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Fibonacci and Lucas numbers of the form $-2^a - 3^b - 5^c + 7^d$

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Abstract. In this note we find all Fibonacci and Lucas numbers of the form $-2^a - 3^b - 5^c + 7^d$ where a, b, c, d are non-negative integers, with $0 \leq \max\{a, b, c\} \leq d$. This result gives an answer to a question posed by Qu, Zeng and Cao.

Keywords: Fibonacci and Lucas sequences, linear form in logarithms, reduction method.

MSC2020: 11B39, 11D04, 11D45.

Números de Fibonacci y Lucas de la forma $-2^a - 3^b - 5^c + 7^d$

Resumen. En esta nota se encuentran todos los números de Fibonacci y de Lucas de la forma $-2^a - 3^b - 5^c + 7^d$, en donde a, b, c y d son enteros no negativos con $0 \leq \text{máx}\{a, b, c\} \leq d$. Este resultado da respuesta a una pregunta de Qu, Zeng y Cao.

Palabras clave: Números de Fibonacci y Lucas, formas lineales en logaritmos, método de reducción.

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci numbers defined by the recurrence $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0,\ F_1 = 1$. Let $(L_n)_{n\geq 0}$ be the Lucas sequence that has the same recurrence formula as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$. The study of diophantine equations that involves Fibonacci or Lucas numbers is a very rich area of research and has attracted the attention of many researchers, see, e.g., [3, 6, 7, 9, 10, 11, 13] and the references therein. For example, Luo [11] proved that $1, 2, 21$, and 55 are the unique Fibonacci numbers that are also triangular numbers, and some years later he also found all Lucas numbers that are also triangular numbers [12]. Marques and Togbé [13] found all the Fibonacci and Lucas numbers that are of the form $2^a+3^b+5^c$, with $0 \leq \max\{a, b\} \leq c$. Later, Qu, Zeng and Cao [8] found all the Fibonacci and Lucas numbers that are of the form $2^a+3^b+5^c+7^d$, with $0 \leq \max\{a, b, c\} \leq d$, and posted the problem of finding Fibonacci and Lucas numbers of the form $-2^a-3^b-5^c+7^d$, with $0 \leq \max\{a, b, c\} \leq d$. In this note we solve this problem. Our main results are the following two theorems:

Theorem 1.1. *All non-negative integer solutions* (n, a, b, c, d) *of the Diophantine equation*

$$F_n = -2^a - 3^b - 5^c + 7^d \tag{1}$$

with $0 \leq \max\{a, b, c\} \leq d$ *belong to the set*

$$\left\{ \begin{array}{llll} (0,0,0,1,1), & (1,1,1,0,1), & (2,1,1,0,1), & (3,0,1,0,1), \\ (4,1,0,0,1), & (7,1,2,2,2), & (8,1,0,2,2), & (9,0,2,1,2), \end{array} \right\}.$$

Theorem 1.2. *All non-negative integer solutions* (n, a, b, c, d) *of the Diophantine equation*

$$L_n = -2^a - 3^b - 5^c + 7^d \tag{2}$$

with $0 \leq \max\{a, b, c\} \leq d$ *belong to the set*

$$\{(0,0,1,0,1), (1,1,1,0,1), (2,1,0,0,1), (3,0,0,0,1), (5,2,2,2,2)\}.$$

2. Preliminaries and tools

In this section we present several known results that we will use in our proofs. First, let's remember some properties of Fibonacci and Lucas sequences.

Let $\gamma := \frac{1+\sqrt{5}}{2}$ and $\mu := \frac{1-\sqrt{5}}{2}$. The numbers γ and μ are the roots of the characteristic polynomial $x^2 - x - 1 = 0$. The well-known Binet's formulas are

$$F_n = \frac{\gamma^n - \mu^n}{\sqrt{5}} \quad \text{and} \quad L_n = \gamma^n + \mu^n, \tag{3}$$

which holds for all $n \geq 0$. Also, the inequalities

$$\gamma^{n-2} \leq F_n \leq \gamma^{n-1} \quad \text{and} \quad \gamma^{n-1} \leq L_n \leq 2\gamma^n \tag{4}$$

holds for all positive integers n.

Let α be an algebraic number of degree d. Let a be the leading coefficient of its minimal polynomial (over $\mathbb{Z}$) and let $\alpha_1, \ldots, \alpha_d$ denote the conjugates of α, with $\alpha_1 = \alpha$. The logarithmic height of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d}\log\max\{1, |\alpha_i|\}\right).$$

The following result is a lower bound for a linear form in logarithms due to Matveev [14].

Lemma 2.1. *Let* $\mathbf{L}$ *be a real number field of degree* $d_{\mathbf{L}}$, $\alpha_1, \ldots, \alpha_\ell \in \mathbf{L}$ *and let* $b_1, \ldots, b_\ell$ *be non-zero integers. Let* $B \geq \max\{|b_1|, \ldots, |b_\ell|\}$. *Let* $A_1, \ldots, A_\ell$ *be real numbers satisfying*

$$A_i \geq \max\{\ d_{\mathbf{L}}\, h(\alpha_i),\ |\log\alpha_i|,\ 0.16\} \qquad \text{for all} \qquad i = 1, \ldots, \ell.$$

If $\alpha_1^{b_1}\cdots\alpha_\ell^{b_\ell} \neq 1$. *Then*

$$|\alpha_1^{b_1}\cdots\alpha_\ell^{b_\ell} - 1| > \exp\left(-1.4 \times 30^{\ell+3} \times \ell^{4.5} \times d_{\mathbf{L}}^2(1 + \log d_{\mathbf{L}})(1 + \log B)\ A_1\cdots A_\ell\right).$$

To reduce even more the bounds obtained with Matveev's result we use a version of Baker-Davenport lemma based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca [2] that is a slightly variation of the one given by Dujella and Petho [4]. For a real number x, we write $\|x\| = \min\{|x - n|\ :\ n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 2.2. *Let* M *be a positive integer. Let* $\alpha,\ \tau,\ A > 0,\ B > 1$ *be given real numbers. Let* p/q *be a convergent of* α *such that* $q > 6M$ *and* $\varepsilon := \|q\tau\| - M\|q\alpha\| > 0$. *Then the inequality*

$$0 < |n\,\alpha - m + \tau| < \frac{A}{B^w}$$

does not have a solution in positive integers $n,\ m$ *and* w *in the ranges*

$$n \leq M \qquad \text{and} \qquad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

We also need the following result (Lemma 7 in [15]).

Lemma 2.3. *If* $m \geq 1$, $T > (4m^2)^m$ *and* $T > x/(\log x)^m$, *then*

$$x < 2^m T(\log T)^m.$$

3. Proof of Theorem 1

In order to simplify some calculations, with a Mathematica's program we have checked all the solutions for equation (1) in the range $0 \leq d \leq n \leq 20$ and $0 \leq n \leq d \leq 20$, that in fact are the solutions that appear in the statement of Theorem 1.1. So in the rest in the proof we assume that $\max\{n, d\} > 20$. We start working with equation (1) and the first inequality of (4). From inequality $\gamma^{n-2} \leq F_n$ we have that $\gamma^{n-2} \leq F_n < 7^d$ which implies that $0.24(n-1) < d$. From $F_n \leq \gamma^{n-1}$ we obtain that $7^d \leq \gamma^{n-1} + 3 \cdot 5^d < 4 \cdot 5^d \cdot \gamma^{n-1}$ and this implies that

$$d < \frac{\log(\gamma)(n-1) + \log(4)}{\log(7/5)} < 1.44(n-1) + 4.13 < 3n,$$

because $n > 2$. So we conclude that

$$0.24n - 0.24 < d < 3n.$$

By using equation (1) and Binet's formula we obtain

$$\frac{\gamma^n}{\sqrt{5}} - 7^d = -(2^a + 3^b + 5^c) + \frac{\mu^n}{\sqrt{5}} < 0,$$

because $|\mu| < 1$ and $2^a \geq 1$. Now

$$\frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1 = -\frac{(2^a + 3^b + 5^c)}{7^d} + \frac{\mu^n}{7^d\sqrt{5}} < 0.$$

By using that $x^d/7^d \leq 1/7^{0.1d}$, for every $x \in \{2,3,5\}$, and that $|\mu^n/(7^d\sqrt{5})| < 1/7^{0.1d}$, we obtain

$$\left|\frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1\right| < \frac{4}{7^{0.1d}}. \tag{5}$$

Notice that this inequality is the same obtained by the authors of [8] and we can obtain equation (6) in the same way as they do. For the reader's convenience we repeated the calculations. We take $\ell := 3$, $\gamma_1 := \gamma$, $\gamma_2 := 7$, $\gamma_3 := \sqrt{5}$ and $b_1 := n$, $b_2 := -d$, $b_3 := -1$. Then $d_{\mathbf{L}} = [\mathbb{Q}(\sqrt{5})\colon \mathbb{Q})] = 2$. Now, $h(\gamma_1) = 1/2 \log \gamma$, $h(\gamma_3) = \log 7$, $h(\gamma_3) = \log \sqrt{5}$, and hence we can take $A_1 := 0.5$, $A_2 := 3.9$ and $A_3 := 1.7$. Let $R := \max\{|b_1|, |b_2|, |b_3|\} = \max\{n, d, 1\}$. By Matveev's result (Lemma 2.1) we have that

$$\left|\frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1\right| > \exp\left(-C\left(1 + \log R\right)\right), \tag{6}$$

where $C = 3.22 \times 10^{12}$. We have two cases

Case 1. $R = n$.

From equations (5) and (6) we obtain

$$\frac{4}{7^{0.1d}} > \exp\left(-C\left(1 + \log n\right)\right). \tag{7}$$

Taking logarithms in equation (7) and using that $\log\gamma/\log 7(n-1) < d$ we obtain, after some straightforward calculations that

$$\frac{n}{\log n} < 2\left(\frac{C}{0.1\log\gamma} + \frac{1}{2}\right)$$

Now we use Lemma 2.3 to obtain that $\max\{d, n\} < 8.8 \times 10^{15}$.

Case 2. $R = d$, that is $n \leq d$.

In this case, after taking logarithms to equation (6) we get

$$\log 4 - 0.1 \log 7d > -C(1 + \log d),$$

that is

$$0.1 \log 7d - \log 4 < C(1 + \log d) < c(2 \log d),$$

because $d \geq 3$. After some straightforward calculations we obtain

$$\frac{d}{2 \log n} < 2 \left(\frac{C}{0.1 \log 7} + 8 \right),$$

and by using Lemma 2.3 we get

$$n \leq d < 2.06052 \times 10^{15}.$$

Now we will reduce the bounds previously obtained. Let

$$\Lambda_F := n \log \gamma - d \log 7 - \log \sqrt{5}.$$

Note that $\Lambda_F < 0$ because

$$e^{\Lambda_F} - 1 = \frac{\gamma^n 7^{-d}}{\sqrt{5}} - 1 = \frac{-(2^a + 3^b + 5^c) + \mu^n}{7^d} < 0.$$

We have that $1 - e^{\Lambda_F} = |e^{\Lambda_F} - 1| < 1/2$ which implies that $e^{\Lambda_F} < 2$. Therefore

$$0 < |\Lambda_F| < e^{|\Lambda_F|} - 1 = e^{|\Lambda_F|} |e^{\Lambda_F} - 1| < 2 \frac{4}{7^{0.1d}}.$$

Thus

$$|n \log \gamma - d \log 7 - \log \sqrt{5}| < \frac{8}{7^{0.1d}}.$$

Dividing by $\log 7$ we get

$$\left| n \frac{\log \gamma}{\log 7} - d - \frac{\log \sqrt{5}}{\log 7} \right| < \frac{8}{\log 7 \times 7^{0.1d}}.$$

Now we use Lemma 2.2 with $\alpha = \frac{\log \gamma}{\log 7}$, $m = d$, $\tau = -\frac{\log \sqrt{5}}{\log 7}$, $A = 8/\log 7$, $B = 1.2$ (in [8] was proved that α is irrational). We take $M = 8.8 \times 10^{15}$, and with the use of *Mathematica* we observe that $q_{39} = 119059818885400441$ (the denominator of the p_{39}/q_{39} convergent of α) satisfies $q_{39} > 6M$ and $\epsilon = 0.419601$. Therefore, if (n, a, b, c, d) is a solution in positive integers to equation 1, then $d < 229$ and hence $n < 928$. Applying again Lemma 2.2, but now with $M = 928$, we obtain $q_8 = 21064$ and $\epsilon = 0.07494895$ which implies that $d < 77$ and hence $n < 314$.

Finally, we use a program written in *Mathematica* to determine all the solutions in the range $d < 77$ and $n < 314$ considering both cases $n > d$ and $d \geq n$. In both cases all the solutions are given in the statement of Theorem 1.

4. Proof of theorem 2

With a *Mathematica*'s program we have checked all the solutions for equation (2) in the range $0 \leq d \leq n \leq 20$ and $0 \leq n \leq d \leq 20$. So, in the rest of the proof we assume that $\max\{n, d\} \geq 20$.

From Binet's formula for the Lucas sequence and equation (2) we obtain

$$\gamma^n - 7^d = -(2^a + 3^b + 5^c) - \mu^n < 0,$$

because $|\mu| < 1$ and $2^a \geq 1$. Now, dividing by 7^d we obtain

$$\gamma^n 7^{-d} - 1 = -\frac{(2^a + 3^b + 5^c)}{7^d} - \frac{\mu^n}{7^d} < 0.$$

From which we deduce that

$$\left|\gamma^n 7^{-d} - 1\right| < \frac{4}{7^{0.1d}}. \tag{8}$$

Now, combining the second inequality of (4) with (2) we obtain $\gamma^{n-1} < 7^d$ and $(7/5)^d < 5\gamma^n$, which together implies that $0.24n - 0.24 < d < 2n$ (for the second inequality we use that $n > 9$). Equation (8) is the same inequality obtained by Qu, Zeng and Cao [8] and hence we apply Matveev's result (Lemma 2.1) exactly as in [8] by taking $\ell := 2$, $\gamma_1 := \gamma$, $\gamma_2 := 7$, and $b_1 := n, b_2 := -d$. Thus $d_{\mathbf{L}} = [\mathbb{Q}(\sqrt{5})\colon \mathbb{Q}] = 2$ and $h(\gamma_1) = 1/2 \log \gamma$, $h(\gamma_2) = \log 7$. We can take $A_1 := 0.5$, $A_2 := 3.9$ and $B := \max\{n, d\}$. By using Lemma 2.1 and after some calculations we obtain

$$\left|\gamma^n 7^{-d} - 1\right| > \exp\left(-C(1 + \log B)\right). \tag{9}$$

where $C = 1.02 \times 10^{10}$. We use equations (8) and (9) to obtain

$$d < \frac{C(1 + \log B) + \log 4}{0.1 \log 7}. \tag{10}$$

Now, we proceed by cases.

Case 1: $d \geq n$

To simplify equation (10) we use that $1 + \log d < 2 \log d$ (because $d > 3$). After some calculations we obtain $d/\log d < (2C + \log 4)/(0.1 \log 7)$ and by using Lemma 2.2 we obtain $d < 5.33 \times 10^{12}$.

Case 2: $n \geq d$**.**

We have that

$$0.24n - 0.24 < d < \frac{C(1 + \log n) + \log 4}{0.1 \log 7}$$

After some straightforward calculations (and using that $n > 3$) we apply Lemma 2.2 to obtain $n < 2.27 \times 10^{13}$.

Now we will reduce the bounds. As in [8], we define

$$\Lambda_L := n \log \gamma - d \log 7.$$

Notice that $\Lambda_L < 0$. For $d > 3$, $|e^{\Lambda_L} - 1| < 1/2$, which implies that $|e^{\Lambda_L}| < 2$. Let $\Lambda := e^{\Lambda_L} - 1$. Then we have

$$0 < |\Lambda_L| < |e^{|\Lambda_L|} - 1| = e^{|\Lambda_L|}|\Lambda| < 2 \times \frac{4}{7^{0.1}d}.$$

Dividing by $\log 7$ we obtain

$$0 < \left|\frac{n \log \gamma}{\log 7} - d\right| < \frac{4.2}{1.2^d}. \tag{11}$$

Now we proceed in a similar way that in [8]. Let $[a_0, a_1, \ldots,] = [0, 4, 22, 1, 5, \ldots]$ be the continued fraction of $\log \gamma / \log 7$, and let p_i/q_i be its ith convergent. We have obtained that $n < 2.27 \times 10^{13}$ and using *Mathematica* we observe that $q_{29} < 2.27 \times 10^{13} < q_{30}$. If $a_M := \max\{a_i \colon i = 0, 1, \ldots, 30\}$, then $a_M = a_{14} = 35$. Now we use properties of continued fractions similarly as in [8] and [5, page 10] to obtain

$$\left|\frac{n \log \gamma}{\log 7} - d\right| > \frac{1}{(a_M + 2)n} = \frac{1}{37n}. \tag{12}$$

From equations (11) and (12) we have

$$\frac{1}{37n} < \left|\frac{n \log \gamma}{\log 7} - d\right| < \frac{4.2}{1.2^d},$$

and hence

$$\frac{1}{37n} < \frac{4.2}{1.2^d},$$

from which, by using $n < 2.27 \times 10^{13}$, we obtain

$$d < \frac{\log(4.2 \times 37n)}{\log 1.2} < 197. \tag{13}$$

Now we proceed by cases.

Case 1: If $n \le d$, then $n < 197$, and using that $d < 2n$ we obtain $98 < n < 197$.

Case 2: When $d \le n$ we use $0.24n - 0.24 < d$ to obtain $n < 822$. And using this last inequality we obtain

$$d < \frac{\log(4.2 \times 37n)}{\log 1.2} < 65. \tag{14}$$

and hence $136 < n < 272$. We write a program in *Mathematica* to obtain all the solutions in the ranges obtained in Case 1 and Case 2 and the results are showed in the statement of Theorem 2.

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