Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 42, N° 2, 2024, pág. 11–21

An approach to derivatives for non-monogenic functions

Julio Cesar Jaramillo-Quiceno ✉,
Universidad Nacional de Colombia, Bogotá, Colombia.
Departamento de Física y Facultad de Matemáticas, Edificio Yu-Takeuchi.

Abstract. In this paper we introduce the derivatives for non-monogenic functions. We establish the derivative for non-monogenic functions on the Dirac operator. We also propose a new type of difference operator for non-monogenic function and new type of derivative.

Keywords: Non-monogenic functions, Dirac operator, new difference operator, new type of derivative.

MSC2020: 81Q99, 46E99, 35A24, 15A66, 16T99, 17B37.

Una aproximación a las derivadas para las funciones no-monogénicas

Resumen. En este artículo introducimos las derivadas para las funciones no monogénicas. Establecemos la derivada para las funciones no-monogénicas para el operador de Dirac. También proponemos un nuevo tipo de operador diferencial para las funciones no monogénicas y un nuevo tipo de derivada.

Palabras clave: Funciones no-monogénicas, operador de Dirac, nuevo operador diferencial, nuevo tipo de derivada.

E-mail: jcjaramilloq@unal.edu.co✉.
Received: 10 June 2024, Accepted: 24 July 2024.
To cite this article: J.C. Jaramillo-Quiceno, An approach to derivatives for non-monogenic functions, *Rev. Integr. Temas Mat.,* 42 (2024), No. 2, 11-21. doi: 10.18273/revint.v42n2-2024002

1. Introduction

Definition 1.1. [2] The real Clifford algebra $\mathcal{A}_m$ is a real vector space with 2^m basis elements $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{2^m-1}$, defined by

$$\begin{aligned} \mathbf{e}_0 \equiv e_0 = 1, \mathbf{e}_1 = &\ e_1, \dots, \mathbf{e}_m = e_m, \\ \mathbf{e}_{12} = e_1e_2, \mathbf{e}_{13} = e_1e_3, \dots, \mathbf{e}_{m-1,m} = &\ e_{m-1}e_m, \dots, \mathbf{e}_{12\dots m} = e_1e_2\dots e_m, \end{aligned}$$

and let $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{12}, \mathbf{e}_{13}, \dots, \mathbf{e}_{m-1,m}, \mathbf{e}_{12\dots m}\}$ be a basis of $\mathbb{R}^m$. The multiplication in $\mathcal{A}_m$ is given by the rule

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}e_0, \quad \alpha, \beta = 1, 2, \dots, m. \tag{1}$$

Definition 1.2. [3] Every element $a = \sum\limits_\alpha a_\alpha \mathbf{e}_\alpha$ $(a_\alpha \in \mathbb{R})$ is called a Clifford number. A product of two Clifford numbers $a = \sum\limits_\alpha a_\alpha \mathbf{e}_\alpha, b = \sum\limits_\beta b_\beta \mathbf{e}_\beta$ is defined by the formula

$$ab = \sum_\alpha \sum_\beta a_\alpha b_\beta \mathbf{e}_\alpha \mathbf{e}_\beta, \tag{2}$$

and their conjugate complex $\overline{a} = \sum\limits_\alpha a_\alpha \overline{\mathbf{e}}_\alpha, \overline{b} = \sum\limits_\beta b_\beta \overline{\mathbf{e}}_\beta$.

In a standar way, we can define a Clifford algebra valued function $f : \mathbb{R}^{m+1} \longrightarrow \mathcal{A}_m$ by the formula (see [2] for more details)

$$f(x) = \sum_\alpha f_\alpha(x)\mathbf{e}_\alpha, \tag{3}$$

with $f_\alpha : \Omega \longrightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}$ is a open set, and we denote by $\mathbf{C}^n(\Omega, \mathcal{A})$ the space of all Clifford algebra valued functions (3) which are n times differentiable in some open connected set $\Omega \subseteq \mathbb{R}^{m+1}$ [2]. The conjugate of $f(x)$ to be the function $\overline{f}(x)$ given by the formula

$$\overline{f}(x) = \sum_{k=0}^{m} (-1)^k \sum_{\alpha=k} f_\alpha(x)\mathbf{e}_\alpha. \tag{4}$$

On other hand, the generalization of Cauchy-Riemann operator is given by

$$\mathcal{D} = \sum_{\beta=0}^{m} \mathbf{e}_\beta \frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial x_0} + \sum_{\beta=1}^{m} \mathbf{e}_\beta \frac{\partial}{\partial x_\beta}, \tag{5}$$

the second term correspond to *Dirac Operator* [1] , [4], which we will denote by $\boldsymbol{D}$.

Definition 1.3. [4] The Clifford algebra valued function $f(x) \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ is called monogenic (Left monogenic function) in Ω if and only if $\boldsymbol{D}f = 0$, that is to say

$$\boldsymbol{D}f = \sum_{\beta,\alpha=1}^{m} \mathbf{e}_\beta \mathbf{e}_\alpha \frac{\partial f_\alpha}{\partial x_\beta} = 0, \tag{6}$$

and while g is called right monogenic function

$$g\boldsymbol{D} = \sum_{\beta,\alpha=1}^{m} \mathbf{e}_\alpha \mathbf{e}_\beta \frac{\partial g_\alpha}{\partial x_\beta} = 0. \tag{7}$$

Remark 1.4. The expressions (6) and (7) are defined on the Dirac operator.

This paper are organized as follows: in section 2, we present the motivation. In section 3, we present the derivative for non-monogenic functions on the Dirac operator. In the section 4, we present a new type of difference operator for non-monogenic function. In the section 5, we introduce the derivative for non-monogenic function and in the final section, we presents some suggestions for further studies are presented.

2. Motivation

The topic of this article are derivatives for non-monogenic functions and the motivation comes from the study of the monogenic function in the Dirac operator (see the references [1], [2]). In accordance with the above, our interest here is to define the derivatives for non-monogenic functions in the Dirac operator $\boldsymbol{D}f \neq 0$ ($f\boldsymbol{D} \neq 0$), and introduce a new type of difference operator similary to Dirac operator such that satisfies the property $D^2 = -\nabla^2$ subject to some commutation relations different to Clifford relation (1) and consequently study their respective derivatives.

3. Derivative for non-monogenic functions on the Dirac operator

Remark 3.1. Let be given a open $\Omega \subset \mathbb{R}^{m+1}$. A function $f \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ is said to be left (right) non-monogenic in Ω if and only if $\boldsymbol{D}f \neq 0 \quad (f\boldsymbol{D} \neq 0)$ in Ω, more exactly

$$\boldsymbol{D}f = \sum_{\beta,\alpha=1}^{m} \mathbf{e}_\beta \mathbf{e}_\alpha \frac{\partial f_\alpha}{\partial x_\beta} \neq 0, \tag{8}$$

$$\overline{\boldsymbol{D}}f = \sum_{\beta,\alpha=1}^{m} \overline{\mathbf{e}}_\beta \overline{\mathbf{e}}_\alpha \frac{\partial f_\alpha}{\partial x_\beta} \neq 0, \tag{9}$$

Definition 3.2. Let $f \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ be a non-monogenic function in Ω. The derivative is defined by

$$\frac{\partial f_\alpha}{\partial x_\beta}(\mathbf{e}_\sigma) = \frac{f_\alpha(\mathbf{e}_\alpha \mathbf{e}_\sigma x_\beta) - f_\alpha(x_\beta)}{\mathbf{e}_\alpha \mathbf{e}_\sigma x_\beta - x_\beta}, \tag{10}$$

and its conjugate complex

$$\frac{\overline{\partial} f_\alpha}{\partial x_\beta}(\mathbf{e}_\sigma) = \frac{f_\alpha(\overline{\mathbf{e}}_\alpha\overline{\mathbf{e}}_\sigma x_\beta) - f_\alpha(x_\beta)}{\overline{\mathbf{e}}_\alpha\overline{\mathbf{e}}_\sigma x_\beta - x_\beta}. \tag{11}$$

for all $\sigma = 1, 2, \ldots, m$.

Lemma 3.3. *For the Dirac operator we have*

$$\boldsymbol{D}f = \sum_{\beta,\alpha=1}^{m} \mathbf{e}_\beta\mathbf{e}_\alpha \left[\frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) - f_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta}\right], \tag{12}$$

and its conjugate complex

$$\overline{\boldsymbol{D}}f = \sum_{\beta,\alpha=1}^{m} \overline{\mathbf{e}}_\beta\overline{\mathbf{e}}_\alpha \left[\frac{f_\alpha(\overline{\mathbf{e}}_\alpha\overline{\mathbf{e}}_\sigma x_\beta) - f_\alpha(x_\beta)}{\overline{\mathbf{e}}_\alpha\overline{\mathbf{e}}_\sigma x_\beta - x_\beta}\right]. \tag{13}$$

Proof. It is sufficient to replace (10) into (8), and (11) into (9). ☑

Theorem 3.4. *Assume that $f_\alpha \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ and $g_\alpha \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ are a non-monogenic functions in Ω for all $\alpha = 1, 2, \ldots, m$ and are differentiable in $x_\beta \in \Omega$ then*

i The product $f_\alpha g_\alpha \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ is differentiable at x_β and

$$\frac{\partial}{\partial x_\beta}(f_\alpha g_\alpha)(\mathbf{e}_\sigma) = f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)\frac{\partial g_\alpha}{\partial x_\beta} + g_\alpha(x_\beta)\frac{\partial f_\alpha}{\partial x_\beta}.$$

ii f_α/g_α is differentiable at x_β and

$$\frac{\partial}{\partial x_\beta}(f_\alpha/g_\alpha)(\mathbf{e}_\sigma) = \frac{g_\alpha(x_\beta)\frac{\partial f_\alpha}{\partial x_\beta} - f_\alpha(x_\beta)\frac{\partial g_\alpha}{\partial x_\beta}}{g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta)}, \quad g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta) \neq 0$$

iii

$$\frac{\partial}{\partial x_\beta}[(x_\beta)^n] = \sum_{k=1}^{n}(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)^{n-k}(x_\beta)^{k-1},$$

and similar arguments apply to conjugate complex.

Proof. *i*

$$\begin{aligned}
\frac{\partial}{\partial x_\beta}(f_\alpha g_\alpha)(\mathbf{e}_\sigma) &= \frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) - f_\alpha(x_\beta)g_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta} \\
&= \frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) + f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta) - f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta) - f_\alpha(x_\beta)g_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta} \\
&= f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)\left[\frac{g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) - g_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta}\right] + g_\alpha(x_\beta)\left[\frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) - f_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta}\right], \\
&= f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)\frac{\partial g_\alpha}{\partial x_\beta} + g_\alpha(x_\beta)\frac{\partial f_\alpha}{\partial x_\beta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_\beta}(f_\alpha/g_\alpha)(\mathbf{e}_\sigma) &= \frac{\frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)}{g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)} - \frac{f_\alpha(x_\beta)}{g_\alpha(x_\beta)}}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta} \\
&= \frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta) - g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)f_\alpha(x_\beta)}{(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta)[g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta)]} \\
&= \frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta) - f_\alpha(x_\beta)g_\alpha(x_\beta) + f_\alpha(x_\beta)g_\alpha(x_\beta) - g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)f_\alpha(x_\beta)}{(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta)[g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta)]} \\
&= \frac{g_\alpha(x_\beta)\left[\frac{f_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) - f_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta}\right] - f_\alpha(x_\beta)\left[\frac{g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta) - g_\alpha(x_\beta)}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta}\right]}{g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta)} \\
&= \frac{g_\alpha(x_\beta)\frac{\partial f_\alpha}{\partial x_\beta} - f_\alpha(x_\beta)\frac{\partial g_\alpha}{\partial x_\beta}}{g_\alpha(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)g_\alpha(x_\beta)}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_\beta}[(x_\beta)^n] &= \frac{(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)^n - x_\beta^n}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta - x_\beta} \\
&= (\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)^{n-1} + (\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)^{n-2}x_\beta + \cdots + x_\beta^{n-1} \\
&= \sum_{k=1}^{n}(\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta)^{n-k}x_\beta^{n-k}.
\end{aligned}$$

☑

Example 3.5. *1. For $x_\beta \neq 0$, $\frac{\partial}{\partial x_\beta}(1/x_\beta) = -\frac{1}{\mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta^2}$.*

2. If $f \in \mathbf{C}^1(\Omega, \mathcal{A}_m)$ defined by $f_\alpha(x_\beta) = (x_\beta)^2 + x_\beta$, then

$$\frac{\partial f_\alpha}{\partial x_\beta} = \mathbf{e}_\alpha\mathbf{e}_\sigma x_\beta + x_\beta + 1.$$

Now, we introduce a new type of difference operator for non monogenic functions as follows.

4. A new type of difference operator for non-monogenic function

Definition 4.1. Let us consider the real algebra $\mathcal{B}_p$ is a real vector space with 2^p basis elements $\mathrm{e}_1, \ldots, \mathrm{e}_{2^p-1}$, and let $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \ldots, \mathrm{e}_p\}$ be a basis of $\mathbb{R}^p$. The multiplication in $\mathcal{B}_p$ are given by the rules

$$\mathrm{e}_n\mathrm{e}_m + q_{nm}\mathrm{e}_m\mathrm{e}_n = \delta_{mn} \tag{14}$$

$$\mathrm{e}_n\mathrm{e}_m + \mathrm{e}_m\mathrm{e}_n = 2(1 + q_{mn}) \quad m, n = 1, 2, \ldots, p, \tag{15}$$

being q_{nm}

$$q_{nm} = \begin{cases} -1 & m \neq n, \\ 0 & m = n \end{cases}. \tag{16}$$

Lemma 4.2. *For $m = n$ we have $\mathrm{e}_n^2 = 1$ and $\mathrm{e}_m = 1$, and for $m \neq n$ also we have* $\mathrm{e}_n\mathrm{e}_m - \mathrm{e}_m\mathrm{e}_n = 0$.

As a direct consequence we obtain the following definition.

Definition 4.3. Let $e_1, e_2, e_3, \ldots, e_p$, be elements that satisfy (14),
Thus the difference operator D is defined as

$$D = \left[e_n \frac{\partial}{\partial x_m} + e_m \frac{\partial}{\partial x_n} \right], \tag{17}$$

which is subject to

$$\begin{aligned} & e_n \frac{\partial}{\partial x_m} e_m \frac{\partial}{\partial x_n} + e_m \frac{\partial}{\partial x_n} e_n \frac{\partial}{\partial x_m} = \\ & - e_n \frac{\partial}{\partial x_m} \left[\left(e_n \frac{\partial}{\partial x_m} \right) \delta_{mn} \right] - e_m \frac{\partial}{\partial x_n} \left[\left(e_m \frac{\partial}{\partial x_n} \right) \delta_{mn} \right] - (1 - q_{mn}) \left(\frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial x_n^2} \right), \end{aligned} \tag{18}$$

and its conjugate

$$\overline{D} = \left[\overline{e}_n \frac{\overline{\partial}}{\partial x_m} + \overline{e}_m \frac{\overline{\partial}}{\partial x_n} \right]. \tag{19}$$

Theorem 4.4. *Let us consider the differential operator*

$$\nabla^2 = \frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial x_n^2} \tag{20}$$

on $\mathbb{R}^p$. *If we try to find a square root of this operator of the form (17), the* $D^2 = -\nabla^2$ *leads to equations* $e_n^2 = 1, e_m^2 = 1$, *(14), (15) and (18).*

Proof. For $m = n$ we have

$$\begin{aligned} D^2 = & \left(e_n \frac{\partial}{\partial x_m} + e_m \frac{\partial}{\partial x_n} \right) \left(e_n \frac{\partial}{\partial x_m} + e_m \frac{\partial}{\partial x_n} \right), \\ = & \; e_n^2 \frac{\partial^2}{\partial x_m^2} + e_n \frac{\partial}{\partial x_m} e_m \frac{\partial}{\partial x_n} + e_m \frac{\partial}{\partial x_n} e_n \frac{\partial}{\partial x_m} + e_m^2 \frac{\partial^2}{\partial x_n^2}, \end{aligned}$$

using (18) we get

$$\begin{aligned} D^2 = & \frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial x_n^2} - e_m \frac{\partial}{\partial x_n} \left[e_\alpha \delta_{m\alpha} \frac{\partial}{\partial x_\beta} \delta_{\beta n} \right] - e_n \frac{\partial}{\partial x_m} \left[e_\beta \delta_{n\beta} \frac{\partial}{\partial x_\alpha} \delta_{m\alpha} \right] \\ & - (1 - 0) \left(\frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_m^2} \right), \end{aligned}$$

and from Lemma 4.2 and (15), finally we obtain

$$D^2 = -\frac{\partial^2}{\partial x_m^2} - \frac{\partial^2}{\partial x_n^2}.$$

For $m \neq n$ we have

$$\begin{aligned} D^2 = {} & \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_m^2} - \mathrm{e}_n \frac{\partial}{\partial x_m}\left(\mathrm{e}_m \frac{\partial}{\partial x_n}\right) - \\ & \mathrm{e}_m \frac{\partial}{\partial x_n}\left(\mathrm{e}_n \frac{\partial}{\partial x_m}\right) - 2\left(\frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial x_n^2}\right), \\ D^2 = {} & \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_m^2} - \\ & (\mathrm{e}_n\mathrm{e}_m + \mathrm{e}_m\mathrm{e}_n)\frac{\partial^2}{\partial x_m \partial x_n} - 2\left(\frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial x_n^2}\right), \end{aligned}$$

and from (15) finally we obtain

$$D^2 = -\frac{\partial^2}{\partial x_m^2} - \frac{\partial^2}{\partial x_n^2}.$$

☑

The next definition gives us an analogous formula of functions according to expressions (14) and (15) to the Clifford algebra valued function.

Definition 4.5. Let $\mathbf{C}^p(\Psi, \mathcal{B}_p)$ be a space of all algebra valued functions following the structure of the relation (14). According to the above , we can stablish the valued functions of the following form

$$f(x) = \sum_{\alpha,\beta} \left[f_\gamma(x)\mathrm{e}_\sigma + f_\sigma(x)\mathrm{e}_\gamma\right], \tag{21}$$

for all $\sigma, \gamma \in \mathbb{N}$. Where Ψ denotes the open such that $\Psi \subset \mathbb{R}^{p+2}$.

One can see their respective conjugate in the following remark.

Remark 4.6. The conjugate of f is given by

$$\overline{f}(x) = \sum_{\sigma,\gamma} \left[f_\gamma(x)\overline{\mathrm{e}}_\sigma + f_\sigma(x)\overline{\mathrm{e}}_\gamma\right]. \tag{22}$$

and therefore we can propose the derivative for non-monogenic function in the following section.

5. A new type of derivative for non-monogenic function

In this section, for simplicity we will denote the functions (21) and (22) as

$$f(x) = f_\gamma(x)\mathrm{e}_\sigma + f_\sigma(x)\mathrm{e}_\gamma, \tag{23}$$
$$\overline{f}(x) = f_\gamma(x)\overline{\mathrm{e}}_\sigma + f_\sigma(x)\overline{\mathrm{e}}_\gamma. \tag{24}$$

and now let us consider the following theorem.

Theorem 5.1. *Let us consider a function $f:\Psi \longrightarrow \mathbb{R}^p$. The left non-monogenic differential operator Df is given by*

$$Df = \mathrm{e}_n\mathrm{e}_\sigma\frac{\partial f_\gamma}{\partial x_m} + \mathrm{e}_n\mathrm{e}_\gamma\frac{\partial f_\sigma}{\partial x_m} + \mathrm{e}_m\mathrm{e}_\sigma\frac{\partial f_\gamma}{\partial x_n} + \mathrm{e}_m\mathrm{e}_\gamma\frac{\partial f_\sigma}{\partial x_n}, \tag{25}$$

and its conjugate

$$\overline{Df} = \overline{\mathrm{e}_n\mathrm{e}_\sigma}\frac{\overline{\partial} f_\gamma}{\partial x_m} + \overline{\mathrm{e}_n\mathrm{e}_\gamma}\frac{\overline{\partial} f_\sigma}{\partial x_m} + \overline{\mathrm{e}_m\mathrm{e}_\sigma}\frac{\overline{\partial} f_\gamma}{\partial x_n} + \overline{\mathrm{e}_m\mathrm{e}_\gamma}\frac{\overline{\partial} f_\sigma}{\partial x_n}. \tag{26}$$

Proof. It is sufficient to apply (17) and (19) into (23) and (24). ☑

Remark 5.2. The right non-monogenic operator can be written as

$$fD = \mathrm{e}_\sigma\mathrm{e}_n\frac{\partial f_\gamma}{\partial x_m} + \mathrm{e}_\gamma\mathrm{e}_n\frac{\partial f_\sigma}{\partial x_m} + \mathrm{e}_\sigma\mathrm{e}_m\frac{\partial f_\gamma}{\partial x_n} + \mathrm{e}_\gamma\mathrm{e}_m\frac{\partial f_\sigma}{\partial x_n}, \tag{27}$$

and its respective conjugate

$$\overline{fD} = \overline{\mathrm{e}_\sigma\mathrm{e}_n}\frac{\overline{\partial} f_\gamma}{\partial x_m} + \overline{\mathrm{e}_\gamma\mathrm{e}_n}\frac{\overline{\partial} f_\sigma}{\partial x_m} + \overline{\mathrm{e}_\sigma\mathrm{e}_m}\frac{\overline{\partial} f_\gamma}{\partial x_n} + \overline{\mathrm{e}_\gamma\mathrm{e}_m}\frac{\overline{\partial} f_\sigma}{\partial x_n}. \tag{28}$$

Consequently, we can state the derivatives $\frac{\partial f_\gamma}{\partial x_m}$, $\frac{\partial f_\sigma}{\partial x_m}$, $\frac{\partial f_\gamma}{\partial x_n}$ and $\frac{\partial f_\sigma}{\partial x_n}$ in the following definition.

Definition 5.3. For a function $f:\Psi \longrightarrow \mathbb{R}^p$, the *derivatives for non - monogenic functions* $\frac{\partial f_\gamma}{\partial x_m}$, $\frac{\partial f_\sigma}{\partial x_m}$, $\frac{\partial f_\gamma}{\partial x_n}$ and $\frac{\partial f_\sigma}{\partial x_n}$ are defined as

$$\frac{\partial f_\gamma}{\partial x_m} = \frac{f_\gamma(\mathrm{e}_\gamma\mathrm{e}_\sigma x_m) - f_\gamma(x_m)}{\mathrm{e}_\gamma\mathrm{e}_\sigma x_m - x_m}, \quad \frac{\partial f_\sigma}{\partial x_m} = \frac{f_\sigma(\mathrm{e}_\sigma\mathrm{e}_\gamma x_m) - f_\sigma(x_m)}{\mathrm{e}_\sigma\mathrm{e}_\gamma x_m - x_m}, \tag{29}$$
$$\frac{\partial f_\gamma}{\partial x_n} = \frac{f_\gamma(\mathrm{e}_\gamma\mathrm{e}_\sigma x_n) - f_\gamma(x_n)}{\mathrm{e}_\gamma\mathrm{e}_\sigma x_n - x_n}, \quad \frac{\partial f_\sigma}{\partial x_n} = \frac{f_\sigma(\mathrm{e}_\sigma\mathrm{e}_\gamma x_n) - f_\sigma(x_n)}{\mathrm{e}_\sigma\mathrm{e}_\gamma x_n - x_n}, \tag{30}$$

and their respective conjugates

$$\frac{\overline{\partial} f_\gamma}{\partial x_m} = \frac{f_\gamma(\overline{\mathrm{e}_\gamma\mathrm{e}_\sigma} x_m) - f_\gamma(x_m)}{\overline{\mathrm{e}_\gamma\mathrm{e}_\sigma} x_m - x_m}, \quad \frac{\overline{\partial} f_\sigma}{\partial x_m} = \frac{f_\sigma(\overline{\mathrm{e}_\sigma\mathrm{e}_\gamma} x_m) - f_\sigma(x_m)}{\overline{\mathrm{e}_\sigma\mathrm{e}_\gamma} x_m - x_m}, \tag{31}$$
$$\frac{\overline{\partial} f_\gamma}{\partial x_n} = \frac{f_\gamma(\overline{\mathrm{e}_\gamma\mathrm{e}_\sigma} x_n) - f_\gamma(x_n)}{\overline{\mathrm{e}_\gamma\mathrm{e}_\sigma} x_n - x_n}, \quad \frac{\overline{\partial} f_\sigma}{\partial x_n} = \frac{f_\sigma(\overline{\mathrm{e}_\sigma\mathrm{e}_\gamma} x_n) - f_\sigma(x_n)}{\overline{\mathrm{e}_\sigma\mathrm{e}_\gamma} x_n - x_n}. \tag{32}$$

Theorem 5.4. *Assume that $f_\gamma : \Psi \longrightarrow \mathbb{R}^p$ and $g_\gamma : \Psi \longrightarrow \mathbb{R}^p$ are non-monogenic differential functions at x_m. Then*

1. *the sum $f_\gamma + g_\gamma : \Psi \longrightarrow \mathbb{R}^p$ is differentiable at x_m and*
$$\frac{\partial}{\partial x_m}(f_\gamma + g_\gamma) = \frac{\partial f_\gamma}{\partial x_m} + \frac{\partial g_\gamma}{\partial x_m},$$
2. *the product $f_\gamma g_\gamma : \Psi \longrightarrow \mathbb{R}^p$ is differentiable at x_m and*
$$\frac{\partial (f_\gamma g_\gamma)}{\partial x_m} = f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)\frac{\partial g_\gamma}{\partial x_m} + g_\gamma(x_m)\frac{\partial f_\gamma}{\partial x_m}.$$
3. *$(x_m)^n : \Psi \longrightarrow \mathbb{R}^p$ is differentiable at x_m and*
$$\frac{\partial}{\partial x_m}(x_m)^n = \sum_{k=1}^{n}(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{n-k}(x_m)^{k-1}.$$

Proof. 1.
$$\begin{aligned}\frac{\partial}{\partial x_m}(f_\gamma + g_\gamma) = &\ \frac{f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) + g_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) - f_\gamma(x_m) - g_\gamma(x_m)}{\mathrm{e}_\gamma \mathrm{e}_\sigma x_m - x_m},\\ = &\ \frac{f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) - f_\gamma(x_m)}{\mathrm{e}_\gamma \mathrm{e}_\sigma x_m - x_m} + \frac{g_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) - g_\gamma(x_m)}{\mathrm{e}_\gamma \mathrm{e}_\sigma x_m - x_m}\\ = &\ \frac{\partial f_\gamma}{\partial x_m} + \frac{\partial g_\gamma}{\partial x_m}.\end{aligned}$$

2.
$$\begin{aligned}\frac{\partial}{\partial x_m}(f_\gamma g_\gamma) &= \frac{f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)g_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) - f_\gamma(x_m)g_\gamma(x_m)}{\mathrm{e}_m \mathrm{e}_\sigma x_m - x_m}\\ &= \frac{f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)g_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) + f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)g_\gamma(x_m) - f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)g_\gamma(x_m) - f_\gamma(x_m)g_\gamma(x_m)}{\mathrm{e}_m \mathrm{e}_\sigma x_m - x_m}\\ &= f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)\left[\frac{g_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) - g_\gamma(x_m)}{\mathrm{e}_\gamma \mathrm{e}_\sigma x_m - x_m}\right] + g_\gamma(x_m)\left[\frac{f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m) - f_\gamma(x_m)}{\mathrm{e}_\gamma \mathrm{e}_\sigma x_m - x_m}\right]\\ &= f_\gamma(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)\frac{\partial g_\gamma}{\partial x_m} + g_\gamma(x_m)\frac{\partial f_\gamma}{\partial x_m}.\end{aligned}$$

3.
$$\begin{aligned}\frac{\partial}{\partial x_m}(x_m)^n = &\ \frac{(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^n - x^n}{\mathrm{e}_\gamma \mathrm{e}_\sigma x_m - x_m}\\ = &\ (\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{n-1} + (\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{n-2}x_m + (\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{n-3}x_m^2 + \ldots + x_m^{n-1}\\ = &\ \sum_{k=1}^{n}(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{n-k}x_m^{k-1}.\end{aligned}$$

☑

Remark 5.5. The proof for conjugates is similar.

Now, let us consider the following example:

Example 5.6. *For* $f_\gamma(x_m) = 3(x_m)^3$

$$\frac{\partial}{\partial x_m}(3(x_m)^3) = 3\sum_{k=1}^{3}(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{3-k}(x_m)^{k-1},$$

thus the left non-monogenic operator Df

$$D(3(x_m)^3) = 3\mathrm{e}_n\mathrm{e}_\sigma \sum_{k=1}^{3}(\mathrm{e}_\gamma \mathrm{e}_\sigma x_m)^{3-k}(x_m)^{k-1}.$$

Now, for the right non-monogenic operator

$$(3(x_m)^3)D = 3\mathrm{e}_\sigma\mathrm{e}_n \sum_{k=1}^{3}(\mathrm{e}_\sigma \mathrm{e}_\gamma x_m)^{3-k}(x_m)^{k-1}.$$

and its conjugate we have

$$\overline{D}(3x_m^3) = 3\overline{\mathrm{e}_n\mathrm{e}_\sigma} \sum_{k=1}^{3}(\overline{\mathrm{e}_\gamma \mathrm{e}_\sigma} x_m)^{3-k}(x_m)^{k-1}.$$

6. Suggestions for further works

The main point of this paper has been to show of the definition the derivatives for non-monogenic functions. There are two further topics arising from this paper which are worth investigation. First, one might consider the $q-$ *quadratic derivatives for non-monogenic functions* based in (see [5] for more details)

$$\begin{aligned}\frac{\partial_q f}{\partial_q x_m} &= \frac{f[(x_m + q^2\mathrm{e}_m x_n)x_n] - f(x_m x_n)}{q x_n},\\ \frac{\partial_q f}{\partial_q x_n} &= \frac{f[(x_n + q^2\mathrm{e}_n x_m)x_m] - f(x_n x_m)}{q x_m},\end{aligned}$$

assuming that $x_n x_m$ are not commutative, and considering the deformed version analogous to (17)

$$D_q f = \mathrm{e}_n \frac{\partial_q f}{\partial_q x_m} + \mathrm{e}_m \frac{\partial_q f}{\partial_q x_m}.$$

and secondly there is the problem of describing the solution of the differential equations of the form

$$Df = \mathrm{e}_n \frac{\partial f}{\partial x_m} + \mathrm{e}_m \frac{\partial f}{\partial x_n} = g, \tag{33}$$

for $g : \Psi \longrightarrow \mathbb{R}^p$.

References

[1] Brackx F.R., Delanghe R., and Sommen F., *Clifford Analysis*, Pitman Advanced Pub. Program, Pitman, London, 1982.

[2] Krasnov Y., “The Structure for Monogenic Functions," in *Clifford Algebras and their Applications in Mathematical Physics: Volume 2: Clifford Analysis* (Ryan, John and Sprößig, Wolfgang), irkhäuser Boston (2000), pp. 247–260. doi: 10.1007/978−1−4612−1374−1_13

[3] Luong N.C., “On a class of monogenic functions in clifford algebra", *VNU Journal of Science: Natural Sciences and Technology*, 12 (1996), No. 1, 40–47.

[4] Ryan J., *Clifford Algebras in Analysis and Related Topics*, CRC Press, Boca Raton, 1996.

[5] Jaramillo J.C., *q-Difference Operators and Derivations on the Quadratic Relativistc Invariant Algebras* Rev. Momento **69** (2024) (in press).