any more reductions. This is how the other optimal element appears:

$$G_{21} = b_{13}\Gamma_5. \tag{34}$$

3) Assuming $a_4 = a_5 = 0$ and $a_3 = 1$ in (12), we have that $G = a_1\Gamma_1 + a_2\Gamma_2 + \Gamma_3$. Using the adjoint operator to (Γ_1, G) and (Γ_3, G) there is no reduction, but applying the adjoint operator to (Γ_2, G) we obtain

$$G_7 = Ad\left(\exp\left(\lambda_7\Gamma_2\right)\right)G = a_1\Gamma_1 + (a_2 + 2a_1\lambda_7)\Gamma_2 + \Gamma_3. \tag{35}$$

3.1) **Case** $a_1 \neq 0$**.** Using $\lambda_7 = \frac{-a_2}{2a_1}$ with $a_1 \neq 0$, in (35), Γ_2 is eliminated, therefore $G_7 = a_1\Gamma_1 + \Gamma_3$. Now, using the adjoint operator to (Γ_4, G_7), we obtain $G_8 = Ad\left(\exp\left(\lambda_8\Gamma_4\right)\right)G_7 = a_1\Gamma_1 + \Gamma_3 + 2\lambda_8\Gamma_4$. It is clear that is not possible to further reduce, then substituting $\lambda_8 = \frac{b_3}{2}$ then $G_8 = a_1\Gamma_1 + \Gamma_3 + b_3\Gamma_4$. Using the adjoint operator to (Γ_5, G_8), we obtain

$$G_9 = Ad\left(\exp\left(\lambda_9\Gamma_5\right)\right)G_8 = a_1\Gamma_1 + \Gamma_3 + b_3\Gamma_4 + 2\lambda_9(1 + a_1)\Gamma_5.$$

3.1.A) **Case** $1 + a_1 \neq 0$**.** It is clear that it is not possible to further reduce. Then substituting $\lambda_9 = \frac{b_4}{2(1+a_1)}$, this is how the other optimal element appears:

$$G_9 = a_1\Gamma_1 + \Gamma_3 + b_3\Gamma_4 + b_4\Gamma_5. \tag{36}$$

3.1.B) **Case** $1 + a_1 = 0$**.** We have $G_9 = -\Gamma_1 + \Gamma_3 + b_3\Gamma_4$, then we have other element of the optimal algebra

$$G_9 = -\Gamma_1 + \Gamma_3 + b_3\Gamma_4. \tag{37}$$

3.2) **Case** $a_1 = 0$**.** We have $G_7 = a_2\Gamma_2 + \Gamma_3$. Now, using the adjoint operator to (Γ_4, G_7), we get $G_{24} = Ad\left(\exp\left(\lambda_{24}\Gamma_4\right)\right)G_7 = a_2\Gamma_2 + \Gamma_3 + 2\lambda_{24}\Gamma_4 - 2a_2\lambda_{24}\Gamma_5$.

3.2.A_1) **Case** $a_2 \neq 0$**.** It is clear that is not possible to further reduce, then substituting $\lambda_{24} = \frac{-b_{16}}{2a_2}$, we get $G_{24} = a_2\Gamma_2 + \Gamma_3 - \frac{b_{16}}{a_2}\Gamma_4 + b_{16}\Gamma_5$. Now, using the adjoint operator to (Γ_5, G_{24}), we obtain

$$G_{25} = Ad\left(\exp\left(\lambda_{25}\Gamma_5\right)\right)G_{24} = a_2\Gamma_2 + \Gamma_3 - \frac{b_{16}}{a_2}\Gamma_4 + (b_{16} + 2\lambda_{25})\Gamma_5.$$

Using $\lambda_{25} = \frac{-b_{16}}{2}$, Γ_5 is removed, this is how the other optimal element appears:

$$G_{25} = a_2\Gamma_2 + \Gamma_3 - \frac{b_{16}}{a_2}\Gamma_4. \tag{38}$$

3.2.A_2) **Case** $a_2 = 0$**.** We have $G_{24} = \Gamma_3 + 2\lambda_{24}\Gamma_4$. Using $\lambda_{24} = \frac{b_{17}}{2}$, we obtain $G_{24} = \Gamma_3 + b_{17}\Gamma_4$. Now, using the adjoint operator to (Γ_5, G_{24}), we obtain

$$G_{26} = Ad\,(\exp{(\lambda_{26}\Gamma_5)})\, G_{24} = \Gamma_3 + b_{17}\Gamma_4 + 2\lambda_{26}\Gamma_5.$$

Thus, we do not get any more reduction, then using $\lambda_{26} = \frac{b_{18}}{2}$ this is how the other optimal element appears:

$$G_{26} = \Gamma_3 + b_{17}\Gamma_4 + b_{18}\Gamma_5. \tag{39}$$

4) Using $a_3 = a_4 = a_5 = 0$ and $a_2 = 1$ in (12), we obtain that $G = a_1\Gamma_1 + \Gamma_2$. Using the adjoint operator to (Γ_1, G) and (Γ_3, G) we conclude that there is no reduction, but using the adjoint operator to (Γ_2, G) we obtain

$$G_{10} = Ad\,(\exp{(\lambda_{10}\Gamma_2)})\, G = a_1\Gamma_1 + (1 + 2a_1\lambda_{10})\Gamma_2. \tag{40}$$

4.1) **Case** $a_1 \neq 0$**.** Using $\lambda_{10} = \frac{-1}{2a_1}$ with $a_1 \neq 0$, in (40), Γ_2 is eliminated, therefore $G_{10} = a_1\Gamma_1$. Now, applying the operator (Γ_4, G_{10}), it is clear that is not possible to further reduce, but using the adjoint operator to (Γ_5, G_{10}), we obtain $G_{11} = Ad\,(\exp{(\lambda_{11}\Gamma_5)})\, G_{10} = a_1\Gamma_1 + 2a_1\lambda_{11}\Gamma_5$. Thus, we do not get any more reduction, then using $\lambda_{11} = \frac{b_5}{2a_1}$, this is how the other optimal element appears:

$$G_{11} = a_1\Gamma_1 + b_5\Gamma_5. \tag{41}$$

4.2) **Case** $a_1 = 0$**.** We have $G_{10} = \Gamma_2$. Now, using the operator (Γ_5, G_{10}), it is not possible to further reduce; however, by applying the adjoint operator to (Γ_4, G_{10}), we get

$$G_{14} = Ad\,(\exp{(\lambda_{14}\Gamma_4)})\, G_{10} = \Gamma_2 - 2\lambda_{14}\Gamma_5$$

Thus, it is not possible to further reduce it, then using $\lambda_{14} = \frac{-b_8}{2}$, this is how the other optimal element appears:

$$G_{11} = \Gamma_2 + b_8\Gamma_5. \tag{42}$$

5) Using $a_2 = a_3 = a_4 = a_5 = 0$ and $a_1 = 1$ in (12), we obtain that $G = \Gamma_1$. Applying the adjoint (Adj) to (Γ_1, G) , (Γ_3, G) and (Γ_4, G) it is not possible to further reduce, but applying the adjoint operator to (Γ_2, G) we obtain

$$G_{12} = Ad\,(\exp{(\lambda_{10}\Gamma_2)})\, G = \Gamma_1 + 2\lambda_{12}\Gamma_2, \tag{43}$$

it is not possible to further reduce, then substituting $\lambda_{12} = \frac{b_6}{2}$ we have that $G_{12} = \Gamma_1 + b_6\Gamma_2$. Using the adjoint operator to (Γ_5, G_{12}), we obtain

$$G_{13} = Ad\,(\exp{(\lambda_{13}\Gamma_5)})\, G_{12} = \Gamma_1 + b_6\Gamma_2 + 2\lambda_{13}\Gamma_5.$$

It is not possible to further reduce, then substituting $\lambda_{13} = \frac{b_7}{2}$, this is how the other optimal element appears:

$$G_{13} = \Gamma_1 + b_6\Gamma_2 + b_7\Gamma_5. \tag{44}$$

☑

4. About invariant solutions

We will characterize the invariant solutions using the operators from Proposition 3.1. To achieve this objective, we will apply the technique of the invariant curve condition ([17], Section 4.3; see also [22]), which is as follows:

$$0 = \eta - y_x \xi = Q(x, y, y_x). \tag{45}$$

An example will be presented below: by taking the element Γ_4 from Proposition 3.1 in (45), we get $Q = \eta_4 - y_x \xi_4 = 0$, then $(y^3) - y_x(0) = 0$, thus $y(x) = 0$, which is the trivial solution for (1).

Table 3 presents both implicit and explicit solutions obtained following the procedure described in the previous paragraph for each element of Proposition (3.1).

	Elements	$Q(x,y,y_x) = 0$	**Solutions**	**Type Solution**
1	Γ_4	$(y^3) - y_x(0) = 0$	$y(x) = 0$	Trivial
2	Γ_5	$(x^2y^3) - y_x(0) = 0$	$y(x) = 0$	Trivial
3	$\Gamma_1 + \Gamma_3$	$(y) - y_x(x) = 0$	$y(x) = 0$	Trivial
4	$\Gamma_1 + \Gamma_4$	$(y^3) - y_x(x) = 0$	$y(x) = \pm\frac{1}{\sqrt{2}\sqrt{c-\log(x)}}$	Explicit
5	$\Gamma_2 + \Gamma_3$	$(y - x^2y) - y_x(x^3) = 0$	$y(x) = \frac{c_1 e^{-1/\left(2x^2\right)}}{x}$	Explicit
6	$\Gamma_4 + \Gamma_5$	$(y^3 + x^2y^3) - y_x(0) = 0$	$y(x) = 0$	Trivial
7	$\Gamma_2 + \Gamma_4$	$(y^3 - x^2y) - y_x(x^3) = 0$	$y(x) = \pm\frac{\sqrt{2}x}{\sqrt{cx^4+1}}$	Explicit
8	$\Gamma_3 + \Gamma_4$	$(y^3 + y) - y_x(0) = 0$	$y(x) = 0$	Trivial
9	$\Gamma_1 + \Gamma_5$	$(x^2y^3) - y_x(x) = 0$	$y(x) = \pm\frac{1}{\sqrt{c-x^2}}$	Explicit
10	$\Gamma_2 + \Gamma_5$	$(x^2y^3 - x^2y) - y_x(x^3) = 0$	$y(x) = \pm\frac{1}{\sqrt{cx^2+1}}$	Explicit
11	$\Gamma_1 + \Gamma_2 + \Gamma_5$	$(x^2y^3 - x^2y) - y_x(x^3 + x) = 0$	$y(x) = \pm\frac{1}{\sqrt{e^{2c}x^2+e^{2c}+1}}$	Explicit
12	$\Gamma_2 + \Gamma_4 + \Gamma_5$	$(y^3 - x^2y + x^2y^3) - y_x(x^3) = 0$	$y(x) = \pm\frac{\sqrt{2}x}{\sqrt{cx^4+2x^2+1}}$	Explicit
13	$\Gamma_3 + \Gamma_4 + \Gamma_5$	$(y^3 + x^2y^3 + y) - y_x(0) = 0$	$y^2 + x^2y^2 + 1 = 0,\ y(x) \neq 0$	Implicit
14	$\Gamma_1 - \Gamma_3 + \Gamma_5$	$(x^2y^3 - y) - y_x(x) = 0$	$y(x) = \pm\frac{1}{\sqrt{cx^2-2x^2\log(x)}}$	Explicit
15	$-\Gamma_1 + \Gamma_3 + \Gamma_4$	$(y^3 + y) - y_x(-x) = 0$	$y(x) = \pm\left(\left(\frac{x^2+4}{4x^2}\right)^{\frac{1}{2}} - \frac{1}{2}\right)^{\frac{1}{2}}$	Explicit
16	$\Gamma_2 + \Gamma_3 - \Gamma_4$	$(y - y^3 - x^2y) - y_x(x^3) = 0$	$y(x) = \pm\frac{1}{\sqrt{ce^{\frac{1}{x^2}}x^2+x^2+1}}$	Explicit
17	$\Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$	$(x^2y^3 + y^3 + y - x^2y) - y_x(x^3) = 0$	$y(x) = \pm\frac{1}{\sqrt{ce^{\frac{1}{x^2}}x^2-2x^2-1}}$	Explicit
18	$\Gamma_1 + \Gamma_3 + \Gamma_4 + \Gamma_5$	$(y^3 + x^2y^3 + y) - y_x(x) = 0$	$y(x) = \pm\frac{\sqrt{2}x}{\sqrt{c-x^2(x^2+2)}}$	Explicit

Table 3. Explicit an implicit solutions for (1), with c being a constant.

Remarks : Note that the solution in numeral 10 coincides with a particular solution presented in [11]. The solution in numeral 9 coincides with the solution presented in [12] and [14].

5. On the calculation of variational symmetries and the presentation of conserved quantities

We will calculate the variational symmetries of (1), and using Noether's theorem [2], we will present the conserved quantities.

According to Nucci [23], our first step will be to use the Jacobi Last Multiplier method to calculate the inverse of the determinant Δ with the ultimate goal of obtaining the

Lagrangian.

$$\Delta = \begin{vmatrix} x & y_x & y_{xx} \\ \Gamma_{3,x} & \Gamma_{3,y} & \Gamma_3^{(1)} \\ \Gamma_{4,x} & \Gamma_{4,y} & \Gamma_4^{(1)} \end{vmatrix} = \begin{vmatrix} x & y_x & y_{xx} \\ 0 & y & y_x \\ 0 & y^3 & 3y^2 y_x \end{vmatrix},$$

where $\Gamma_{4,x}, \Gamma_{4,y}, \Gamma_{3,x}$, and $\Gamma_{3,y}$ are the components of the symmetries Γ_3 and Γ_4 presented in Proposition 5, and $\Gamma_4^{(1)}, \Gamma_3^{(1)}$ as their first prolongations. Then, we have $\Delta = 2y^3 y_x$, thus $M = \frac{1}{\Delta} = \frac{y^{-3}}{2y_x}$. According to [23], $M = L_{y_x y_x}$, so $L_{y_x y_x} = \frac{y^{-3}}{2y_x}$. After integrating twice with respect to y_x, we obtain the following Lagrangian:

$$L(x, y, y_x) = \frac{y^{-3}}{2} y_x \ln(y_x) - \frac{y^{-3}}{2} y_x + y_x f_1(x, y) + f_2(x, y), \tag{46}$$

with arbitrary functions f_1 and f_2. In (46), consider $f_1 = f_2 = 0$. (**Note:** other Lagrangians can be calculated for (1) using different vector fields in Proposition 5). Thus, we obtain the following:

$$D_x[f(x, y)] = L_x \xi(x, y) + L\xi_x(x, y) + L_y \eta(x, y) + L_{y_x} \eta_{[x]}(x, y),$$

Applying (46) and (9), we have the following:

$$\begin{aligned} &\xi_x \left(\frac{y^{-3}}{2} y_x \ln(y_x) - \frac{y^{-3}}{2} y_x \right) + \eta \left(\frac{-3y^{-4}}{2} y_x \ln(y_x) + \frac{3y^{-4}}{2} y_x \right) \\ + \quad &\left(\eta_x + (\eta_y - \xi_x) y_x - \xi_y y_x^2\right) \left(\frac{y^{-3}}{2} \ln(y_x) \right) - y_x f_y - f_x = 0. \end{aligned}$$

In the last equation, associating terms in regard to $1, y_x, y_x^2$ and y_x^3, and simplifying some terms we obtain the following determinant equations:

$$\xi_y = \eta_x = f_x = 0, \tag{47a}$$

$$-3\eta + y\eta_y = 0, \tag{47b}$$

$$-y\xi_x + 3\eta - 2y^4 f_y = 0. \tag{47c}$$

If we solve the system (47a), (47b) and (47c) for ξ, η and f, we obtain the generators of the variational Noether's symmetries, these solutions are

$$\eta = a_1 y^3, \qquad \xi = a_2 x + a_3, \qquad \text{and} \qquad f(y) = \frac{a_2 y^{-2}}{4} + \frac{3a_1 \ln(y)}{2} + a_4 \tag{48}$$

where a_1, a_2, a_3 and a_4 are constants. Thus, the Noether symmetry group or variational symmetries are

$$V_1 = y^3 \frac{\partial}{\partial y}, \qquad V_2 = x \frac{\partial}{\partial x}, \qquad \text{and} \qquad V_3 = \frac{\partial}{\partial x}. \tag{49}$$

Remarks : Note that $V_1 = \Gamma_4$ and $V_2 = \Gamma_1$, this implies that, two of the symmetries of equation (1), are variational symmetries. If we follow what is proposed in [24], the way to calculate the conserved quantities is to solve the expression

$$I = (X y_x - Y) L_{y_x} - XL + f,$$

so, using (46), (48) and (49). Therefore, the conserved quantities are given by

$$\begin{aligned} I_1 &= \frac{-\ln(y_x)}{2} + \frac{a_2 y^{-2}}{4} + \frac{3a_1 \ln(y)}{2} + a_4, \\ I_2 &= \frac{y^{-3}(x-1)y_x \ln(y)}{2} + \frac{y^{-3}}{2} y_x + \frac{a_2 y^{-2}}{4} + \frac{3a_1 \ln(y)}{2} + a_4, \\ I_3 &= \frac{y^{-3} y_x}{2} + \frac{a_2 y^{-2}}{4} + \frac{3a_1 \ln(y)}{2} + a_4. \end{aligned} \tag{50}$$

6. About Nonlinear Self-Adjointness

In this section, we present the main definitions in N. Ibragimov's approach to nonlinear self−adjointness of differential equations adopted to our specific case. For further details the interested reader is directed to [25, 26, 27].

Consider second order differential equation

$$\mathfrak{F}\left(x, y, y_{(1)}, y_{(2)}, \cdots, y_{(s)}\right) = 0, \tag{51}$$

with independent variables x and a dependent variable y, where $y_{(1)}, y_{(2)}, \cdots y_{(s)}$ denote the collection of $1, 2, \cdots, s-$th order derivatives of y.

Definition 6.1. Let $\mathfrak{F}$ be a differential function and $\nu = \nu(x)$-the new dependent variable, known as the adjoint variable or nonlocal variable [27]. The formal Lagrangian for $\mathfrak{F} = 0$ is the differential function defined by

$$\mathfrak{L} := \nu \mathfrak{F}. \tag{52}$$

Definition 6.2. Let $\mathfrak{F}$ be a differential function and for the differential equation (51), denoted by $\mathfrak{F}[y] = 0$, we define the adjoint differential function to $\mathfrak{F}$ by

$$\mathfrak{F}^* := \frac{\delta \mathfrak{L}}{\delta y} \tag{53}$$

and the adjoint differential equation by

$$\mathfrak{F}^*[y, \nu] = 0, \tag{54}$$

where the Euler operator

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \sum_{m=1}^{\infty} (-1)^m D_{x_i} \cdots D_{x_i, m} \frac{\partial}{\partial y_{x_{i_1} x_{i_2} \cdots x_{i_m}}} \tag{55}$$

and D_{x_i} is the total derivative operator with respect to x_i defined by

$$D_{x_i} = \partial_{x_i} + y_{x_i} \partial_y + y_{x_i x_j} \partial_{y_{x_j}} + \cdots + y_{x_i x_{i_1} x_{i_2} \cdots x_{i_n}} \partial_{y_{x_{i_1}} x_{i_2} \cdots x_{i_n}} \cdots$$

Definition 6.3. The differential equation (51) is said to be nonlinearly selfadjoint if there exists a substitution

$$\nu = \phi(x, y) \neq 0 \tag{56}$$

such that

$$\tilde{\mathfrak{F}}^*\Big|_{\nu=\phi(x,y)} = \lambda\mathfrak{F} \tag{57}$$

for some undetermined coefficient $\lambda = \lambda(x, y, \cdots)$. If $\nu = \phi(y)$ in (56) and (57), the equation (51) is called quasi self-adjoint. If $\nu = y$, we say that the equation (51) is strictly self-adjoint.

Now we shall obtain the adjoint equation to the eq. (1). For this purpose we write (1) in the form (51), where

$$\mathfrak{F} := y_{xx} - 3y_x^2y^{-1} - x^{-1}y_x = 0. \tag{58}$$

Then the corresponding formal Lagrangian (52) is given by

$$\mathfrak{L} := \nu\left(y_{xx} - \frac{3y_x^2}{y} - \frac{y_x}{x}\right) = 0, \tag{59}$$

and the Euler operator (55) transfomed into:

$$\frac{\delta\mathfrak{L}}{\delta y} = \frac{\partial\mathfrak{L}}{\partial y} - D_x\frac{\partial\mathfrak{L}}{\partial y_x} + D_x^2\frac{\partial\mathfrak{L}}{\partial y_{xx}}. \tag{60}$$

Now, the explicit form of the operator (60), which was applied to $\mathfrak{L}$, have the form (59). Thus, we get the adjoint equation (54) for (1):

$$\mathfrak{F}^* = \nu\left(-3\frac{y_x^2}{y^2} - \frac{1}{x^2} + 6\frac{y_{xx}}{y}\right) + \nu_x\left(6\frac{y_x}{y} + \frac{1}{x}\right) + \nu_{xx} = 0. \tag{61}$$

The following proposition presents the most important result of the current section.

Proposition 6.4. *The following substitution $\phi(x, y)$ makes equation (1) nonlinearly self-adjoint*

$$\phi(x, y) = \frac{1}{y^3}\left(k_1x^{-1} + k_2x\right), \tag{62}$$

where k_1, k_2 are arbitrary constants.

Proof. Substituting in (61), and then in (58), $\nu = \phi(x, y)$ and its respective derivatives, and comparing the corresponding coefficients we get five equations:

$$-\phi_y = \lambda, \tag{63a}$$

$$6y^{-1}\phi + 2\phi_y = 0, \tag{63b}$$

$$6y^{-1}\phi_x + 2\phi_{xy} = 0, \tag{63c}$$

$$-\phi + x\phi_x + x^2\phi_{xx} = 0, \tag{63d}$$

$$-3y^{-1}\phi + 3\phi_y + y\phi_{yy} = 0. \tag{63e}$$

Now, we studies only Eqs. (63b),(63d) and (63e) because the Eq.(63c) is obtaned from Eq.(63b) by differentiation with respect to x.

When solving the system (63a)-(63e) for ϕ we obtain

$$\phi(x, y) = \frac{k_1x^{-1} + k_2x}{y^3}, \tag{64}$$

Then the proposition is proved. ☑

7. About conservation laws

In this section, using the conservation theorem of N. Ibragimov in [27], we will establish some conservation laws for (1). Since the Eq. (1) is of second order, the formal Lagrangian contains derivatives up to order two. Thus, the general formula in [27] for the component of the conserved vector is reduced to

$$C^x = W^j \left[\frac{\partial \mathfrak{L}}{\partial_{y_x}} - D_x \left(\frac{\partial \mathfrak{L}}{\partial_{y_{xx}}} \right) \right] + D_x[W^j] \left[\frac{\partial \mathfrak{L}}{\partial_{y_{xx}}} \right], \tag{65}$$

where

$$W^j = \eta^j - \xi^j y_x$$

$j = 1, \cdots, 5$ the formal Lagrangian (59)

$$\mathfrak{L} := \nu \left(y_{xx} - \frac{3y_x^2}{y} - \frac{y_x}{x} \right)$$

and η^j, ξ^j are the infinitesimals of a Lie group symmetry admitted by Eq. (1), stated in (5). From (1), (5) and (62) into (65) we get the following conservation vectors for each symmetry indicated in (5).

$$\begin{gathered} C_1^x = \nu \left(3xy^{-1}y_x^2 - y_x \right) - \nu_x(xy_x), \\ C_2^x = \nu \left(3x^3y^{-1}y_x^2 + x^2y_x - xy \right) + \nu_x(x^2y + x^3y_x), \\ C_3^x = \nu(-5y_x) - \nu_x(y), \\ C_4^x = \nu(9y^2y_x^2 + y_x(x^{-1}y^3)) + \nu_x(y^3y_x), \\ C_5^x = \nu(y_x(-3x^2y^2) + xy^3) - \nu_x(x^2y^3), \end{gathered} \tag{66}$$

where $\nu = y^{-3} \left(k_1x^{-1} + k_2x \right)$ and $\nu_x = y^{-3} \left(k_1x^{-1} + k_2x \right)$.

8. Classification of Lie algebra

Using Levi's theorem, it is possible to classify a finite-dimensional Lie algebra with characteristic 0. Also, this theorem makes it possible to deduce the existence of two important classes of Lie algebras: The solvable algebra and the semisimple algebra. As it is known, for the mentioned algebras, there are certain particularities, for example, within the soluble ones there are the Nilpotent Lie algebras.

If we look at the group of generators present in the Table 1, we get a five dimensional Lie algebra. A basic and classical way to classify a Lie algebra is by means of the Cartan-Killing form, which is denote as $K(.,.)$, this form is expressed in accordance with the following propositions [28].

Proposition 8.1. *(Cartan's theorem) A Lie algebra is semisimple if and only if its Killing form is nondegenerate.*

Proposition 8.2. *A Lie subalgebra $\mathfrak{g}$ is solvable if and only if $K(X,Y) = 0$ for all $X \in [\mathfrak{g},\mathfrak{g}]$ and $Y \in \mathfrak{g}$. Other way to write that is $K(\mathfrak{g},[\mathfrak{g},\mathfrak{g}]) = 0$.*

We also need the next statements to make the classification.

Definition 8.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k. Choose a basis e_j, $1 \leq i \leq n$, in $\mathfrak{g}$ where $n = \dim \mathfrak{g}$ and set $[e_i, e_j] = C_{ij}^k e_k$. Then the coefficients C_{ij}^k are called structure constants.

Proposition 8.4. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras of dimension n. Suppose each has a basis with respect to which the structure constant are the same. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

Let be $\mathfrak{g}$ the Lie algebra related to the group of infinitesimal generators of the equation (1). Analyzing the table of the commutators (Table 1), it is enough to consider the next relations:

$[\Gamma_1, \Gamma_2] = 2\Gamma_2, \quad [\Gamma_1, \Gamma_5] = 2\Gamma_5, \quad [\Gamma_2, \Gamma_4] = -2\Gamma_5, \quad [\Gamma_3, \Gamma_4] = 2\Gamma_4, \quad [\Gamma_3, \Gamma_5] = -2\Gamma_5.$
The matrix form of the Cartan-Killing K representation is:

$$K = \begin{bmatrix} 8 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which the determinant vanishes, and thus by Cartan criterion it is not semisimple, (see Proposition 8.1). Since a nilpotent Lie algebra has a Cartan-Killing form that is identically zero, we conclude, using the counter-reciprocal of the last claim, that the Lie algebra $\mathfrak{g}$ is not nilpotent. We verify that the Lie algebra is solvable using the Cartan criteria to solvability, (Proposition 8.2), and then we have a solvable nonnilpotent Lie algebra. The Nilradical of the Lie algebra $\mathfrak{g}$ is generated by $\Gamma_2, \Gamma_4, \Gamma_5$, that is, we have a Solvable Lie algebra with three dimensional Nilradical. Let m the dimension of the Nilradical M of a Solvable Lie algebra. In this case, in fifth dimensional Lie algebra we have that $3 \leq m \leq 5$. Mubarakzyanov in [29] classified the 5-dimensional solvable nonlilpotent Lie algebras, in particular the solvable nonnilpotent Lie algebra with three dimensional Nilradical, this Nilradical is isomorphic to $\mathfrak{h}_3$ the Heisenberg Lie algebra. Then by the Proposition 8.4, and consequently we establish a isomorphism of Lie algebras with $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}_{5,34}$. In summery we have the next proposition.

Proposition 8.5. *The 5-dimensional Lie algebra $\mathfrak{g}$ related to the symmetry group of the equation (1) is a solvable nonnilpotent Lie algebra with three dimensional Nilridical. Besides that Lie algebra is isomorphich with $\mathfrak{g}_{5,34}$ in the Mubarakzyanov's classification.*

9. Conclusion

Utilizing the Lie symmetry group (see Proposition 2.1), we obtained the optimal algebra (see Proposition 3.1), applying these operators it was possible to determine all the invariant solutions (see Table 3), with the exception of those presented in numerals 9 and 10, the rest of these solutions have not been presented so far in the literature.

We have presented the variational symmetries for (1) as described in (49), along with their respective conservation laws in (50). Additionally, by leveraging nonlinear self-adjointness (refer to Proposition 6.4), we have constructed non-trivial conservation laws using Ibragimov's method (66). The Lie algebra associated with the equation (1) is a solvable non-nilpotent Lie algebra with a three-dimensional nilradical. Furthermore, this Lie algebra is isomorphic to $\mathfrak{g}_{5,34}$ in Mubarakzyanov's classification.

The objectives initially proposed for the Lie algebra classification of (1) have been accomplished.

For future research, it would be valuable to explore the theory of equivalence groups, as it can facilitate preliminary classifications to structure a comprehensive classification of equation (1).

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