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q-Relativistic wave equation of the form $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$

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Abstract. In this paper we introduce a q-relativistic wave equation of the form $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$. We present the q-spinorial solutions using the method of separated variables in the q-relativistic wave equation. Some comments are mentioned at the end of the paper.

Keywords: q-Relativistic wave equation, separation of variables, fermionic and bosonic solutions, Lorentz coordinates

MSC2020: 81Q99, 46E99, 35A24, 15A66, 16T99, 17B37.

q-Ecuación de onda relativista de la forma $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$

Resumen. En este artículo introducimos una q-ecuación de onda relativista de la forma $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$. Presentamos las q-soluciones espinoriales usando el método de separación de variables en la q-ecuación de onda relativista. En el final del artículo se mencionaron algunos comentarios.

Palabras clave: q-Ecuación de onda relativista, separación de variables, soluciones fermionicas y bosonicas, coordenadas de Lorentz.

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1. Introduction

In the years of 1926 and 1927 Klein and Gordon proposed a relativistic wave equation for particles with spin 0, which originally is Schrödinger Relativistic Equation [1, 2]. Duffin - Kemmer - Petiau developed a relativistic theory that described the dynamics of a system of particles of spin 0 and 1 [3, 4, 5, 6]. Proca stated a relativistic wave equation for massive particles with spin 1 [7]. Dirac in 1938 proposed a wave equation that unifies the special relativity with the spin of particles, particulary the electron [8]. Fierz - Pauli proposed a relativistic wave equation for the arbitrary spin particles [9]. Rarita - Schwinger stated a relativistic wave equation which describes the dynamics of the particles with spin 3/2 [10]. Bhabha described at the beginning of his theory the dynamics of protons through a relativistic wave equation as an exact copy of Dirac equation [11]. Niederle defines a wave equation for the massive particles with an arbitrary rational spin number [12]. Silenko verifies the accuracy of the wave equations for particles with spin 1 [13]. Tutik and Kulikov describe the dynamics of fermions with massive bosons with integer spin through a relativistic wave equation [14]. In the q-deformed case, The q-deformed wave equations then have the same properties as the undeformed ones. Pillin describe the q-deformed relativistic wave equations based on the representation theory of the q-deformed Lorentz and Poincaré symmetries [15]. On other hand, the Pauli matrices of q-deformed Minkowski space (e.g. [18] for more details), are defined as

$$
\begin{aligned}
(\sigma_+)^{\alpha}_{\dot\beta} &= \begin{bmatrix} 0 & 0 \\ 0 & kq^{1/2}\lambda_+^{1/2} \end{bmatrix}, \quad (\sigma_3)^{\alpha}_{\dot\beta} = k\begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix}, \\
(\sigma_-)^{\alpha}_{\dot\beta} &= k\begin{bmatrix} q^{1/2}\lambda_+^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\sigma_0)^{\alpha}_{\dot\beta} = k\begin{bmatrix} 0 & -q^{-1} \\ 1 & 0 \end{bmatrix},
\end{aligned} \tag{1}
$$

and their conjugated counterparts

$$
\begin{aligned}
(\overline{\sigma}_+)^{\alpha}_{\dot\beta} &= \begin{bmatrix} 0 & 0 \\ 0 & \overline{k}q^{-1/2}\lambda_+^{1/2} \end{bmatrix}, \quad (\overline{\sigma}_3)^{\alpha}_{\dot\beta} = \overline{k}\begin{bmatrix} 0 & 1 \\ q^{-1} & 0 \end{bmatrix}, \\
(\overline{\sigma}_-)^{\alpha}_{\dot\beta} &= \overline{k}\begin{bmatrix} q^{-1/2}\lambda_+^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\overline{\sigma}_0)^{\alpha}_{\dot\beta} = \overline{k}\begin{bmatrix} 0 & 1 \\ -q & 0 \end{bmatrix},
\end{aligned} \tag{2}
$$

where $k, \overline{k}$ are characteristic parameters associated to bosons ($q = +1$) and fermions ($q = -1$), and $\lambda_+ = q + q^{-1}$. The aim of this work is to show that q-relativistic wave equations of the form $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$ can be constructed with the help of Pauli matrices of q-Minkowski space following [16]. The paper is organized as follows: in Section 2 the q-relativistic wave equation of the form $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$ is presented. In Section 3 we present the q-relativistic wave equation with separated variables. In Section 4 we find the fermionic and bosonic solutions. Motivated by the Schmidke [17] work, we introduce the Lorentz coordinates, and the final section with comments and suggestion for further work.

2. q-Relativistic wave equation of the form $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$

In this article, we investigate the q-relativistic wave equation of the form $i\boldsymbol{\partial}^q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi$, and, at the same time, it is easier to study, as it contains only first-order derivatives.

Definition 2.1. According to above and following [16], we introduce the q-differential spinorial equation of the form

$$i(\partial_x^q \psi_q^x + \partial_y^q \psi_q^y + \partial_z^q \psi_q^z) + m\psi = E\psi_0, \tag{3}$$

being m and E constants, and $\psi_q^x, \psi_q^y, \psi_q^z, \psi_0$ q-deformed spinors in the minkowskian space which are defined by

$$\psi_q^x = \begin{bmatrix} 0 \\ kq^{1/2}\lambda_+ \varphi_{\dot{\beta}} \end{bmatrix}, \psi_q^y = \begin{bmatrix} kq^{1/2}\lambda_+ \psi^\alpha \\ 0 \end{bmatrix}, \psi_q^z = \begin{bmatrix} kq\varphi_{\dot{\beta}} \\ k\psi^\alpha \end{bmatrix}, \psi_0 = \begin{bmatrix} -kq^{-1}\varphi_{\dot{\beta}} \\ k\psi^\alpha \end{bmatrix}, \tag{4}$$

subject to Pauli matrices in the Minkowskian space (1) (see, for instance [17, 18]), and the expression $\psi = \begin{bmatrix} \psi^\alpha \\ \varphi_{\dot{\beta}} \end{bmatrix}$ is called the Dirac spinor, mentioned in the work of Beretetskii et al [19].

Remark 2.2. Respect to solution, is important rewrite (3) as a system of differential equations

$$ikq^{1/2}\lambda_+^{1/2}\partial_y^q \psi^\alpha + ikq\partial_z^q \varphi_{\dot{\beta}} + m\psi^\alpha = -kq^{-1}E\varphi_{\dot{\beta}}, \tag{5}$$

$$ikq^{1/2}\lambda_+^{1/2}\partial_x^q \varphi_{\dot{\beta}} + ik\partial_z^q \psi^\alpha + m\varphi_{\dot{\beta}} = Ek\psi^\alpha. \tag{6}$$

In following section we will consider the method of separation of variables to obtain the solutions of (5) and (6).

3. A q-relativistic wave equation with separated variables

Theorem 3.1. *Using the separation of variables, the solutions of equations (5) and (6) are given by:*

$$\psi_q^x = \begin{bmatrix} 0 \\ kq^{1/2}\lambda_+ d_{\dot{\beta}} \exp_q \left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}} \right] \end{bmatrix}, \tag{7}$$

$$\psi_q^y = \begin{bmatrix} kq^{1/2}\lambda_+ c^\alpha \exp_q \left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}} \right] \\ 0 \end{bmatrix}, \tag{8}$$

$$\psi_q^z = \begin{bmatrix} kqd_{\dot{\beta}} \exp_q \left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}} \right] \\ kc^\alpha \exp_q \left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}} \right] \end{bmatrix}, \tag{9}$$

$$\psi_0 = \begin{bmatrix} -kq^{-1} d_{\dot{\beta}} \exp_q \left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}} \right] \\ kc^{\alpha} \exp_q \left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}} \right] \end{bmatrix}, \tag{10}$$

and

$$\psi = \begin{bmatrix} c^{\alpha} \exp_q \left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}} \right] \\ d_{\dot{\beta}} \exp_q \left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}} \right] \end{bmatrix}, \tag{11}$$

where c^{α} and $d_{\dot{\beta}}$ are spinors respectively.

Proof. First, let us consider that ψ^{α} can be expressed as $f^{\alpha}(y)h^{\alpha}(z)$ and $\varphi_{\dot{\beta}}$ as $F_{\dot{\beta}}(x)H_{\dot{\beta}}(z)$. Therefore (6) and (5) can be written in the form

$$ikq^{1/2}\lambda_+^{1/2} h^{\alpha}(z) \frac{\mathrm{d}^q f^{\alpha}(y)}{\mathrm{d}^q y} + m f^{\alpha}(y) h^{\alpha}(z) = -ikq F_{\dot{\beta}}(x) \frac{\mathrm{d}^q H_{\dot{\beta}}(z)}{\mathrm{d}^q z} - kq^{-1} E F_{\dot{\beta}}(x) H_{\dot{\beta}}(z), \tag{12}$$

and

$$ikq^{1/2}\lambda_+^{1/2} H_{\dot{\beta}}(z) \frac{\mathrm{d}^q F_{\dot{\beta}}(x)}{\mathrm{d}^q x} + m F_{\dot{\beta}}(x) H_{\dot{\beta}}(z) = -ik f^{\alpha}(y) \frac{\mathrm{d}^q h^{\alpha}(z)}{\mathrm{d}^q z} + Ek f^{\alpha}(y) h^{\alpha}(z). \tag{13}$$

Equaling both equations to $af^{\alpha}(y)h^{\alpha}(z) + bF_{\dot{\beta}}(x)H_{\dot{\beta}}(z)$ and considering the cases $a = 0$, $b = 0$, we have

$$ikq^{1/2}\lambda_+^{1/2} h^{\alpha}(z) \frac{\mathrm{d}^q f^{\alpha}(y)}{\mathrm{d}^q y} + m f^{\alpha}(y) h^{\alpha}(z) = a f^{\alpha}(y) h^{\alpha}(z), \quad (b = 0), \tag{14}$$

$$-ikq F_{\dot{\beta}}(x) \frac{\mathrm{d}^q H_{\dot{\beta}}(z)}{\mathrm{d}^q z} - kq^{-1} E F_{\dot{\beta}}(x) H_{\dot{\beta}}(z) = b F_{\dot{\beta}}(x) H_{\dot{\beta}}(z), \quad (a = 0), \tag{15}$$

$$ikq^{1/2}\lambda_+^{1/2} H_{\dot{\beta}}(z) \frac{\mathrm{d}^q F_{\dot{\beta}}(x)}{\mathrm{d}^q x} + m F_{\dot{\beta}}(x) H_{\dot{\beta}}(z) = b F_{\dot{\beta}}(x) H_{\dot{\beta}}(z), \quad (a = 0), \tag{16}$$

and

$$-ik f^{\alpha}(y) \frac{\mathrm{d}^q h^{\alpha}(z)}{\mathrm{d}^q z} + Ek f^{\alpha}(y) h^{\alpha}(z) = a f^{\alpha}(y) h^{\alpha}(z) \quad (b = 0), \tag{17}$$

thus we express the general solutions in the form

$$
\begin{aligned}
f^{\alpha}(y) = &\ u^{\alpha} \exp_q\left[-\frac{i(m-a)}{kq^{1/2}\lambda_+}y\right], \quad (b=0), && (18)\\
H_{\dot\beta}(z) = &\ v_{\dot\beta} \exp_q\left[i\frac{(b+kq^{-1}E)z}{kq}\right], \quad (a=0), && (19)\\
F_{\beta}(x) = &\ w_{\dot\beta} \exp_q\left[-\frac{i(m-b)}{kq^{1/2}\lambda_+^{1/2}}x\right], \quad (a=0), && (20)\\
h^{\alpha}(z) = &\ t^{\alpha} \exp_q\left[\frac{i(a-E)z}{k}\right], \quad (b=0). && (21)
\end{aligned}
$$

These latter formulas imply that (4) can be written as

$$
\psi_q^x = \begin{bmatrix} 0 \\ kq^{1/2}\lambda_+ d_{\dot\beta} \exp_q\left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}}\right] \end{bmatrix},
$$

$$
\psi_q^y = \begin{bmatrix} kq^{1/2}\lambda_+ c^{\alpha} \exp_q\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}}\right] \\ 0 \end{bmatrix},
$$

$$
\psi_q^z = \begin{bmatrix} kqd_{\dot\beta} \exp_q\left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}}\right] \\ kc^{\alpha} \exp_q\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}}\right] \end{bmatrix},
$$

$$
\psi_0 = \begin{bmatrix} -kq^{-1}d_{\dot\beta} \exp_q\left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}}\right] \\ kc^{\alpha} \exp_q\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}}\right] \end{bmatrix},
$$

and

$$
\psi = \begin{bmatrix} c^{\alpha} \exp_q\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{kq^{1/2}\lambda_+^{1/2}}\right] \\ d_{\dot\beta} \exp_q\left[\frac{i(b+kq^{-1}E)z}{kq} - \frac{i(m-b)x}{kq^{1/2}\lambda_+^{1/2}}\right] \end{bmatrix}.
$$

which corresponds to (7), (8), (9), (10) and (11). ☑

In the following section shows the solutions for $q=-1$ (which correspond to fermionic case) and $q=+1$ (Bosonic case) (see footnothe of the Section 7.1 of reference [20]).

Theorem 3.2. *The conjugated form of the equation (3) is given by*

$$
-i(\partial_x^q \overline{\psi}_q^x + \partial_y^q \overline{\psi}_q^y + \partial_z^q \overline{\psi}_q^z) + m\overline{\psi} = E\overline{\psi}_0, \tag{22}
$$

and the q-conjugated spinors $\overline{\psi}^x_q, \overline{\psi}^y_q, \overline{\psi}^z_q, \overline{\psi}$ *and* $\overline{\psi}_0$

$$\overline{\psi}^x_q = \begin{bmatrix} 0 & \overline{k}q^{-1/2}\lambda_+^{1/2}\varphi^\alpha \end{bmatrix}, \overline{\psi}^y_q = \begin{bmatrix} \overline{k}q^{-1/2}\lambda_+^{1/2}\psi_{\dot{\beta}} & 0 \end{bmatrix}, \overline{\psi}^z_q = \begin{bmatrix} \overline{k}\varphi^\alpha & q^{-1}\overline{k}\psi_{\dot{\beta}} \end{bmatrix}, \tag{23}$$

$$\overline{\psi} = \begin{bmatrix} \psi_{\dot{\beta}} & \varphi^\alpha \end{bmatrix}, \overline{\psi}_0 = \begin{bmatrix} \overline{k}\varphi^\alpha & -q\overline{k}\psi_{\dot{\beta}} \end{bmatrix}. \tag{24}$$

Proof. The same reasoning applies to the expressions (5) and (6), therefore we can stablish the following system of differential equations

$$-i\overline{k}q^{-1/2}\lambda_+^{1/2}\partial_y^q\psi_{\dot{\beta}} - i\overline{k}\partial_z^q\varphi^\alpha + m\psi_{\dot{\beta}} = \overline{k}E\varphi^\alpha, \tag{25}$$

and

$$-i\overline{k}q^{-1/2}\lambda_+^{1/2}\partial_x^q\varphi^\alpha - iq^{-1}\overline{k}\partial_z^q\psi_{\dot{\beta}} + m\varphi^\alpha = -qE\overline{k}\psi_{\dot{\beta}}. \tag{26}$$

Now, equaling (25) and (26) to $\overline{a}f_{\dot{\beta}}(y)h_{\dot{\beta}}(z) + \overline{b}F^\alpha(z)H^\alpha(x)$, considering the cases $\overline{a} = 0$, $\overline{b} = 0$, and performing same calculations to obtain (7), (8), (9), (10) and (11), we obtain the following conjugated spinors

$$\overline{\psi}^x_q = \left[0 \quad \overline{k}q^{-1/2}a^\alpha \exp_q\left[\tfrac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \tfrac{i(\overline{b}-m)x}{\overline{k}q^{-1/2}\lambda_+^{1/2}} \right] \right], \tag{27}$$

$$\overline{\psi}^y_q = \left[\overline{k}q^{-1/2}\lambda_+^{1/2}b_{\dot{\beta}} \exp_q\left[\tfrac{i(\overline{a}-m)y}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \tfrac{i(\overline{a}-qE\overline{k})z}{\overline{k}q^{-1}} \right] \quad 0 \right], \tag{28}$$

$$\overline{\psi}^z_q = \left[\overline{k}a^\alpha \exp_q\left[\tfrac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \tfrac{i(\overline{b}-m)x}{\overline{k}q^{-1/2}\lambda_+^{1/2}} \right] \quad q^{-1}\overline{k}b_{\dot{\beta}} \exp_q\left[\tfrac{i(\overline{a}-m)y}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \tfrac{i(\overline{a}-qE\overline{k})z}{\overline{k}q^{-1}} \right] \right], \tag{29}$$

$$\overline{\psi} = \left[b_{\dot{\beta}} \exp_q\left[\tfrac{i(\dot{a}-m)y}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \tfrac{i(\overline{a}-qE\overline{k})z}{\overline{k}q^{-1}} \right] \quad a^\alpha \exp_q\left[\tfrac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \tfrac{i(\overline{b}-m)x}{\overline{k}q^{-1/2}\lambda_+^{1/2}} \right] \right], \tag{30}$$

and

$$\overline{\psi}_0 = \left[\overline{k}a^\alpha \exp_q\left[\tfrac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \tfrac{i(\overline{b}-m)x}{\overline{k}q^{-1/2}\lambda_+^{1/2}} \right] \quad -q\overline{k}b_{\dot{\beta}} \exp_q\left[\tfrac{i(\dot{a}-m)y}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \tfrac{i(\overline{a}-qE\overline{k})z}{\overline{k}q^{-1}} \right] \right]. \tag{31}$$

The q-conjugated spinors (23) and (24) has been obtained from the relations

$$\overline{\psi}^x_q = (\overline{\sigma}_+)^\alpha_{\dot{\beta}}\overline{\psi}, \quad \overline{\psi}^y_q = (\overline{\sigma}_-)^\alpha_{\dot{\beta}}\overline{\psi}, \quad \overline{\psi}^z_q = (\overline{\sigma}_3)^\alpha_{\dot{\beta}}\overline{\psi}, \quad \overline{\psi}_0 = (\overline{\sigma}_0)^\alpha_{\dot{\beta}}\overline{\psi},$$

where $(\overline{\sigma}_+)^\alpha_{\dot{\beta}}, (\overline{\sigma}_-)^\alpha_{\dot{\beta}}, (\overline{\sigma}_3)^\alpha_{\dot{\beta}}$ and $(\overline{\sigma}_0)^\alpha_{\dot{\beta}}$ are the conjugated Pauli matrices on the q-minkowskian space (2). ☑

On other hand, according to mentioned in [20, p. 229], we can formulate the solutions for the cases $q = -1$ and $q = +1$ that corresponds to fermionic and bosonic cases, which we will show in the following section.

4. Fermionic and bosonic solutions

Remark 4.1. For the fermionic case ($q = -1$), the Eq. (3) can be expressed in the form

$$i(\partial_x^{-1}\psi_{-1}^x + \partial_y^{-1}\psi_{-1}^y + \partial_z^{-1}\psi_{-1}^z) + m\psi = E\psi_0. \tag{32}$$

Thus for $q = -1$, the Eqs. (7), (8), (9), (10) and (11) can be written as

$$\psi_{-1}^x = \begin{bmatrix} 0 \\ -2kid_{\dot\beta}\exp_{-1}\left[\frac{-i(b-kE)z}{k} + \frac{i(m-b)x}{k\sqrt{2}}\right] \end{bmatrix}, \tag{33}$$

$$\psi_{-1}^y = \begin{bmatrix} -k\sqrt{2}c^\alpha \exp_{-1}\left[\frac{i(a-E)z}{k} + \frac{i(m-a)y}{k\sqrt{2}}\right] \\ 0 \end{bmatrix}, \tag{34}$$

$$\psi_{-1}^z = \begin{bmatrix} -kd_{\dot\beta}\exp_{-1}\left[\frac{-i(b+kq^{-1}E)z}{k} + \frac{i(m-b)x}{k\sqrt{2}}\right] \\ kc^\alpha \exp_{-1}\left[\frac{i(a-E)z}{k} + \frac{i(m-a)y}{k\sqrt{2}}\right] \end{bmatrix}, \tag{35}$$

$$\psi_0 = \begin{bmatrix} kd_{\dot\beta}\exp_{-1}\left[\frac{-i(b-kE)z}{k} + \frac{i(m-b)x}{k\sqrt{2}}\right] \\ kc^\alpha \exp_{-1}\left[\frac{i(a-E)z}{k} + \frac{i(m-a)y}{k\sqrt{2}}\right] \end{bmatrix}, \tag{36}$$

and

$$\psi = \begin{bmatrix} c^\alpha \exp_{-1}\left[\frac{i(a-E)z}{k} + \frac{i(m-a)y}{k\sqrt{2}}\right] \\ d_{\dot\beta}\exp_{-1}\left[\frac{-i(b-kE)z}{k} + \frac{i(m-b)x}{k\sqrt{2}}\right] \end{bmatrix}. \tag{37}$$

Respect to bosonic case, the Eq. (3) is given by

$$i(\partial_x^{+1}\psi_{+1}^x + \partial_y^{+1}\psi_{+1}^y + \partial_z^{+1}\psi_{+1}^z) + m\psi = E\psi_0. \tag{38}$$

Therefore, the solutions corresponding to $q = +1$ in Eqs. (7), (8), (9), (10) and (11) are given by

$$\psi_{+1}^x = \begin{bmatrix} 0 \\ 2kd_{\dot\beta}\exp_{+1}\left[\frac{i(b+kE)z}{k} - \frac{i(m-b)x}{k\sqrt{2}}\right] \end{bmatrix}, \tag{39}$$

$$\psi_{+1}^y = \begin{bmatrix} kc^\alpha \exp_q\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{k\sqrt{2}}\right] \\ 0 \end{bmatrix}, \tag{40}$$

$$\psi^z_{+1} = \begin{bmatrix} kd_{\dot{\beta}} \exp_{+1}\left[\frac{i(b+kE)z}{k} - \frac{i(m-b)x}{k\sqrt{2}}\right] \\ kc^{\alpha} \exp_{+1}\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{k\sqrt{2}}\right] \end{bmatrix}, \tag{41}$$

$$\psi_0 = \begin{bmatrix} -kd_{\dot{\beta}} \exp_{+1}\left[\frac{i(b+kE)z}{k} - \frac{i(m-b)x}{k\sqrt{2}}\right] \\ kc^{\alpha} \exp_{+1}\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{k\sqrt{2}}\right] \end{bmatrix}, \tag{42}$$

and

$$\psi = \begin{bmatrix} c^{\alpha} \exp_{+1}\left[\frac{i(a-E)z}{k} - \frac{i(m-a)y}{k\sqrt{2}}\right] \\ d_{\dot{\beta}} \exp_{+1}\left[\frac{i(b+kE)z}{k} - \frac{i(m-b)x}{k\sqrt{2}}\right] \end{bmatrix}. \tag{43}$$

According to above, physically we can say that the Eqs (32) and (38) are called the *relativistic wave equations for q-boson and q-fermion*, being E and m the energy and mass respectively. On other hand, the conjugated solutions are obtained substituying $q = -1$ into Eqs. (27), (28), (29), (30) and (31) resulting

$$\overline{\psi}^x_{-1} = \begin{bmatrix} 0 & \overline{k} i a^{\alpha} \exp_{-1}\left[\frac{i(\overline{b}+E\overline{k})z}{\overline{k}} - \frac{i(\overline{b}-m)x}{\overline{k}\sqrt{2}}\right] \end{bmatrix}, \tag{44}$$

$$\overline{\psi}^y_{-1} = \begin{bmatrix} -\overline{k}\sqrt{2} b_{\dot{\beta}} \exp_{-1}\left[\frac{-i(\overline{a}-m)y}{\overline{k}\sqrt{2}} - \frac{i(\overline{a}+E\overline{k})z}{\overline{k}}\right] & 0 \end{bmatrix}, \tag{45}$$

$$\overline{\psi}^z_{-1} = \begin{bmatrix} \overline{k} a^{\alpha} \exp_{-1}\left[\frac{i(\overline{b}+E\overline{k})z}{\overline{k}} - \frac{i(\overline{b}-m)x}{\overline{k}\sqrt{2}}\right] & -\overline{k} b_{\dot{\beta}} \exp_{-1}\left[\frac{-i(\overline{a}-m)y}{\overline{k}\sqrt{2}} - \frac{i(\overline{a}+E\overline{k})z}{\overline{k}}\right] \end{bmatrix}, \tag{46}$$

$$\overline{\psi} = \begin{bmatrix} b_{\dot{\beta}} \exp_{-1}\left[\frac{-i(\dot{a}-m)y}{\overline{k}\sqrt{2}} - \frac{i(\overline{a}+E\overline{k})z}{\overline{k}}\right] & a^{\alpha} \exp_{-1}\left[\frac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \frac{-i(\overline{b}-m)x}{\overline{k}\sqrt{2}}\right] \end{bmatrix}, \tag{47}$$

and

$$\overline{\psi}_0 = \begin{bmatrix} \overline{k} a^{\alpha} \exp_{-1}\left[\frac{i(\overline{b}+E\overline{k})z}{\overline{k}} - \frac{i(\overline{b}-m)x}{\overline{k}\sqrt{2}}\right] & -q\overline{k} b_{\dot{\beta}} \exp_{-1}\left[\frac{-i(\dot{a}-m)y}{\overline{k}\sqrt{2}} - \frac{i(\overline{a}+E\overline{k})z}{\overline{k}}\right] \end{bmatrix}. \tag{48}$$

For the bosonic case

$$\overline{\psi}^x_{+1} = \begin{bmatrix} 0 & \overline{k} a^{\alpha} \exp_{+1}\left[\frac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \frac{i(\overline{b}-m)x}{\overline{k}\sqrt{2}}\right] \end{bmatrix}, \tag{49}$$

$$\overline{\psi}^y_{+1} = \begin{bmatrix} \overline{k}\sqrt{2} b_{\dot{\beta}} \exp_q\left[\frac{i(\overline{a}-m)y}{\overline{k}\sqrt{2}} + \frac{i(\overline{a}-E\overline{k})z}{\overline{k}}\right] & 0 \end{bmatrix}, \tag{50}$$

$$\overline{\psi}^z_{+1} = \begin{bmatrix} \overline{k} a^{\alpha} \exp_{+1}\left[\frac{i(\overline{b}+E\overline{k})z}{\overline{k}} + \frac{i(\overline{b}-m)x}{\overline{k}\sqrt{2}}\right] & \overline{k} b_{\dot{\beta}} \exp_{+1}\left[\frac{i(\overline{a}-m)y}{\overline{k}\sqrt{2}} + \frac{i(\overline{a}-E\overline{k})z}{\overline{k}}\right] \end{bmatrix}, \tag{51}$$

$$\overline{\psi} = \left[b_{\dot\beta} \exp_{+1} \left[\tfrac{i(\dot a - m)y}{\overline{k}\sqrt{2}} + \tfrac{i(\overline{a} - qE\overline{k})z}{\overline{k}q^{-1}} \right] \quad a^\alpha \exp_{+1} \left[\tfrac{i(\overline{b} + E\overline{k})z}{\overline{k}} + \tfrac{i(\overline{b} - m)x}{\overline{k}\sqrt{2}} \right] \right], \tag{52}$$

and

$$\overline{\psi}_0 = \left[\overline{k} a^\alpha \exp_{+1} \left[\tfrac{i(\overline{b} + E\overline{k})z}{\overline{k}} + \tfrac{i(\overline{b} - m)x}{\overline{k}\sqrt{2}} \right] \quad -\overline{k} b_{\dot\beta} \exp_{+1} \left[\tfrac{i(\dot a - m)y}{\overline{k}\sqrt{2}} + \tfrac{i(\overline{a} - E\overline{k})z}{\overline{k}} \right] \right]. \tag{53}$$

Remark 4.2. It is important to mention that ∂_μ^{+1} and ∂_μ^{-1} for all $\mu = x, y, z$, not correspond to derivatives of order +1 and -1.

5. Lorentz coordinates

Based on [17], our aim now is to construct quantum Lorentz coordinates (q-four vectors) with the spinors (7), (8), (27), and (28). Let us consider the bilinear combinations

$$\begin{aligned} A &= \overline{\psi}^x_q \psi^y_q, \quad B = \overline{\psi}^y_q \psi^x_q, \\ C &= \overline{\psi}^x_q \psi^x_q, \quad D = \overline{\psi}^y_q \psi^y_q. \end{aligned} \tag{54}$$

Following [17], we can define the Lorentz coordinates of the following manner

$$\begin{aligned} \Psi^0 &= \frac{1}{\sqrt{2}}(C + D) \\ &= \frac{1}{\sqrt{2}} \left\{ k\overline{k} a^\alpha d_{\dot\beta} \exp_q \left[\frac{i(\overline{b} + E\overline{k})z}{\overline{k}} + \frac{i(\overline{b} - m)x}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \frac{i(b + kq^{-1}E)z}{kq} - \frac{i(m - b)x}{kq^{1/2}\lambda_+^{1/2}} \right] \right\} \\ &+ \frac{1}{\sqrt{2}} \left\{ \overline{k} k \lambda_+^{3/2} b_{\dot\beta} c^\alpha \exp_q \left[\frac{i(\overline{a} - m)y}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \frac{i(\overline{a} - qE\overline{k})z}{\overline{k}q^{-1}} + \frac{i(a - E)z}{k} - \frac{i(m - a)}{kq^{1/2}\lambda_+^{1/2}} \right] \right\} \\ \Psi^1 &= \frac{1}{\sqrt{2}}(A + B) = 0, \\ \Psi^2 &= \frac{1}{\sqrt{2}}(C - D) \\ &= \frac{1}{\sqrt{2}} \left\{ k\overline{k} a^\alpha d_{\dot\beta} \exp_q \left[\frac{i(\overline{b} + E\overline{k})z}{\overline{k}} + \frac{i(\overline{b} - m)x}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \frac{i(b + kq^{-1}E)z}{kq} - \frac{i(m - b)x}{kq^{1/2}\lambda_+^{1/2}} \right] \right\} \\ &- \frac{1}{\sqrt{2}} \left\{ \overline{k} k \lambda_+^{3/2} b_{\dot\beta} c^\alpha \exp_q \left[\frac{i(\overline{a} - m)y}{\overline{k}q^{-1/2}\lambda_+^{1/2}} + \frac{i(\overline{a} - qE\overline{k})z}{\overline{k}q^{-1}} + \frac{i(a - E)z}{k} - \frac{i(m - a)y}{kq^{1/2}\lambda_+^{1/2}} \right] \right\}, \\ \Psi^4 &= \frac{i}{\sqrt{2}}(A - B) = 0. \end{aligned} \tag{55}$$

However, these coordinates are not real. Therefore, to obtain the real coordinates we will make need the following assumptions:

$$\begin{aligned} b + kq^{-1}E = 0, \quad m = b, \quad a = E, \quad m = a, \\ \overline{b} + E\overline{k} = 0, \quad \overline{b} = m, \quad \overline{a} - qE\overline{k} = 0, \\ \overline{a} - m = 0, \end{aligned}$$

and substituying into (55), we get

$$\begin{aligned} \Psi^0 = \frac{1}{\sqrt{2}}(C+D) = \frac{1}{\sqrt{2}}\left\{k\overline{k}a^\alpha d_{\dot{\beta}} + \overline{k}k\lambda_+^{3/2} b_{\dot{\beta}}c^\alpha\right\} \\ \Psi^1 = \frac{1}{\sqrt{2}}(A+B) = 0, \\ \Psi^2 = \frac{1}{\sqrt{2}}(C-D) = \frac{1}{\sqrt{2}}\left\{k\overline{k}a^\alpha d_{\dot{\beta}} - \overline{k}k\lambda_+^{3/2} b_{\dot{\beta}}c^\alpha\right\}, \\ \Psi^4 = \frac{i}{\sqrt{2}}(A-B) = 0, \end{aligned} \tag{56}$$

and consequently we can obtain an *invariant Minkowski length* (see [17])

$$L = q^{-2}k^2\overline{k}^2\lambda_+^{3/2}a^\alpha d_{\dot{\beta}} b_{\dot{\beta}} c^\alpha. \tag{57}$$

We shall now prove the following theorem.

Theorem 5.1. *The following conditions are equivalent*

$$a^\alpha d_{\dot{\beta}} b_{\dot{\beta}} c^\alpha = b_{\dot{\beta}} c^\alpha a^\alpha d_{\dot{\beta}}, \tag{58}$$

$$a^\alpha d_{\dot{\beta}} = q^2\lambda_+^{3/2} b_{\dot{\beta}} c^\alpha. \tag{59}$$

Proof. Following [17], we introduce the q-four vectors by

$$AB = BA - q^{-1}\lambda_+ CD + qD^2, \tag{60}$$

$$BC = CB - q^{-1}\lambda_+ BD, \tag{61}$$

$$AC = CA + q^2 AD, \tag{62}$$

$$BD = q^2 DB, \tag{63}$$

$$AD = q^{-2} DA, \tag{64}$$

$$CD = DC. \tag{65}$$

Substituting (54) into (60), (61), (62), (63), (64) and (65) yields

$$\begin{aligned} 0 = & -q^{-1}\lambda_+ k\bar{k}a^\alpha d_{\dot\beta} + q^2\bar{k}k\lambda_+^{3/2} b_{\dot\beta}c^\alpha, & (66)\\ BC = & \ 0, & (67)\\ AC = & \ 0, & (68)\\ BD = & \ 0, & (69)\\ AD = & \ 0, & (70)\\ k\bar{k}a^\alpha d_{\dot\beta}\bar{k}k\lambda_+^{3/2} b_{\dot\beta}c^\alpha\lambda_+ = & \ \bar{k}k\lambda_+^{3/2} b_{\dot\beta}c^\alpha k\bar{k}a^\alpha d_{\dot\beta}, & (71)\end{aligned}$$

an easy computation shows that yields (58) and (59). ☑

6. Comments and suggestion for further work

There are further topics arising from this paper which are worth investigation. There is the problem of describing the differential equation

$$i\boldsymbol{D}_q \cdot \boldsymbol{\psi}_q + m\psi_0 = E\psi, \tag{72}$$

where $\boldsymbol{D}_q = \boldsymbol{\partial}_q - ie\boldsymbol{f}_q$ and $\boldsymbol{f}_q$ is a arbitrary function and e parameter. The differential operator $\boldsymbol{D}_q$ is called the *q-covariant derivative*. Other suggestion for further work, is the problem of describing the q-Dirac differential and integral operators for the spinorial variable function motivated by [16]

$$\begin{aligned} D^q_\mu \psi = & \ \frac{\partial^q \psi}{\partial^q u^\alpha_{\dot\beta}} D^q_\mu u^\alpha_{\dot\beta}, \\ \oint_{\Gamma_q} \frac{\psi((qu)^\alpha_{\dot\beta}(x_\mu)) D^q_\mu u^\alpha_{\dot\beta}}{(qu)^\alpha_{\dot\beta}(x_\mu) - qu^\alpha_{\dot\beta}(x_0)} = & \ \sum_{n=0}^{\infty} \left[\gamma_\mu \psi(qu^\alpha_{\dot\beta}(x_0))\right]^n, \\ \oint_{\Gamma_q} \frac{\psi(u^\alpha_{\dot\beta}(x_\mu)) D^q_\mu u^\alpha_{\dot\beta}}{qu^\alpha_{\dot\beta}(x_\mu) - (qu)^\alpha_{\dot\beta}(x_0)} = & \ \frac{1}{q}\sum_{n=0}^{\infty} \left[\gamma_\mu \psi((qu)^\alpha_{\dot\beta}(x_0))\right]^n, \end{aligned}$$

and the solution for the differential equation in q-spinorial variables

$$D^q_\mu \psi(u^\alpha_{\dot\beta}) - e\gamma^\mu A^q_\mu(x)\psi(u^\alpha_{\dot\beta}) - m\varphi(u^\alpha_{\dot\beta}) = 0, \quad e \in R,$$

where Γ_q is a q-complex closed contour, and γ^μ the Dirac matrices, and m a constant.

References

[1] Gordon W., “Der Comptoneffekt nach der Schrödingerschen Theorie”, *Z.* Physik, 40 (1926), No. 1–2, 117–133. doi: 10.1007/BF01390840

[2] Klein O., “Quantentheorie und fünfdimensionale Relativitätstheorie”, *Z.* Physik, 37 (1926), No. 12, 895–906. doi: 10.1007/BF01397481

[3] Petiau G., Thesis (Ph.D.), University of Paris, Paris, 1936, *A*cad. R. Belg. Cl. Sci. Mem, Collect. (8) 16, No. 2. (1936).

[4] Duffin R., “On The Characteristic Matrices of Covariant Systems”, *P*hys. Rev. C, 54 (1938), No. 12, 1114. doi: 10.1103/PhysRev.54.1114

[5] Kemmer N., “The Particle Aspect of Meson Theory”, *P*roc. R. Soc. Lond. A, 173 (1939), 91–116. doi: 10.1098/rspa.1939.0131

[6] Okninski A., “Duffin-Kemmer-Petiau and Dirac Equations-A Supersymmetric Connection”, *S*ym, 4 (2012), No. 3, 427–440. doi: 10.3390/sym4030427.

[7] Proca A.J., “Sur la théorie ondulatoire des électrons positifs et négatifs”, *J.* Phys. Radium, 7 (1936), No. 8, 347–353. doi: 10.1051/jphysrad:0193600708034700

[8] Dirac P.A.M., “The quantum theory of the electron”, *P*roc. Roy. Soc. Lond. A, 117 (1928), 610–624. doi: 10.1098/rspa.1928.0023.

[9] Fierz M. and Pauli W., “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field”, *P*roc. Roy. Soc. Lond. A, 173 (1939), 211–232. doi: 10.1098/rspa.1939.0140.

[10] Rarita W. and Schwinger J., “On a Theory of Particles with Half-Integral Spin”, *P*hys. Rev., 60 (1941), No. 1, 61. doi: 10.1103/PhysRev.60.61

[11] Bhabha H.J., “Relativistic Wave Equations for the Elementary Particles”, *R*ev. Mod. Phys., 17 (1945), No. 2–3, 200–216. doi: 10.1103/RevModPhys.17.200

[12] Niederle J. and Nikitin A., “Relativistic wave equations for interacting massive particles with arbitrary half-intreger spins”, *P*hys. Rev. D, 64 (2001), 125013–125024. doi: 10.1103/PhysRevD.64.125013.

[13] Silenko A., “Verification of Relativistic Wave Equations for Spin-1 Particles”, arXiv:hep-th/0401183v1.

[14] Kulikov D. and Titik R., “Relativistic wave equation for one spin-1/2 and one spin-0 particle”, International School-Seminar on New Physics and Quantum Chromodynamics of External Conditions, Dnipropetrovsk, Ukraine, 148–151, May, 2007.

[15] Pillin M., “q-Deformed Relativistic Wave Equations”, *J.* Math. Phys., 35 (1994), No. 6, 2804–2817. doi: 10.1063/1.530487

[16] Jaramillo J.C., “An Introduction to Spinor Differential and Integral Calculus from $q-$ Lorentzian Algebra”, *R*ev. Integr. Temas Mat., 41 (2023), No. 1, 45–60. doi: 10.18273/revint.v41n1-2023003

[17] Schmidke W.B., Wess J. and Zumino B., “A $q-$ deformed Lorentz algebra”, *Z.* Phys. C Particles and Fields, 52 (1991), No. 3, 471–476. doi: 10.1007/BF01559443

[18] Schmidt A. and Wachter H., “Spinor calculus for q-deformed quantum spaces I”, arXiv:0705.1640.

[19] Beretetskii V.B., Lifshitz E.M. and Pitaekskii L.P., *R*elativistic Quantum Theory, course of theoretical physics part 1, Pergamon press, vol. 4, 1971. doi: 10.1088/0031-9112/22/9/025

[20] Mansour T. and Schork M., *Commutation Relations, Normal Ordering, and Stirling Numbers*, CRC press Taylor and Francis group, 2016, 229. doi: 10.1201/b18869