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Boundedness of the Hilbert Transform on Rearrangement Invariant Spaces

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Abstract. In this self-contained review, the aspects about the applications of decreasing rearrangement techniques to the analysis of pointwise estimates for the Hilbert transform are analyzed. We make a consistent revision of these techniques in the proof of the L_p-boundedness of the Hilbert transform. This is a celebrated theorem due to M. Riesz.

Keywords: Hilbert transform, distribution function, decreasing rearregement, Hardy's inequality.

MSC2020: 44A15, 47B38, 28-02.

Acotación de la transformada de Hilbert en espacios invariantes por reordenamiento

Resumen. En esta revisión autocontenida, se analizan los aspectos sobre las aplicaciones de técnicas de reordenamiento decreciente para el análisis de estimaciones punto a punto para la transformada de Hilbert. Realizamos una revisión consistente de estas técnicas en la demostración de la acotación en L_p de la transformada de Hilbert. Este es un teorema célebre debido a M. Riesz.

Palabras clave: Transformada de Hilbert, función distribución, reordenamiento decreciente, desigualdad de Hardy.

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1. The Hilbert transform

The Hilbert transform of a sufficiently well-behaved function $f(x)$ is defined to be

$$
\begin{aligned}
Hf(x) =& \text{p.v}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,dy\\
=& \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int\limits_{|x-y|>\epsilon}\frac{f(y)}{x-y}\,dy.
\end{aligned}
\tag{1}
$$

The central idea behind the definition of transform is quite simple, namely to transform $f(x)$ by convolving it with the kernel $\frac{1}{\pi x}$. It is doing so rigorously that one finds technical difficulties, the kernel fails to be absolutely integrable due to its slow decay and more importantly, due the singularity at the origin. The limiting argument in (1) is used to avoid the singularity by truncating the kernel around the origin in a systematic fashion. The aim of this review is to address the L_p-boundedness of H. We provide this analysis in Section 4. Now, we are going to provide some references about further aspects of the theory of singular integrals.

The theory of singular integrals has its roots in the works of Calderón and Zygmund [3] and Mihlin [15]. The Hilbert transform is a fundamental example in the theory of singular integrals. Two celebrated results about the boundedness properties of H can be remarked: the L_p-boundedness result of H due to M. Riesz, and the weak-$(1,1)$ boundedness of H, due to Kolmogorov. For the general aspects of the theory of singular integrals as well as the modern developments about the theory of pseudo-differential operators, which are important generalizations of the singular integrals, we refer the reader to [7, 8, 11, 12, 17]. For a substantial treatment about the subject we refer the reader to [14], and for a concise review on the subject we refer to [9].

This paper is organized as follows. In Section 2 we present some preliminaries dedicated to the rearrangements of functions. In Section 3, we present the definition of Calderón-Zygmund operators and we motivate the Hilbert transform as a fundamental example of this definition. Finally, in Section 4, we make a review of the L_p-boundedness of the Hilbert transform.

2. Preliminaries

Let us begin by presenting some definitions and properties regarding to the decreasing rearrangement. As usual $(\mathbb{R},\mathcal{L},m)$ stand for the one-dimensional Euclidean space endowed with the Lebesgue measure and $\mathcal{F}(\mathbb{R},\mathcal{L})$ denote the set of all $\mathcal{L}$-measurable functions on R.

Definition 2.1. The distribution function D_f of a function $f\in\mathcal{F}(\mathbb{R},\mathcal{L})$ is given by

$$D_f(\lambda)=m(\{x\in R:|f(x)|>\lambda\}),\tag{2}$$

for all $\lambda\geq 0$.

Observe that the distribution function D_f depends only on the absolute value of the function f and its global behavior. Moreover, notice that D_f may even assume the value $+\infty$.

It should be pointed out that the notation for the distribution function in (2) is not standard, other authors use the notations f_*, μ_f, d_f, λ_f, among others.

The distribution function D_f enjoy the following properties.

Theorem 2.2. *Let f and g be two functions in $\mathcal{F}(\mathbb{R}, \mathcal{L})$. Then for all $\lambda, \lambda_1, \lambda_2, \lambda_3 \geq 0$ we have:*

a) D_f is decreasing and continuous from the right;

b) $|g| \leq |f|$ m-a.e implies that $D_g(\lambda) \leq D_f(\lambda)$;

c) $D_{cf}(\lambda_2) = D_f\left(\frac{\lambda_1}{|c|}\right)$ for all $c \in C \setminus \{0\}$;

d) $D_{f+g}(\lambda_1 + \lambda_2) \leq D_f(\lambda_1) + D(\lambda_2)$;

e) $D_{fg}(\lambda_1 \lambda_2) \leq D_f(\lambda_1) + D(\lambda_2)$;

f) If $|f| \leq \liminf_{n\to\infty} |f_n|$ m-a.e, then

$$D_f(\lambda) \leq \liminf_{n\to\infty} D_{f_n}(\lambda);$$

g) If $|f_n| \uparrow |f|$, then $\lim_{n\to\infty} D_{f_n}(\lambda) = D_f(\lambda)$.

For the proof of all this properties see [6, 5].

With the notation of the distribution function we are ready to introduce the decreasing rearrangement function and its important properties.

Definition 2.3. Let $f \in \mathcal{F}(\mathbb{R}, \mathcal{L})$. The decreasing rearrangement of f is the function

$$f^* : [0, \infty) \longrightarrow [0, \infty],$$

defined by

$$f^*(t) = \inf\{\lambda \geq 0 : D_f(\lambda) \leq t\},$$

taking the usual convention that $\inf(\emptyset) = \infty$.

The next theorem establishes some basic properties of the decreasing rearrangement function.

Theorem 2.4. *The decreasing rearrangement function has the following properties:*

(a) f^* is decreasing;

(b) $f^*(t) > \lambda$ if and only if $D_f(\lambda) > t$;

(c) f *and* f^* *are equimeasurable, that is* $D_f(\lambda) = D_{f^*}(\lambda)$ *for all* $\lambda \geq 0$;

(d) $(\alpha f)^*(t) = |\alpha| f^*(t)$, $\alpha \in R$;

(e) *If* $|f_n| \uparrow |f|$, *then* $f_n^* \uparrow f^*$;

(f) *If* $|f| \leq \liminf_{n\to\infty} |f_n|$, *then* $f^* \leq \liminf_{n\to\infty} f_n^*$;

(g) *For* $0 < p < \infty$, $(|f|^p)^*(t) = [f^*(t)]^*$;

(h) *If* $|f| \leq |g|$, *then* $f^*(t) \leq g^*(t)$;

(i) *If* $E \in \mathcal{L}$, *then* $(\chi_E)^*(t) = \chi_{(0,m(E))}(t)$;

(j) *If* $\in \mathcal{L}$, *then* $(f\chi_E)^*(t) \leq f^*(t)\chi_{(0,m(E))}(t)$;

(k) *Let* $E_f(\lambda) = \{x \in R : |f(x)| > \lambda\}$, *If* $f \in \mathcal{F}(\mathbb{R}, \mathcal{L})$, $\lambda > 0$ *and* $F = \chi_{E_f(\lambda)}$ *then* $F^* = \chi_{E_{f^*(\lambda)}}(t)$.

For the proof of all these properties see [6, 5]. The next result tells us that a function cannot have two different decreasing rearrangements, see [6] and also [5, Theorem 1.8].

Theorem 2.5. *There exists only one right–continuous decreasing function* f^* *equimeasurable with* f.

For a positive strictly decreasing function f, it is itself its decreasing rearrangement, as the next result shows.

Theorem 2.6. *Let* f *be a strictly decreasing and non-negative function on* $(0, \infty)$ *then* $f^*(t) = f(t)$.

Proof. Consider

$$\begin{aligned} m(\{x \in (0,\infty) : f(x) > \lambda\}) =& m(\{x \in (0,\infty) : f^{-1}(f(x)) < f^{-1}(\lambda)\}) \\ =& m(\{x \in (0,\infty) : 0 < x < f^{-1}(\lambda)\}) \\ =& f^{-1}(\lambda). \end{aligned}$$

Take $t = f^{-1}(\lambda)$ then $\lambda = f(t)$. Thus

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \leq t\} = f(t),$$

that is $f^*(t) = f(t)$, as we claim. The proof of Theorem 2.6 is complete. ☑

The following theorem is quite important since it allows us to calculate an integral in a general space via an one-dimensional integral. The formula (3) below is sometimes called the Cavalieri principle.

Theorem 2.7. *Let* $(X, \mathscr{A}, \mu)$ *be a* σ*-finite measure space and let* f *be an* $\mathscr{A}$*–measurable function. Then*

$$\int_X |f|^p \, d\mu = p \int_0^\infty \lambda^{p-1} \mu(\{x \in X : |f(x)| > \lambda\}). \tag{3}$$

For the proof of this statement we refer to [6]. The next theorem, known as Minkowski integral inequality, will be useful in proving latter results in this paper.

Theorem 2.8 (Minkowski integral inequality)**.** *Let $(X, \mathscr{A}_1, \mu)$ and $(Y, \mathscr{A}_2, \nu)$ be σ-finite measure spaces. Suppose that f is a $\mathscr{A}_1 \times \mathscr{A}_2$-measurable function and $f(\cdot, y) \in L_p(\mu)$ for all $y \in Y$. Then for $1 \le p \le \infty$ we have*

$$\left(\int\limits_X \left| \int\limits_Y f(x,y)\, d\nu \right|^p d\mu \right)^{\frac{1}{p}} \le \int\limits_Y \left(\int\limits_X |f(x,y)|^p \, d\mu \right)^{\frac{1}{p}} d\nu.$$

For the proof of this statement see [6]. A function and its decreasing rearrangement has the same L_p-norm. Indeed, we have the following L_p-identity.

Theorem 2.9. *Let $f \in L_p(\mathbb{R})$, $1 \le p < \infty$. Then*

$$\int\limits_{\mathbb{R}} |f|^p \, dm = \int_0^\infty (f^*(t))^p \, dt.$$

Proof. By Theorem 2.7 and Theorem 2.4 (c), we have that

$$\begin{aligned} \int\limits_{\mathbb{R}} |f|^p \, dm =& p \int_0^\infty \lambda^{p-1} m(\{x \in R : |f(x)| > \lambda\}) \, d\lambda \\ =& p \int_0^\infty \lambda^{p-1} m(\{t \in [0,\infty) : f^*(t) > \lambda\}) \, d\lambda \\ =& p \int_0^\infty \lambda^{p-1} \left(\int_0^\infty \chi_{(0,f^*(t))\, dt} \right) d\lambda. \end{aligned}$$

Next, by applying the Fubini theorem, we have that

$$\begin{aligned} \int_0^\infty p\lambda^{p-1} \left(\int_0^\infty \chi_{(0,f^*(t))}(\lambda)\, dt \right) d\lambda &= \int_0^\infty \lambda^{p-1} \chi_{(0,f^*(t))} \, d\lambda dt \\ &= \int_0^\infty \int_0^{f^*(t)} p\lambda^{p-1} \, d\lambda dt \\ &= \int_0^\infty (f^*(t))^p \, dt. \end{aligned}$$

The proof is complete. ☑

The following inequality is due to Hardy and Littlewood.

Theorem 2.10. *If f and g, both belong to $\mathcal{F}(\mathbb{R}, \mathcal{L})$, we have the identity*

$$\int\limits_{\mathbb{R}} |fg| \, dm \le \int_0^\infty f^*(t) g^*(t) \, dt. \tag{4}$$

Proof. Assume first that $f = \chi_A$ and $g = \chi_B$ are characteristic functions where A and B are sets in $\mathcal{L}$. We suppose without loss of generality that $m(A)$ and $m(B)$ are finite. Then it follows from Theorem 2.4(i) that

$$\begin{aligned}\int_{\mathbb{R}} |fg|\, dm &= \int_{\mathbb{R}} \chi_{A\cap B}\, dm \\ &= m(A\cap B) \\ &\leq \int_0^{\min(m(A),m(B))} dt \\ &= \int_0^{m(A)} \chi_{(0,m(B))}(t)\, dt \\ &= \int_0^{m(A)} g^*(t)\, dt \\ &= \int_0^{\infty} \chi_{(0,m(A))}(t) g^*(t)\, dt \\ &= \int_0^{\infty} f^*(t)g^*(t)\, dt.\end{aligned}$$

In general let f and g be two functions belonging to $\mathcal{F}(\mathbb{R},\mathcal{L})$. Then

$$\begin{aligned}\int_{\mathbb{R}} |fg|\, dm &= \int_{\mathbb{R}} \left(\int_0^{|f|} d\alpha\right)\left(\int_0^{|g|} d\beta\right) dm \\ &= \int_{\mathbb{R}} \left(\int_0^{|f|} \chi_{E_f(\alpha)} d\alpha\right)\left(\int_0^{|g|} \chi_{E_f(\beta)} d\beta\right) dm.\end{aligned}$$

It follows from Fubini's theorem and Theorem 2.4 (h) and the property (k) of this theorem that

$$\begin{aligned}\int_{\mathbb{R}} |fg|\, dm &= \int_0^{\infty}\int_0^{\infty}\int_{\mathbb{R}} \chi_{E_f(\alpha)}(\alpha)\chi_{E_f(\beta)}(\beta)\, dm d\alpha d\beta \\ &\leq \int_0^{\infty}\int_0^{\infty}\left\{\int_0^{\infty} \left(\chi_{E_f(\alpha)}\right)^*(t)\left(\chi_{E_f(\beta)}\right)^*(t)\, dt\right\} d\alpha d\beta \\ &= \int_0^{\infty}\int_0^{\infty}\left\{\int_0^{\infty} \chi_{E_{f^*}(\alpha)}(t)\chi_{E_{f^*}(\beta)}(t)\, dt\right\} d\alpha d\beta \\ &= \int_0^{\infty}\left(\int_0^{\infty} \chi_{E_{f^*}(\alpha)}(t)\, d\alpha\right)\left(\int_0^{\infty} \chi_{E_{f^*}(\beta)}(t)\, d\beta\right) dt \\ &= \int_0^{\infty}\left(\int_0^{f^*} d\alpha\right)\left(\int_0^{g^*} d\beta\right) dt = \int_0^{\infty} f^*(t)g^*(t)\, dt.\end{aligned}$$

The proof is complete. ☑

Theorem 2.11. *Let* $f \in L_1(\mathbb{R})$. *Then*

$$\sup\left\{\int\limits_E |f|\,dm : m(E) = t\right\} = \int_0^t f^*(s)\,ds.$$

Proof. Given the real number $t > 0$, we have that

$$\begin{aligned}\int\limits_E |f|\,dm &= \int_0^\infty m(E \cap E_f(\lambda))\,d\lambda \\ &= \int\limits_{\{\lambda : D_f(\lambda) \le t\}} m(E \cap E_f(\lambda))\,d\lambda + \int\limits_{\{\lambda : D_f(\lambda) > t\}} m(E \cap E_f(\lambda))\,d\lambda.\end{aligned}$$

The following is a well known fact from the measure theory: if $m(E_f(\lambda)) > t > 0$, then there exists a set $E \in \mathcal{L}$ such that $E \subseteq E_f(\lambda)$ and $m(E) = t$. Hence

$$\begin{aligned}\sup\left\{\int\limits_E |f|\,dm : m(E) = t\right\} &= \int_{f^*(t)}^\infty m(E_f(\lambda))\,d\lambda + \int_0^{f^*(t)} t\,d\lambda \\ &= \int_{f^*(t)}^\infty D_f(\lambda)\,d\lambda + \int_0^{f^*(t)} t\,d\lambda \\ &= \int_{f^*(t)}^\infty D_{f^*}(\lambda)\,d\lambda + tf^*(t) \\ &= \int_0^t \int_{f^*(t)}^{f^*(s)} d\lambda ds + tf^*(t) \\ &= \int_0^t f^*(s)\,ds.\end{aligned}$$

The proof is complete. ☑

3. Calderon–Zygmund singular integral operators

Taking into account the Calderón–Zygmund theory we are going to have the Hilbert transform as a fundamental example. We recall the following definition.

Definition 3.1. Suppose that $K(x) \in L_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and satisfies the following conditions:

(a) $|K(x)| \le B|x|^{-n}$ for all $x \ne 0$.

(b) $\displaystyle\int\limits_{r \le |x| \le R} |K(x)| = 0$ for all $0 < r < R < \infty$.

(c) $\displaystyle\int\limits_{|x| \ge 2y} |K(x - y) - K(x)| \le B$ for $y \ne 0$.

A kernel as in Definition 3.1 is called the Calderón–Zygmund Kernel where B is a constant independent of x and y. The condition (c) is called Hörmander condition..

Now, we present the following fundamental theorem.

Theorem 3.2. *Suppose that K is the Calderón–Zygmund kernel. For $\epsilon > 0$ and $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, let*

$$T_\epsilon f(x) = \int\limits_{|y|\geq\epsilon} f(x-y)K(y)\,dy.$$

Then the following statements holds:

(1) $\|T_\epsilon f\|_{L_p} \leq A_p\|f\|_{L_p}$ where A_p is independent of ϵ and f.

(2) For any $f \in L_p(\mathbb{R}^n)$, $\lim\limits_{\epsilon\to 0} T_\epsilon f$ exists in the sense of the L_p–norm. That is, there exists an operator T such that

$$Tf(x) = p.v\int\limits_{\mathbb{R}^n} f(x-y)K(y)\,dy,$$

holds for almost every $x \in \mathbb{R}^n$.

In addition, one can show that $K(x) = \dfrac{1}{\pi x}$, that is, the kernel of the Hilbert transform satisfies the hypothesis in Theorem 3.2. We prove that it satisfies the properties in Definition 3.1 below. Note that

(a) $|K(x)| = \left|\dfrac{1}{\pi x}\right| \leq \dfrac{B}{|x|}$ where $B = \dfrac{2}{\pi}$.

(b)

$$\begin{aligned}\left|\int\limits_{r\leq|x|\leq R} K(x)\,dx\right| &= \left|\int\limits_{r\leq|x|\leq R} \frac{dx}{\pi x}\right| \\ &\leq \frac{1}{\pi}\int\limits_{r\leq|x|\leq R} \frac{dx}{|x|} \\ &\leq \frac{1}{\pi}\int\limits_{|x|\leq R} \frac{dx}{|x|} \\ &\leq \frac{1}{\pi}\int_{-R}^{R} \frac{dx}{x} = 0.\end{aligned}$$

Since $\dfrac{1}{x}$ is an odd function, we have

$$\int\limits_{r\leq|x|\leq R} K(x) = 0.$$

(c) Note that

$$|K(x-y)-K(x)| = \frac{1}{\pi}\left|\frac{1}{x-y}-\frac{1}{x}\right| = \frac{1}{\pi}\left|\frac{y}{x(x-y)}\right|.$$

Now, if $|x| \geq 2|y|$ then $|x-y| \geq \dfrac{|x|}{2}$. Indeed

$$\begin{aligned} &|x|-|y| \leq |x-y| \\ \Longrightarrow &|x| - \frac{|x|}{2} \leq |y-x| \\ \Longrightarrow &|x-y| \geq \frac{|x|}{2}. \end{aligned}$$

Consequently,

$$\int\limits_{|x|\geq 2|y|} |K(x-y)-K(x)|\,dx \leq \frac{2|y|}{\pi} \int\limits_{|x|\geq 2|y|} \frac{dx}{|x|^2} \leq \frac{2}{\pi}.$$

Hence $K(x) = \frac{1}{\pi x}$ is a Calderon–Zygmund kernel. Now, let us define

$$\begin{aligned} H_\epsilon f(x) = &\int\limits_{|y|\geq\epsilon} f(x-y)K(y)\,dy \\ = &\frac{1}{\pi}\int\limits_{|y|\geq\epsilon} \frac{f(x-y)}{y}\,dy. \end{aligned}$$

By Theorem 3.2, we have that

$$Hf(x) = \text{p.v}\frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{f(x-y)}{y}\,dy.$$

So we might say roughly that Hilbert's transformation of a function f is the convolution of f with the Calderón–Zygmund kernel $K(x) = \dfrac{1}{\pi x}$.

4. A theorem due to E. M. Stein and G. Weiss

This section is based on [1, 4]. The following lemmas will be helpful for the proof of Theorem 4.3. We present these lemmas and their proofs below.

Lemma 4.1. *Let $P(x) = x^n + a_n x^{n+1} + \cdots + a_2 x + a_1$ be polynomial of degree n. Let $r_1, r_2, \cdots, r_n$ be the roots of $P(x) = 0$, then*

$$\sum_{k=1}^{n} r_k = -a_n.$$

Proof. The proof uses the mathematical induction. When $n = 2$, the polynomial $x^2 + a_2 x + a_1 = 0$ has two roots, r_1 and r_2, such that

$$(x-r_1)(x-r_2) = 0,$$

then

$$x^2 - r_1x - r_2x + r_1r_2 = 0.$$

In consequence

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0.$$

Hence $r_1 + r_2 = -a_2$. Now, for $n = 3$, the polynomial $x^3 + a_3x^2 + a_2x + a_1 = 0$ has three roots named r_1, r_2 and r_3 such that

$$\begin{aligned}(x - r_1)(x - r_2)(x - r_3) &= 0,\\ (x^2 - (r_1 + r_2)x + r_1r_2)(x - r_3) &= 0,\\ x^3 - (r_+r_2)x^2 + (r_1r_2)x - r_3x^2 + (r_1 + r_2)r_3x - r_1r_2r_3 &= 0,\\ x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 &= 0.\end{aligned}$$

From this last equality we have that

$$r_1 + r_2 + r_3 = -a_3.$$

Next, suppose that for

$$x^n + a_nx^{n+1} + \cdots + a_2x + a_1 = 0,$$

the property

$$\sum_{k=1}^{n} r_k = -a_n \qquad \text{holds.}$$

Now, the polynomial $x^{n+1} + a_{n+1}x^n + \cdots + a_2x + a_1 = 0$, has $n+1$ roots $r_1, r_2, \cdots, r_{n+1}$ such that

$$\begin{aligned}(x - r_1)(x - r_2)\cdots(x - r_n)(x - r_{n+1}) &= 0,\\ \left[x^n - \left(\sum_{k=1}^{n} r_k\right)x^{n-1} + \cdots\right](x - r_{n+1}) &= 0,\\ \left[x^{n+1} - \left(\sum_{k=1}^{n} r_k\right)x^n + \cdots\right] + \left[-r_{n+1}x^n + \left(\sum_{k=1}^{n} r_k\right)r_{n+1}x^{n-1} + \cdots\right] &= 0,\\ x^{n+1} - \left(\sum_{k=1}^{n+1} r_k\right)x^n + \cdots &= 0.\end{aligned}$$

Therefore

$$\sum_{k=1}^{n+1} r_k = -a_{n+1}.$$

The proof of Lemma 4.1 is complete. ☑

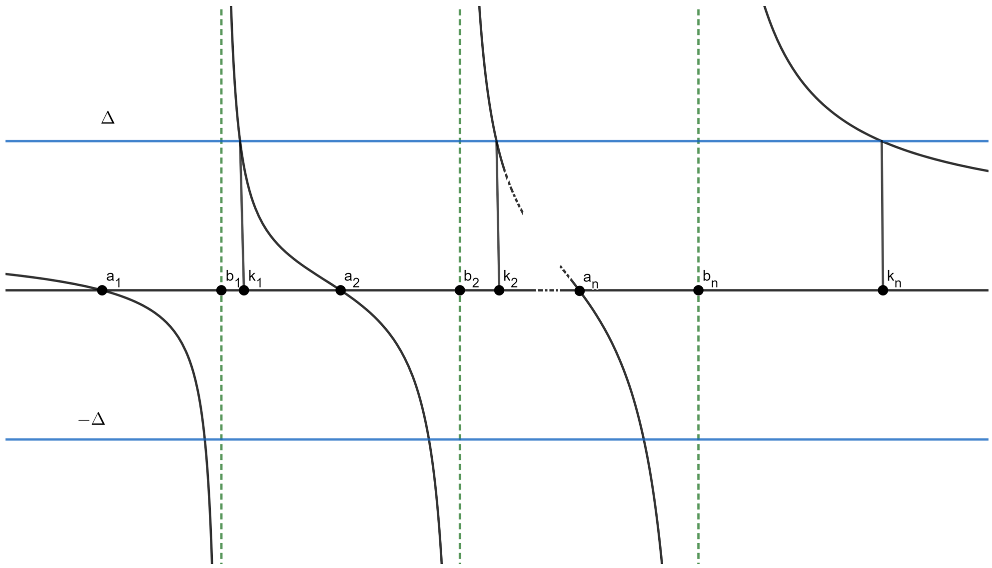

Figure 1. $\prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = g(x)$

Lemma 4.2. *Suppose that a_i, b_i $(i = 1, 2, 3, \cdots, n)$ are real numbers satisfying that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ and let g be the rational function*

$$g(x) = \prod_{k=1}^{n} \frac{x-a_k}{x-b_k} \qquad (x \in \mathbb{R}). \tag{5}$$

If $\Delta \neq 1$, then the equation $g(x) = |\Delta|$ has n different roots $r_1, r_2, \cdots, r_n$ which satisfy that

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (1-\Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k). \tag{6}$$

Furthermore, if $\Delta > 1$, then

$$(\Delta - 1) m(\{g > \Delta\}) = (\Delta + 1) m(\{g < -\Delta\}) = \sum_{k=1}^{n} (b_k - a_k). \tag{7}$$

Proof. Since g has a simple pole at each b_k, $(k = 1, 2, 3, \cdots, n)$ and

$$\lim_{|x|\to\infty} g(x) = \lim_{|x|\to\infty} \prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = \prod_{k=1}^{n} \lim_{|x|\to\infty} \frac{x-a_k}{x-b_k} = 1, \tag{8}$$

there are exactly n different solutions, say $r_1, r_2, \cdots, r_n$ to the equation $g(x) = |\Delta|$ $(\Delta \neq 1)$. Then

$$\prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = \Delta \qquad \text{and} \qquad \prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = -\Delta.$$

For $\prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = \Delta$, we have that

$$\prod_{k=1}^{n}(x-a_k) = \Delta \prod_{k=1}^{n}(x-b_k),$$

and so

$$\prod_{k=1}^{n}(x-a_k) - \Delta \prod_{k=1}^{n}(x-b_k) = 0,$$

where

$$P(x) = \sum_{k=0}^{n} p_k x^k = \prod_{k=1}^{n}(x-a_k) - \Delta \prod_{k=1}^{n}(x-b_k) = 0. \tag{9}$$

Then

$$\prod_{k=1}^{n}(x-a_k) - \Delta \prod_{k=1}^{n}(x-b_k) = 0,$$

which implies that

$$\begin{aligned} \left[x^n - \left(\sum_{k=1}^{n} a_k\right) x^{n+1} + \cdots\right] - \Delta \left[x^n - \left(\sum_{k=1}^{n} b_k\right) x^{n+1} + \cdots\right] &= 0 \\ (1-\Delta)x^n + \left(-\sum_{k=1}^{n} a_k + \Delta \sum_{k=1}^{n} b_k\right) x^{n-1} + \cdots &= 0 \\ x^n + \frac{-\sum_{k=1}^{n} a_k + \Delta \sum_{k=1}^{n} b_k}{(1-\Delta)} x^{n-1} + \cdots &= 0. \end{aligned}$$

Since $r_1, r_2, \cdots, r_n$ are the roots of the polynomial $P(x) = 0$, then by Lemma 4.1 we have that

$$\sum_{k=1}^{n} r_k = -\frac{-\sum_{k=1}^{n} a_k + \Delta \sum_{k=1}^{n} b_k}{(1-\Delta)}.$$

In consequence,

$$\begin{aligned} \sum_{k=1}^{n} r_k &= (1-\Delta)^{-1} \sum_{k=1}^{n} a_k - \frac{\Delta}{(1-\Delta)} \sum_{k=1}^{n} b_k \\ &= (1-\Delta)^{-1} \sum_{k=1}^{n} a_k + \left(\frac{1-\Delta-1}{(1-\Delta)}\right) \sum_{k=1}^{n} b_k \\ &= (1-\Delta)^{-1} \sum_{k=1}^{n} a_k + \left(1 - \frac{1}{(1-\Delta)}\right) \sum_{k=1}^{n} b_k \\ &= (1-\Delta)^{-1} \sum_{k=1}^{n} a_k + \left(1 - (1-\Delta)^{-1}\right) \sum_{k=1}^{n} b_k \\ &= (1-\Delta)^{-1} \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k - (1-\Delta)^{-1} \sum_{k=1}^{n} b_k. \end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{n} b_k &= \sum_{k=1}^{n} r_k + (1-\Delta)^{-1}\sum_{k=1}^{n} b_k - (1-\Delta)^{-1}\sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{n} r_k + (1-\Delta)^{-1}\sum_{k=1}^{n}(b_k - a_k).\end{aligned}$$

If $\Delta > 1$, then $\{g > \Delta\} = \bigcup\limits_{k=1}^{n}(b_k, r_k)$, (see figure 1) and so

$$\begin{aligned}m(\{g > \Delta\}) &= m\left(\bigcup_{k=1}^{n}(b_k, r_k)\right) \\ &= \sum_{k=1}^{n}(r_k - b_k).\end{aligned}$$

Since

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (1-\Delta)^{-1}\sum_{k=1}^{n}(b_k - a_k),$$

we have that

$$\sum_{k=1}^{n}(b_k - r_k) = (1-\Delta)^{-1}\sum_{k=1}^{n}(b_k - a_k),$$

and putting these equations together we have that

$$\begin{aligned}-\sum_{k=1}^{n}(r_k - b_k) &= (1-\Delta)^{-1}\sum_{k=1}^{n}(b_k - a_k) \\ -m(\{g > \Delta\}) &= (1-\Delta)^{-1}\sum_{k=1}^{n}(b_k - a_k) \\ (\Delta - 1)m(\{g > \Delta\}) &= \sum_{k=1}^{n}(b_k - a_k).\end{aligned}$$

Moreover, if $-\Delta < -1$, then $\{g < -\Delta\} = \bigcup_{k=1}^{n}(r_k, b_k)$, (see figure 1). Hence

$$\begin{aligned}m(\{g < -\Delta\}) &= m\left(\bigcup_{k=1}^{n}(r_k, b_k)\right) \\ &= \sum_{k=1}^{n}(b_k - r_k).\end{aligned}$$

Now, we have

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (\Delta + 1)^{-1}\sum_{k=1}^{n}(b_k - a_k).$$

Also,

$$\sum_{k=1}^{n}(b_k - r_k) = (\Delta+1)^{-1}\sum_{k=1}^{n}(b_k - a_k).$$

In consequence

$$m(\{g < -\Delta\}) = (\Delta+1)^{-1}\sum_{k=1}^{n}(b_k - a_k).$$

Finally,

$$(\Delta+1)m(\{g < -\Delta\}) = \sum_{k=1}^{n}(b_k - a_k).$$

In view of the analysis above, we conclude that

$$(\Delta-1)m(\{g > \Delta\}) = (\Delta+1)m(\{g < -\Delta\}) = \sum_{k=1}^{n}(b_k - a_k). \tag{10}$$

The proof of Lemma 4.2 is complete. ☑

In the following result we can observe that the distribution function of $H\chi_E$ depends only on the measure of E and not on the way in which E happens to be distributed over the real line.

Theorem 4.3 (Stein-Weiss [19]). *Let E be the union of finitely many disjoint intervals, each of finite length. Then*

$$D_{H_{\chi_E}}(\lambda) = \frac{2m(E)}{\sinh(\pi\lambda)}; \qquad \lambda > 0. \tag{11}$$

Where $D_{H_{\chi_E}}(\lambda) = m(\{|H\chi_E| > \lambda\})$.

Proof. We may express the set E in the form

$$E = \bigcup_{j=1}^{n}(a_j, b_j), \tag{12}$$

where $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. We already know that

$$\begin{aligned} H_{\chi_E}(x) &= \frac{1}{\pi}\left[\sum_{i=1}^{n}\int_{a_i}^{b_i}\frac{dy}{x-y}\right] \\ &= \frac{1}{\pi}\left[\sum_{i=1}^{n}\log\left|\frac{x-a_i}{x-b_i}\right|\right] \\ &= \frac{1}{\pi}\log\left|\prod_{i=1}^{n}\frac{x-a_i}{x-b_i}\right|. \end{aligned}$$

Fix $\lambda > 0$, and let $F = \{|H_{\chi_E}| > \lambda\}$. Then $m(F) = D_{H_{\chi_E}}(\lambda)$. Since

$$H_{\chi_E}(x) = \frac{1}{\pi}\left[\log\left|\prod_{i=1}^{n}\frac{x-a_i}{x-b_i}\right|\right],$$

we have that

$$e^{\pi H_{\chi_E}(x)} = \left|\prod_{i=1}^{n}\frac{x-a_i}{x-b_i}\right|.$$

If we set $g(x) = \prod_{i=1}^{n}\frac{x-a_i}{x-b_i}$, F can be decompose as

$$F = \{|g| > e^{\pi\lambda}\} \cup \{|g| < e^{-\pi\lambda}\} = F_1 \cup F_2. \tag{13}$$

Now, by applying Lemma 4.2 to g we obtain,

$$\begin{aligned}
m(F_1) &= m(\{|g| > e^{\pi\lambda}\})\\
&= m(\{g > e^{\pi\lambda}\}) + m(\{g < -e^{\pi\lambda}\})\\
&= \frac{\sum_{i=1}^{n}(b_i-a_i)}{e^{\pi\lambda}-1} + \frac{\sum_{i=1}^{n}(b_i-a_i)}{e^{\pi\lambda}+1}\\
&= \frac{m(E)}{e^{\pi\lambda}-1} + \frac{m(E)}{e^{\pi\lambda}+1}\\
&= \frac{2e^{\pi\lambda}m(E)}{e^{2\pi\lambda}-1}\\
&= \frac{m(E)}{\sinh(\pi\lambda)}.
\end{aligned}$$

Next, for F_2, we have that,

$$\begin{aligned}
m(F_2) &= m(\{|g| < e^{-\pi\lambda}\})\\
&= m(\{g > -e^{-\pi\lambda}\}) + m(\{g < e^{-\pi\lambda}\})\\
&= \frac{\sum_{i=1}^{n}(b_i-a_i)}{-e^{-\pi\lambda}-1} + \frac{\sum_{i=1}^{n}(b_i-a_i)}{-e^{-\pi\lambda}+1}\\
&= \frac{m(E)}{-e^{-\pi\lambda}-1} + \frac{m(E)}{-e^{-\pi\lambda}+1}\\
&= \frac{-2e^{\pi\lambda}m(E)}{e^{-2\pi\lambda}-1}\\
&= \frac{-m(E)}{\sinh(-\pi\lambda)}\\
&= \frac{m(E)}{\sinh(\pi\lambda)}.
\end{aligned}$$

Finally

$$m(\{|H_{\chi_E}| > \lambda\}) = m(F_1) + m(F_2) = \frac{2m(E)}{\sinh(\pi\lambda)} \tag{14}$$

i.e.

$$m(\{|H_{\chi_E}| > \lambda\}) = \frac{2m(E)}{\sinh(\pi\lambda)}.$$

The proof of Theorem 4.3 is complete. ☑

A short proof of the Stein-Weiss theorem using complex-variable methods can be found in Calderón [2] and Garnett [13]. Also, an additional discussion can found in Sagher and Xiang [18]. The Stein-Weiss theorem has been discussed for the ergodic Hilbert transform by Ephremidze [10]. As a corollary of Theorem 4.3, given that f^* is essentially the inverse function of $D_f(\lambda)$, we compute $(H\chi_E)^*(t)$ as follows.

Corollary 4.4. *Let $E \subseteq$ with $m(E) < \infty$. Then*

$$(H\chi_E)^*(t) = \frac{1}{\pi}\sinh^{-1}\left(\frac{2m(E)}{t}\right).$$

The next theorem shows that the operator H is anticommutative.

Theorem 4.5. *Let $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$ $(1 < p < \infty)$. Then*

$$\int_{-\infty}^{\infty} f(x)Hg(x)\,dx = -\int_{-\infty}^{\infty} Hf(x)g(x)\,dx.$$

Proof. Let us start by defining

$$H_n f(x) = \int\limits_{|x-y|>\frac{1}{n}} \frac{f(y)}{x-y}\,dy.$$

Note that

$$H_n f(x) \leq H_{n+1} f(x).$$

Consequently,

$$Hf(x) = \lim_{n\to\infty} \int\limits_{|x-y|>\frac{1}{n}} \frac{f(y)}{x-y}\,dy.$$

An application of Fubini Theorem gives

$$\begin{aligned}
\int_{\mathbb{R}} H_n f(x)g(x) &= \int_{\mathbb{R}} \left(\int_{|x-y|>\frac{1}{n}} \frac{f(y)}{x-y} \right) g(x)\,dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{f(y)\chi_{\{|x-y|>\frac{1}{n}(y)\}}}{x-y}\,dy \right) g(x)\,dx \\
&= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} \frac{g(x)\chi_{\{|y-x|>\frac{1}{n}(x)\}}}{y-x}\,dx \right) dy \\
&= -\int_{\mathbb{R}} f(y) \left(\int_{|y-x|>\frac{1}{n}} \frac{g(x)}{y-x}\,dx \right) dy \\
&= -\int_{E} f(y) H_n g(y)\,dy.
\end{aligned}$$

Finally by the monotone convergence Theorem we have that

$$\begin{aligned}
\int_{\mathbb{R}} Hf(x)g(x) &= \int_{\mathbb{R}} \lim_{n\to\infty} H_n f(x)g(x)\,dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} H_n f(x)g(x)\,dx \\
&= \lim_{n\to\infty} -\int_{\mathbb{R}} f(x)H_n g(x)\,dx \\
&= -\int_{\mathbb{R}} f(x) \lim_{n\to\infty} H_n g(x)\,dx \\
&= -\int_{\mathbb{R}} f(x)Hg(x)\,dx.
\end{aligned}$$

The proof is complete. ☑

The following result, due to O'Neil, provides a bound of $(Hf)^{**}$. We provide a proof with enough details.

Theorem 4.6 (O'Neil-Weiss [16]). *If*

$$\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{2m(E)}{t}\right) dt < +\infty,$$

The function $Hf(x)$ exists almost everywhere and for each $s>0$ one has that

$$\begin{aligned}(Hf)^{**}(s) \leq & \frac{2}{\pi s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)\,dt\\ =&\frac{2}{\pi}\int_0^\infty \frac{f^{**}(t)}{\sqrt{s^2+t^2}}\,dt,\end{aligned}$$

where

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds.$$

Proof. By Theorem 2.11, the statement will be established if we can show

$$\int\limits_E |Hf(x)|\,dx \leq \frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)\,dt=\frac{2s}{\pi}\int_0^\infty \frac{f^{**}(t)}{\sqrt{s^2+t^2}}\,dt,$$

for each set E of measure s, that is $m(E)=s$.

Given such a set, let $E_1\subset E$ be the subset where $Hf(x)\geq 0$ and let $E_2=E\setminus E_1$. Using Theorem 2.10, Corollary 4.4 and Theorem 4.5 we have that

$$\begin{aligned}\int\limits_E |Hf(x)|\,dx =& \int\limits_{E_1} Hf(x)\,dx-\int\limits_{E_2} Hf(x)\,dx\\ =&\int_{-\infty}^\infty Hf(x)\chi_{E_1}(x)\,dx-\int_{-\infty}^\infty Hf(x)\chi_{E_2}(x)\,dx\\ =&-\int_{-\infty}^\infty f(x)H\chi_{E_1}(x)\,dx+\int_{-\infty}^\infty f(x)H\chi_{E_2}(x)\,dx\\ \leq&\int_{-\infty}^\infty |f(x)H\chi_{E_1}(x)|\,dx+\int_{-\infty}^\infty |f(x)H\chi_{E_2}(x)|\,dx\\ \leq&\int_0^\infty f^*(t)\left(H\chi_{E_1}\right)^*(t)\,dt+\int_0^\infty f^*(t)\left(H\chi_{E_2}\right)^*(t)|\,dt.\end{aligned}$$

By Corollary 4.4, the previous expression is equal to

$$\begin{aligned}&\frac{1}{\pi}\int_0^\infty f^*(t)\left\{\sinh^{-1}\left(\frac{2m(E_1)}{t}\right)+\sinh^{-1}\left(\frac{2m(E_2)}{t}\right)\right\}\,dt\\ \leq&\frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{m(E_1)+m(E_2)}{t}\right)\,dt\\ =&\frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{m(E)}{t}\right)\,dt=\frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)\,dt.\end{aligned}$$

So the desired inequality holds. The equality

$$\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)\,dt=\int_0^\infty \frac{f^{**}(t)}{\sqrt{s^2+t^2}}\,dt$$

follows by integration by parts. The proof of Theorem 4.6 is complete. ☑

Corollary 4.7 (O'Neil–Weiss's Inequality). *Let f be a measurable function on $(-\infty, \infty)$. Then*

$$H^{**}(t) \leq \frac{2}{\pi}\left[\frac{1}{t}\int_0^t f^{**}(s)\,ds + \int_t^\infty f^{**}(s)\frac{ds}{s}\right],$$

for each $t > 0$.

We shall need the following form of the Hardy inequality (see [6]).

Lemma 4.8. *If $p > 1$ and f is a nonnegative function defined on $(0, \infty)$. Then:*

a) $\left(\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty (f(x))^p\,dx\right)^{\frac{1}{p}}$;

b) $\left(\int_0^\infty \left(\int_x^\infty f(t)\frac{dt}{t}\right)^p\right)^{\frac{1}{p}} \leq p\left(\int_0^\infty (f(x))^p\,dx\right)^{\frac{1}{p}}$.

Now we present a theorem due to M. Riesz.

Theorem 4.9 (M. Riesz Theorem). *If $1 < p < \infty$ and $f \in L_p(\mathbb{R})$. Then there exists A_p independent of $f \in L_p(\mathbb{R})$ such that*

$$\|Hf\|_{L_p(\mathbb{R})} \leq A_p \|f\|_{L_p(\mathbb{R})}.$$

Proof. By Corollary 4.7 we have that

$$\begin{aligned}\|Hf\|_{L_p(\mathbb{R})} \leq &\left(\int_0^\infty ((Hf)^{**}(s))^p\,ds\right)^{\frac{1}{p}}\\ \leq &\left\{\int_0^\infty \left[\frac{2}{\pi}\left(\frac{1}{s}\int_0^s f^{**}(t)\,dt + \int_s^\infty f^{**}(t)\frac{dt}{t}\right)\right]^p ds\right\}^{\frac{1}{p}}\\ = &A + B.\end{aligned}$$

Now, by Lemma 4.8(a), we have that

$$A \leq \frac{2p}{\pi(p-1)}\left(\int_0^\infty (f^{**}(s))^p\,ds\right)^{\frac{1}{p}} \leq \frac{2}{\pi}\left(\frac{p}{p-1}\right)\|f\|_{L_p(\mathbb{R})}.$$

And by part (b) of Lemma 4.8 we have that

$$B \leq \frac{2}{\pi}p\left(\int_0^\infty (f^{**}(s))^p\,ds\right)^{\frac{1}{p}} \leq \frac{2p^2}{\pi(p-1)}\|f\|_{L_p(\mathbb{R})}.$$

The analysis above shows the theorem. ☑

We can also give another proof of Theorem 4.9 using Minkowski's integral inequality, as it is shown below. The proof is due to O'Neil-Weiss [16].

Proof. By making an appropriate change of variables and by employing Minkowski's integral inequality (see Theorem 2.8), we have that

$$\begin{aligned}
\|Hf\|_{L_p} \leq & \left(\int_0^\infty [(Hf)^{**}(s)]^p \, ds\right)^{\frac{1}{p}} \\
\leq & \frac{2}{\pi}\left[\int_0^\infty \left(\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt\right)^p ds\right]^{\frac{1}{p}} \\
= & \frac{2}{\pi}\left[\int_0^\infty \left(\frac{1}{s}\int_0^\infty f^*\left(\frac{s}{u}\right)\sinh^{-1}(u)\left(\frac{s}{u^2}\right)du\right)^p ds\right]^{\frac{1}{p}} \\
= & \frac{2}{\pi}\left[\int_0^\infty \left(\int_0^\infty f^*\left(\frac{s}{u}\right)\sinh^{-1}(u)\frac{du}{u^2}\right)^p ds\right]^{\frac{1}{p}} \\
\leq & \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty \left(f^*\left(\frac{s}{u}\right)\frac{\sinh^{-1}(u)}{u^2}\right)^p ds\right)^{\frac{1}{p}} du \\
= & \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^2}\left(\int_0^\infty \left[f^*\left(\frac{s}{u}\right)\right]^p ds\right)^{\frac{1}{p}} du \\
= & \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^2}u^{\frac{1}{p}}\left(\int_0^\infty [f^*(\omega)]^p \, d\omega\right)^{\frac{1}{p}} du \\
= & \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^{1+\frac{1}{q}}}\, du \left(\int_0^\infty [f^*(\omega)]^p \, d\omega\right)^{\frac{1}{p}} \\
= & \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^{1+\frac{1}{q}}}\, du \left(\int_{\mathbb{R}} |f(x)|^p \, dm\right)^{\frac{1}{p}} \\
= & A_p\|f\|_{L_p}.
\end{aligned}$$

Thus we proved Theorem 4.9 with

$$A_p = \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^{1+\frac{1}{q}}}\, du.$$

☑

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