Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 42, N° 1, 2024, pág. 23–30

The Cantor-Schröder-Bernstein Theorem in some categories of modules

Diego Peralta[a]✉, Héctor Pinedo[b]
[a,b]Universidad Industrial de Santander, Escuela de matemáticas, Bucaramanga, Colombia.

Abstract. The Cantor-Schröder-Bernstein theorem has been studied in several categories throughout mathematics. In this article, we prove that this theorem holds in some relevant categories of modules, such as noetherian, and artinian, and prove that some strong versions of it also hold the category of finitely generated modules over a principal ideal domain.

Keywords: Cantor-Schröder-Bernstein theorem, Dedekind finite, orthogonal modules, chain conditions.

MSC2020: 18D70, 13E05, 13E10.

El teorema de Cantor-Schröder-Bernstein en ciertas categorías de módulos

Resumen. El teorema de Cantor-Schröder-Bernstein se ha estudiado en varias categorías a lo largo de las matemáticas. En este artículo, demostramos que este teorema se cumple en algunas categorías relevantes de módulos, como las de noetherianos y artinianos, y demostramos que algunas versiones más fuertes de este también se aplican a la categoría de módulos finitamente generados sobre un dominio de ideales principales.

Palabras clave: teorema de Cantor-Schröder-Bernstein, Dedekind finito, módulos ortogonales, condiciones de cadena.

1. Introduction

In the category of sets, the Cantor-Schröder-Bernstein theorem (CSB Theorem for short) states that if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$. In this context, the sets A and B are *isomorphic*. Therefore, it seems natural to strengthen the result in other structures. If A and B are objects in some category such as groups, rings, modules, topological spaces, manifolds, etc, and there are monic maps from each other, then A and B are isomorphic. However, this result is not true in every category. For instance in the category of abelian groups when taking a prime number p, and the groups $G = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p^i}$ and $H = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{2p^i}$, then it is observed in Example 3.1 of [2] that there are monomorphism $f : G \to H$

E-mail: [a]dandresreyes88@gmail.com✉ [b]hpinedot@uis.edu.co .
Received: 10 October 2023, Accepted: 16 February 2024.
To cite this article: D. Peralta, H. Pinedo, The Cantor-Schröder-Bernstein Theorem in some categories of modules, *Rev. Integr. Temas Mat.,* 42 (2024), No. 1, 23-30. doi: 10.18273/revint.v42n1-2024002

and $g : H \to G$, but the groups G and H are not isomorphic. Also, in [2] the author shows that the CSB Theorem holds in the category of finitely generated abelian groups. Besides, in [1] it was proved that if A and B are injective modules over a ring that can be embedded in each other, then A and B are isomorphic, and as a consequence one obtains that if two modules over a ring embed in one another, then their injective hulls are isomorphic. The CSB Theorem also holds for other algebraic structures as the so-called MV-algebras. Indeed in [3], the author gives an abstract version of the CSB Theorem that is applied to these structures and to σ-complete Boolean algebras. Moreover, in [6] Hötzel presents a version for the category of ∞-groupoids in which the author concludes that the CSB Theorem holds in any boolean ∞-topos. In this work, we contribute to the study of the CSB Theorem and we shall concentrate on some relevant subcategories of the category of modules over an arbitrary ring. To do so, we structured the work as follows. After the introduction in Section 3 we establish some basic notations and facts that allow us to recognize when in a category the CSB Theorem holds. Later in Section 4 we analyze modules endowed with some finiteness property, such as descending and ascending chain conditions, and we prove in Proposition 4.1 and Proposition 4.3 that some finiteness condition on the modules is enough to extend the CSB to some relevant families of modules. Also in Theorem 4.5 we show that a strong version of the CSB theorem holds for finitely generated free modules over a PID. At the end of this work, we treat the notion of Dedekind finite modules and we provide in Theorem 4.9 a characterization of when a direct sum of modules is Dedekind finite in terms of the summands of the family.

2. Preliminaries

We shall use basic notions from category theory. For a pair of objects A and B in a category $\mathcal{C}$. For more notions in category theory, the interested reader may consult [5].

- The collection of morphisms between A and B is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$, or simply $\mathrm{Hom}\,(A, B)$ if $\mathcal{C}$ is clear.
- A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is called a *split monomorphism*, if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = 1_{\mathrm{A}}$, if moreover $f \circ g = 1_{\mathrm{B}}$, we say that f is an *isomorphism* and the objects A and B are called *isomorphic*.
- Recall that a (covariant) functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is *faithful*, if for any A, B objects in $\mathcal{C}$ the mapping $\mathbf{F}_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}A, \mathbf{F}B)$ is injective.
- A monic endomorphism over an object A is a monomorphism in $\mathrm{Hom}\,(A, A)$.

In category theory, the study of the set $\mathrm{Hom}\,(A, A)$ for a given object A holds a special significance, because by examining it, we get a deeper understanding of how endomorphisms interact with and define the object itself. This exploration allows us to uncover essential structural characteristics and symmetries within the category. Moreover, the study of $\mathrm{Hom}\,(A, A)$ becomes even more crucial when we consider its role in the context of the strong CSB (Cantor-Schröder-Bernstein) property, a fundamental concept in category theory. Understanding the behavior of $\mathrm{Hom}\,(A, A)$ is often a key step in proving or establishing properties related to the strong CSB property, which deals with the cardinality of morphism sets.

3. The CSB property

For the reader's convenience, we start this section by recalling some notions and facts from [2].

Definition 3.1. Let $\mathcal{C}$ be an arbitrary category, we say that:

- $\mathcal{C}$ has *the CSB property* if each pair of objects A, B in $\mathcal{C}$ are isomorphic provided that there are monomorphisms $f : A \to B$ and $g : B \to A$.
- $\mathcal{C}$ has *the strong CSB property* if whenever there are monomorphisms $f : C \to D$ and $g : D \to C$, then both f and g are isomorphisms.

We denote by **Set** the category of sets, whose objects are sets and morphisms are functions. We also, denote by **FinSet** the subcategory of **Set** having as objects the finite sets. The next result gives us a condition for a category to have the CSB property.

Theorem 3.2. ([2, Theorem 2.5]) If a category $\mathcal{C}$ has a *faithful* functor $\mathscr{F} : \mathcal{C} \to \mathbf{FinSet}$, then $\mathcal{C}$ has the CSB property.

Using Theorem 3.2 one constructs several examples of categories where the CSB property holds. Indeed, we have the next.

Proposition 3.3. The following categories have the CSB property.

- [2, Example 2.2] The category of well-ordered sets and their morphisms the order-preserving functions.
- [2, Proposition 3.5] The category of vector spaces over a fixed field $\mathbb{F}$ and linear transformations between them.
- [2, Theorem 3.2] The category of finitely-generated abelian groups and homomorphisms.

Now we recall a notion that will be of great importance in our study of the strong CSB property.

Definition 3.4. An object A in a category $\mathcal{C}$ is called Dedekind finite if all monic endomorphisms $f : A \to A$ are isomorphisms.

Example 3.5. Any object in the category **FinSet** or $\mathbf{FinVect}_{\mathbb{K}}$ of finitely generated spaces over a field $\mathbb{K}$ respectively, is Dedekind finite.

Proposition 3.6. ([2, Proposition 2.8]) A category has the strong *CSB* property, if and only if, all objects are Dedekind finite.

The following is an immediate consequence of Example 3.5 and Proposition 3.6.

Corollary 3.7. The categories of **FinSet** and $\mathbf{FinVect}_{\mathbb{K}}$ have the strong CSB property.

3.1. The Split CSB property

Let us note for a moment that the Cantor-Schröder-Bernstein theorem was initially intended only for the category of sets, in which all monomorphisms turn out to be split monomorphism, but it is well known that this does not hold in any category as Example 3.8 below shows.

Example 3.8. Fix $m \in \mathbb{Z} - \{\pm 1\}$, and consider $m\mathbb{Z} := \{mk : k \in \mathbb{Z}\}$ and $\mathbb{Z}$ as $\mathbb{Z}$-*modules.* Then the inclusion homomorphism $i : m\mathbb{Z} \to \mathbb{Z}$ is a monomorphism, however, it is not difficult to see that i has no left inverse.

Now, we present notions related to the strong CSB property in terms of split monomorphisms. We proceed with the next.

Definition 3.9. Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. Then:

- $\mathcal{C}$ has the *split CSB property* if whenever there is a pair of split monomorphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$, there is an isomorphism $A \xrightarrow{h} B$.
- $\mathcal{C}$ has the *strong split CSB property* if whenever there are split monomorphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$, both f and g are isomorphisms.
- X is *split Dedekind finite* if all split monic endomorphisms $f : C \to C$ are isomorphisms.
- Suppose that $\mathcal{C}$ is an abelian category, we say that X is an *indecomposable object* if $X \cong B \oplus C$ implies $B \cong 0$ or $C \cong 0$.

Remark 3.10. Let $\mathbf{Grp_A}$ be the category of abelian groups. Note that $\mathbb{Z}$ is not Dedekind finite in $\mathbf{Grp_A}$. However, $\mathbb{Z}$ is split Dedekind finite. Indeed, let $f : \mathbb{Z} \to \mathbb{Z}$ be a split monomorphism in $\mathbf{Grp_A}$, then exists $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f = 1_{\mathbb{Z}}$, in particular g is an epimorphism. If $f(1) = n$ and $g(n) = n \cdot g(1) = m$ then $g(f(1)) = 1 = g(n) = n \cdot g(1)$ so that $\frac{1}{n} = g(1)$ and $n \in \{-1, 1\}$, thus $f \in \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\}$ and f is an isomorphism. We conclude that $\mathbb{Z}$ is split Dedekind finite in $\mathbf{Grp_A}$. Further, $\mathbb{Z}$ is not Dedekind finite, to observe this it is enough to consider the monomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(n) = 2n$, for all $n \in \mathbb{Z}$.

Analogous to the strong CSB property, we can study the strong split CSB property by analyzing the split monomorphism in $\mathrm{Hom}_{\mathcal{C}}(A, A)$, for any object A of $\mathcal{C}$.

Proposition 3.11. ([2, Proposition 2.12]) A category $\mathcal{C}$ has the strong split CSB property, if and only if, all its objects are split Dedekind finite.

Definition 3.12. Let $\{C_i\}_{i\in\mathbb{N}}$ be a family of subobjects of a certain object C in a category $\mathcal{C}$. We say that the *chain*

$$\cdots C_2 \subset C_1 \subset C, \tag{1}$$

stabilize, if there exists $n \in \mathbb{N}$ such that $C_n = C_k$ with $n \leq k$. The object C is said to have the descending chain condition on its subobjects, if all chains as (1) stabilize.

Theorem 3.13. ([2, Theorem 4.1]) In a category $\mathcal{C}$, where the bijective endomorphisms are isomorphisms, the objects that satisfy the descending chain condition are Dedekind finite.

Remark 3.14. While the descending chain condition guarantees that the object is Dedekind finite, the converse is not true. Indeed, the additive group $\mathbb{Q}$, which is Dedekind finite since all its injective endomorphisms are of the form $\phi(x) = ax$ with $a \in \mathbb{Q}^*$, but it does not satisfy the descending chain condition.

Regarding the notions of split Dedekind finite and indecomposable, we finish this section with the next.

Theorem 3.15. ([2, Theorem 5.4]) If all objects in a category $\mathcal{C}$ can be written uniquely as a countable direct sum of indecomposable objects, then an object is split Dedekind finite if and only if each indecomposable component of it occurs a finite number of times.

4. The CSB property in some categories of modules

In this section, we use Theorem 3.6 and Proposition 3.11 to study the CSB property in the category of left A-modules and their homomorphisms. As observed in Section 3, the category $\mathbf{FinVect}_{\mathbb{K}}$ has the strong CSB property and we shall see that the property of being finitely generated can be used to study the CSB Theorem in relevant classes of modules endowed with some finiteness conditions.
The following is a consequence of Theorem 3.13

Proposition 4.1. Let A be a ring. Then the category of artinian A-modules has the strong CSB property.

Proof. Let M be an artinian A-module and $f : M \to M$ be a monomorphism. Then the family

$$M \geq f(M) \geq f^2(M) \ldots,$$

stabilizes, thus by Theorem, the map f is an isomorphism. ☑

The following is known, but we present its proof for the sake of completeness.

Proposition 4.2. Let M be a noetherian module then there are non-zero indecomposable submodules $M_1, \ldots, M_n, n \geq 1$, of M such that $M = M_1 \oplus \cdots \oplus M_n$.

Proof. If M is indecomposable we are done. On the contrary, let Γ_0 be the collection of proper direct summands of M, then $\Gamma_0 \neq \emptyset$, and let A_0 be a maximal element of Γ_0 with $M = A_0 \oplus B_0$. Let us see that B_0 is indecomposable (and certainly non-zero). Suppose the contrary; then there exist $X, Y < B_0$ (where $<$ means strict submodule), with $X, Y \neq 0$, such that $B_0 = X \oplus Y$. This results in $M = A_0 \oplus X \oplus Y$ with $A_0 < A_0 + X < M$, contradicting the maximality of A_0.
Let Θ_0 be the set defined by: $B \in \Theta_0$ if and only if $B \leq M$, B is a direct summand of M, and B can be decomposed into a finite direct sum of non-zero indecomposable submodules. Note that $\Theta_0 \neq \emptyset$ since $B_0 \in \Theta_0$. Moreover, each $B \in \Theta_0$ is non-zero. Let N be a maximal element of Θ_0 with $M = N \oplus N_0$, $N = M_1 \oplus \ldots \oplus M_n$, $n \geq 1$, and $M_i \neq 0$ indecomposable, for each $1 \leq i \leq n$. If we assume that $N_0 \neq 0$, we can repeat this process and obtain $N_0 = A_0' \oplus B_0'$, where B_0' is a non-zero indecomposable submodule. Then, we have $M = N \oplus B_0' \oplus A_0'$, and $N \oplus B_0' = B_0' \oplus M_1 \oplus \ldots \oplus M_n$, a sum of indecomposables, contradicting the maximality of N in Θ_0, i.e.,

$$M = N \oplus 0 = N = \bigoplus_{i=1}^{n} M_i,$$

where each M_i is a non-zero indecomposable submodule. ☑

Using Theorem 3.15 and Proposition 4.2 we get the following.

Corollary 4.3. The category of noetherian A-modules over a ring A has the strong split CSB property.

4.2. Finitely generated modules over principal ideal domains

Since any finitely generated module over a PID is Noetherian it follows from Proposition 4.3 that the category $\mathbf{Mod}_R$, of R-modules where R is a PID has the strong split CSB property. In this section, we prove that these modules also have the strong CSB property. For this, we start by observing that like fields, PID's possess the property of the invariant basis number, which can be stated as follows

$$R^m \simeq R^n \quad \text{as} \quad R\text{-modules} \quad \text{implies} \quad m = n.$$

Now in the attempt to obtain an extension of Corollary 3.7 we shall concentrate on rings over PID's. For free modules over these rings, the concept of dimension is well-defined, and it is the number of elements of an arbitrary basis. Next, we state a result which will be useful to our purposes.

Proposition 4.4. Let M be a free finite-dimensional A-module and let $N \leq M$. Then N is free and $\dim(N) \leq \dim(M)$.

Proof. If $M = 0$, then $N = 0$ is a free module of dimension 0. Let M be nonzero; let $X = \{x_1, \ldots, x_n\}$ be a basis for M. For each $1 \leq k \leq n$, we define $N_k := N \cap \langle x_1, \ldots, x_k \rangle$.
Let us prove that N_k is free with dimension $\leq k$. This implies, in particular, that $N_n = N \cap \langle x_1, \ldots, x_n \rangle = N \cap M = N$ is free with dimension $\leq n$. For $k = 1$, we have $N_1 = N \cap \langle x_1 \rangle$; we define $I_1 := \{r \in R \mid x_1 \cdot r \in N\}$. Note that I_1 is an ideal of R, and therefore $I_1 = \langle a_1 \rangle$. Moreover, $N_1 = \langle x_1 \cdot a_1 \rangle$: indeed, let $x \in N_1$, then $x = x_1 \cdot r \in N$, so $r \in I_1$, and thus $r = a_1 \cdot s$. This ensures that $x = (x_1 \cdot a_1) \cdot s \in \langle x_1 \cdot a_1 \rangle$, i.e., $N_1 \subseteq \langle x_1 \cdot a_1 \rangle$. On the other hand, by definition, $x_1 \cdot a_1 \in N$, i.e., $x_1 \cdot a_1 \in N \cap \langle x_1 \rangle = N_1$, which implies $\langle x_1 \cdot a_1 \rangle \subseteq N_1$. Now, suppose that N_k is free of dimension $\leq k$. Let $I_{k+1} := \{r \in R \mid x_{k+1} \cdot r \in N + \langle x_1, \ldots, x_k \rangle\}$. Then, there exists $a_{k+1} \in R$ such that $I_{k+1} = \langle a_{k+1} \rangle$. We then have $x_{k+1} \cdot a_{k+1} = z + x_1 \cdot b_1 + \ldots + x_k \cdot b_k$, where $z \in N$ and $b_i \in R$, $1 \leq i \leq k$. We will show that $N_{k+1} = N_k + \langle z \rangle$.

Indeed, let $x \in N_{k+1}$, then $x \in N$ and $x = x_1 \cdot c_1 + \ldots + x_{k+1} \cdot c_{k+1}$. This implies $x_{k+1} \cdot c_{k+1} = x - (x_1 \cdot c_1 + \ldots + x_k \cdot c_k) \in N + \langle x_1, \ldots, x_k \rangle$, i.e., $c_{k+1} \in I_{k+1}$, so $c_{k+1} = a_{k+1} \cdot d$. Thus,

$$\begin{aligned} x &= x_1 \cdot c_1 + \ldots + x_k \cdot c_k + x_{k+1} \cdot a_{k+1} \cdot d \\ &= x_1 \cdot c_1 + \ldots + x_k \cdot c_k + z \cdot d + x_1 \cdot b_1 \cdot d + \ldots + x_k \cdot b_k \cdot d \\ &= x_1 \cdot c_1 + \ldots + x_k \cdot c_k + x_1 \cdot b_1 \cdot d + \ldots + x_k \cdot b_k \cdot d + z \cdot d \in N_k + \langle z \rangle, N \end{aligned}$$

since $x_1 \cdot c_1 + \ldots + x_k \cdot c_k + x_1 \cdot b_1 \cdot d + \ldots + x_k \cdot b_k \cdot d \in \langle x_1, \ldots, x_k \rangle \cap N$. Conversely, since $N_k \subseteq N_{k+1}$ and $z \in N \cap \langle x_1, \ldots, x_{k+1} \rangle$, we have $z \in N_{k+1}$, and thus $N_k + \langle z \rangle \subseteq N_{k+1}$. This completes the proof of $N_{k+1} = N_k + \langle z \rangle$.

If $z = 0$, then $N_{k+1} = N_k$, and thus $\dim N_{k+1} = \dim N_k = k < k+1$. Suppose now $z \neq 0$; if $a_{k+1} = 0$, then $z \in N \cap \langle x_1, \ldots, x_k \rangle = N_k$ and again $N_{k+1} = N_k$. Therefore, let $a_{k+1} \neq 0$, and we will show that for this case, the sum is direct. Let $x \in N_k \cap \langle z \rangle$, then $x = x_1 \cdot c_1 + \ldots + x_k \cdot c_k \in N$ and $x = z \cdot c$, so

$$x = x_1 \cdot c_1 + \ldots + x_k \cdot c_k = (x_{k+1} \cdot a_{k+1}) \cdot c - (x_1 \cdot b_1 + \ldots + x_k \cdot b_k) \cdot c,$$

and by linear independence, we have $a_{k+1} \cdot c = 0$. Since $a_{k+1} \neq 0$, then $c = 0$, and thus $x = 0$. We have shown that $N_{k+1} = N_k \oplus \langle z \rangle$ with $z \neq 0$. If Y is a basis for N_k, then $Y \cup \{z\}$ is a basis for N_{k+1} (since M is torsion-free, implying $Ann(z) = 0$). This implies that $\dim(N_{k+1}) \leq k+1$. ☑

Theorem 4.5. Let R be a PID. Then the category of finitely generated free R-modules has the CSB property. In particular, the category of finitely generated free abelian groups has the CSB property.

Proof. Let M and N be finitely generated free R-modules such that there exist monomorphisms $\phi : M \to N$ and $\psi : N \to M$. Let $\dim(M) = n$ and $\dim(N) = m$. Then there are R-module isomorphisms $A \cong R^n$ and $B \cong R^m$. On the other hand since $\phi(A) \leq B$ and $\psi(B) \leq A$ one has by Proposition 4.4 that $\dim(A) = \dim(\phi(A)) = n \leq \dim(B) = m$ and $\dim(B) = \dim(\psi(B)) = m \leq \dim(A) = n$, which gives then $M \cong R^n \cong R^m \cong N$, as desired. ☑

4.3. On Dedekind finite modules

We finish this work with the problem of Dedekind finiteness in the category R-mod, the idea is to characterize when a direct sum of modules is Dedekind finite. To that end, we first introduce the following.

Definition 4.6. Let A be a ring and $\{M_i\}_{i\in I}$ be a family of A-modules. We say that it is an *orthogonal family of A-modules*, if $\mathrm{Hom}\,(M_i, M_j) = \{0_{M_i M_j}\}$ for any $i, j \in I$ with $i \neq j$.

Example 4.7. Let $\{e_i\}_{i\in I}$ be a family of orthogonal idempotents in a ring A, then the family $\{e_i A\}_{i\in I}$ is orthogonal. A family of orthogonal $\mathbb{Z}$-modules is given by $\{\mathbb{Z}_{n_i}\}_{i\in I}$ where $\gcd(n_i, n_j) = 1$, for all $i, j \in I$ with $i \neq j$.

The following facts inspire the notion of orthogonality between A-modules:

1. Let V a $\mathbb{F}$-vector space of finite dimension n. If $\{w_i\}_{i=1}^n$ is orthogonal pairwise then $V = \bigoplus_{i=1}^n \langle w_i \rangle$.
2. Let e and f be two orthogonal idempotents in a ring A, then $\mathrm{Hom}_R(eA, fA)$ contains only the zero map. Moreover, if A is unital and $\{e_i\}_{i=1}^n$ is a family of orthogonal idempotents such that $\sum_{i=1}^n e_i = 1$ then $A = \bigoplus_{i=1}^{n} e_i A$.

Remark 4.8. In Example 4.7 noting that while both 1. and 2. illustrate the concept of orthogonality between elements ($\{w_i\}_{i=1}^n$), part 2. demonstrates a categorical manifestation of this property between the objects ($\{e_i A\}_{i=1}^n$). This is primarily attributed to its establishment of orthogonality among objects within the category, achieved by only considering the interplay between the objects and morphisms.

We have the next.

Theorem 4.9. Let $\{M_i\}_{i\in I}$ be a family of R-modules. If $\bigoplus_{i\in I} M_i$ is Dedekind finite then module M_i is Dedekind finite for all $i \in I$. Moreover, if the family is orthogonal, then the converse holds.

Proof. Suppose that there exists $j \in I$ such that M_j is not Dedekind finite and $\bigoplus_{i\in I} M_i$ is Dedekind finite. Then there exists a monomorphism $\phi_j : M_j \to M_j$ such that ϕ_j is not isomorphism, thus we construct a monomorphism $\phi : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} M_i$ such that $\phi|_{M_j} = \phi_j$, and thus $\oplus_{i\in I} M_i$ is not be Dedekind finite. For the converse, suppose that $\{M_i\}_{i\in I}$ is a family of orthogonal A-modules. Let $\phi : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} M_i$ be a monomorphism, note that for each element $m = \sum_{i\in I} m_i \in \bigoplus_{i\in I} M_i$

we have that $\phi(m) = \sum_{i\in I}\phi(m_i)$. Since the family $\{M_i\}_{i\in I}$ is orthogonal, we get that $\phi(m_i) \in M_i$, for any $m_i \in M_i$, then $\phi = \sum_{i\in I}\phi_i$ where $\phi_i = \phi_{|M_i}$, for every $i \in I$, and it is clear that ϕ_i is a monomorphism, hence

$$\phi\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I}\phi(M_i) = \bigoplus_{i\in I}\phi_i(M_i) = \bigoplus_{i\in I} M_i,$$

and $\bigoplus_{i\in I} M_i$ is Dedekind finite. ☑

Regarding Theorem 4.9, we notice the following.

Example 4.10. The family $\{\mathbb{Z}_n, \mathbb{Z}_{2n}\}$ is not orthogonal in $\mathbf{Grp}_A$, however $\mathbb{Z}_n \oplus \mathbb{Z}_{2n}$ is Dedekind finite.

Now, we address the Gram-Schmidt orthogonalization method. This process provides us with an orthogonal basis from a given one. Extending this idea, it becomes natural to ask: Given a family of objects, when does an orthogonal decomposition exist? Inspired by this question, we recall the next.

Theorem 4.11. *([4, Theorem 4.2]) Let X be an object of an additive and suppose there are two decompositions*

$$X_1 \oplus \ldots \oplus X_r = X = Y_1 \oplus \ldots \oplus Y_s$$

into objects with local endomorphism rings. Then $r = s$ and and there exists a permutation σ such that $X_i \cong Y_{\sigma(i)}$ for $1 \leq i \leq r$.

Now we recall the next.

Definition 4.12. Let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is a Krull-Schmidt category if every object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings.

An important example of a Krull-Schmidt category is formed by the category of modules having finite length.

Corollary 4.13. Let $\mathcal{C}$ be a Krull-Schmidt category. Let $M \in Ob(\mathcal{C})$ with $M = \bigoplus_{i=1\in n} M_i$, as a sum of indecomposable objects having local endomorphism rings: Suppose that $\{M_i\}_{i=1}^n$ is orthogonal. Then, if $M = \bigoplus_{i=1}^{m} \hat{M}_i$, is another sum of indecomposable objects having local endomorphism rings, the family $\{\hat{M}_i\}_{i=1}^m$ is orthogonal.

Proof. Since $\oplus_{i=1}^n M_i = \oplus_{i=1}^m \hat{M}i$ we have by Theorem 4.11, that $m = n$ and there exists a permutation $\sigma \in S_m$ such that $M_{\sigma(k)} \cong \hat{M}_k$ for all $k \in \{1, 2, \ldots, m\}$. Thus for any $i, j \in \{1, 2, \ldots, m\}$ we have Consequently, we have $\mathrm{Hom}\,(\hat{M}_i, \hat{M}_j) \simeq \mathrm{Hom}\,(M_{\sigma(i)}, M_{\sigma(j)})$, which implies

that $\mathrm{Hom}\,(\hat{M}_i, \hat{M}_j) = \{0_{\hat{M}_i,\hat{M}_j}\}$, and the family $\{\hat{M}_i\}_{i=1}^m$ is orthogonal. ☑

References

[1] Bumby R.T., "Modules which are isomorphic to submodules of each other", *Arch. Math*, 16 (1965), 184–185.

[2] Don Laackman, "The Cantor-Schröeder-Bernstein property in categories", The University of Chicago, (2010). https://math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Laackman.pdf

[3] Freytes H., "An Algebraic Version of the Cantor-Bernstein-Schröder Theorem", *Czech Math J*, 54 (2004), 609–621. doi: 10.1007/s10587-004-6412-x

[4] Krause H., "Krull-Schmidt categories and projective covers", *Expo. Math.*, 33 (2015), No. 4, 535-549. doi: 10.1016/j.exmath.2015.10.001

[5] Jacobson N., *Basic Algebra II*, W.H. Freeman and Company, New York, 1989.

[6] Martín E., "The Cantor-Schörder-Bernstein Theorem for ∞-groupoids", *J. Homotopy Relat. Struct.*, 16 (2021), No. 3, 363–366.