



Stability and bifurcation analysis in a predator prey model involving additive Allee effect

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Abstract. In this paper we study codimension 1 Hopf bifurcation for a two dimensional autonomous nonlinear ordinary differential equations system, modeling a predator-prey interaction with Holling type II functional response and additive Allee effect in the prey equation. Positivity, dissipation, boundedness and permanence of the solutions are analyzed. Furthermore, stability and bifurcation analysis are carried out to show the existence of periodic orbits due to the occurrence of codimension 1 Hopf bifurcation, involving weak Allee effect as well as strong Allee effect. In the case of strong Allee effect, through computer simulations carried in MAPLE 13, we conjecture that this model may admit a heteroclinic bifurcation. We present some simulations which allow one to verify the analytical results.

Keywords: Predator-prey system, Allee effect, positivity, dissipation, boundedness, permanence, Stability, bifurcation.

MSC2020: 34D20, 37G15, 37N25, 92B05.

Análisis de estabilidad y bifurcación en un modelo presa depredador que involucra efecto Allee aditivo

Resumen. En este artículo estudiamos bifurcación de Hopf de codimensión 1 para un sistema de ecuaciones diferenciales ordinarias bidimensional autónomo no lineal, modelando una interacción depredador-presa con respuesta funcional Holling tipo II y efecto Allee aditivo en la ecuación de la presa. Se

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analiza positividad, disipación, acotación y permanencia de las soluciones. Además, se realizan análisis de estabilidad y bifurcación para mostrar la existencia de órbitas periódicas debido a la ocurrencia de bifurcación de Hopf de codimensión 1, involucrando efecto Allee débil así como efecto Allee fuerte. En el caso de un fuerte efecto Allee, a través de simulaciones realizadas en MAPLE 13, conjeturamos que este modelo puede admitir una bifurcación heteroclínica. Presentamos algunas simulaciones que permiten verificar los resultados analíticos.

Palabras clave: Sistema presa-depredador, efecto Allee, positividad, disipación, acotación, permanencia, estabilidad, bifurcación.

1. Introduction

The predator-prey models have been of crucial importance in the development of the nonlinear systems, which has allowed the researchers interested in this field to accomplish big advances in the study of the spatio-temporal dynamics of these models [23], [15], [11], [31], [32], [4], [3], [18]. It is well known that in nature some species often cooperate amongst themselves in their search for food or when they try to escape from predators. Allee [5], [6] studied extensively the aspects of aggregation and associated cooperative and social characteristics among the members of a species. He proposed that intraspecific cooperation might lead to inverse density dependence (that is, the per capita rate of population growth increases as population sizes become larger) and observed that many animal and plant species suffer a decrease of the per capita growth rate, as their populations reach small sizes or low densities. Furthermore, Allee brought the attention to the possibility of a positive relationship between individual fitness and population size. The phenomenon, broadly referred to as the Allee effect, has attracted a lot of attention with the rise of conservation biology. Generally speaking, a population is said to have an Allee effect, if the per capita growth rate is initially an increasing function for the low density. Moreover, it is called a strong Allee effect if the per capita growth rate in the limit of low density is negative, and a weak Allee effect means that the per capita growth rate is positive at zero density.

According with Stephens [29], the Allee effect occurs when the individual fitness is reduced at low densities of the population and the effect of competition results dominant, so that the variation with respect the population size is not positive for any populational size. That is, the reproductive success of an individual never increases as the population grows, since competition by resources increases as the number of competitors increases. Wertheim [34] states that the Allee effect component reflects only an isolated mechanism that produces a benefit to aggregation, which may or may not be sufficient to offset the associated costs.

Clearly, the above mentioned situations are different from the logistic growth, in which the per capita growth rate is a decreasing function of density (see black line in Figure 1). Logistic growth, also known as Verhulst model - has been modeled first in [30] by the

following differential equation:

$$\dot{N}(t) = rN(t) \left(1 - \frac{N(t)}{K} \right) = Nf_1(N) \quad (1)$$

where $N(t)$ denotes the populational density at time t , $f_1(N) = r \left(1 - \frac{N(t)}{K} \right)$ is the per capita population growth rate, $r > 0$ is the intrinsic growth rate, and $K > 0$ is the carrying capacity the environment. It is worth observing that in logistic growth, an increment in the population density leads to a negative effect in the reproduction and survival of an individual. On the other hand, the Allee effect means that this negative effect occurs when there is an increment in high densities of populations, whereas in lower densities of populations this increment may be beneficial.

It is widely accepted that the Allee effect may be understood as the cause of the increase in extinction risk at low densities, introducing, in some case, a population threshold that has to be exceeded by population to be able to grow. According with [15], [14], a strong Allee effect introduces a per capita population threshold (the minimal size of the population required to survive), and the population must surpass this threshold to grow (see Figure 1 red line); in contrast, a population with a weak Allee effect does not have a threshold (see Figure 1 green line).

Taking into account the carrying capacity of the environment with respect to the prey in the per capita growth rate of the population, the weak Allee effect has been modeled in [10] by the following differential equation:

$$\dot{N}(t) = \frac{r}{K} N^2 \left(1 - \frac{N}{K} \right) = Nf_2(N)$$

in which $N(t)$ denotes the populational density at time t , $f_2(N) = \frac{r}{K} N \left(1 - \frac{N}{K} \right)$ is the per capita population growth rate, $0 < r < 1$ is the species growth rate and $K > 0$ is the carrying capacity of the environment. As mentioned previously, the Allee effect is weak if there exists no critical density population, below which the per capita rate becomes negative. Weak Allee effects are then used to describe cases where the population growth rate is negatively affected by low population sizes, but where the per capita population growth rate cannot go below zero. Therefore populations will grow at low population sizes (see green line in Figure 2).

On the other hand, strong density dependence is used to denote Allee effects where the per capita population growth rate can become negative, which means that there is a critical point in population size below which the population will tend towards extinction (see red line in Figure 2). The strong Allee effect has been modeled in [10] by the following differential equation:

$$\dot{N}(t) = rN \left(1 - \frac{N}{K} \right) \left(\frac{N}{A} - 1 \right) = Nf_3(N)$$

where $N(t)$ denotes the populational density at time t , $f_3(N) = r \left(1 - \frac{N}{K} \right) \left(\frac{N}{A} - 1 \right)$ is the per capita population growth rate, r is the species growth rate, K is the carrying capacity, and A is the threshold for the population to not go extinct ($0 < A < K$).

Based on a widespread evidence in natural populations, several mechanisms have been hypothesized to invoke the Allee effect. Among other phenomena, the most cited and

obvious cause of the Allee effect is the difficulty of finding mates at low population sizes in sexually reproducing species. Other mechanisms concern genetic inbreeding, leading to decreased fitness; reproductive facilitation (or cooperative interaction), and predation [15], [14].

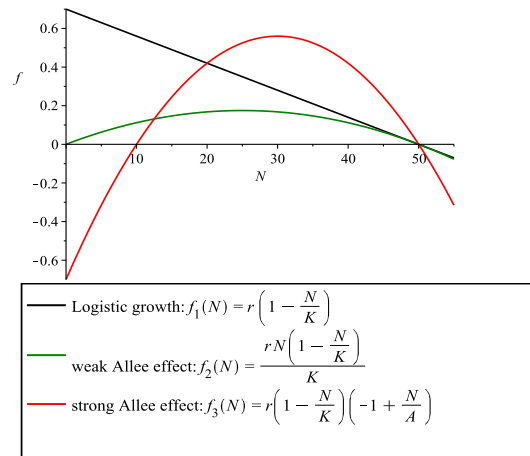


Figure 1. Per capita population growth rate: in the case of logistic growth (black line), we note that the per capita growth rate is a decreasing function of density, an increment in low densities of population leads to a negative effect in the reproduction and survival of an individual; in the case of weak Allee effect (green curve) and strong Allee effect (red curve), we note that unlike to the logistic growth, a negative effect occurs when there is an increment in high densities of populations, whereas in lower densities this increment may be beneficial; the following parameter values $r = 0.7$, $K = 50$ and $A = 10$, were considered for simulations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

For all the aforementioned phenomena, and others, the major consequence of the Allee effect is the existence of a critical density below which the aggregation unit considered (that is, population, colony, or social group) is likely to go extinct. The implications of the Allee effect are potentially very important in most areas of ecology and evolution. Particularly, on population dynamics the practical management of population numbers, whether aiming to increase or reduce them, is strongly affected by this effect. The consequences of Allee effects are also significant for the theory of population dynamics, because most classic models imply a linear decrease of growth with density, as opposed to the non-linear relationship associated with the Allee effect.

It is important to observe that the Allee effect has been modeled in different forms [4], [3]. For instance, if in a predator-prey model we assume that the prey growth is damped by the Allee effect and $N = N(t)$ indicates the population size, which depends on time t , the most usual continuous growth equation to express the Allee effect is represented by the following ordinary differential equation:

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) (N - m) \quad (2)$$

which is the prototypical model for the *multiplicative Allee effect*. The parameter r in model (2) represents the intrinsic growth rate, K represents the carrying capacity of the

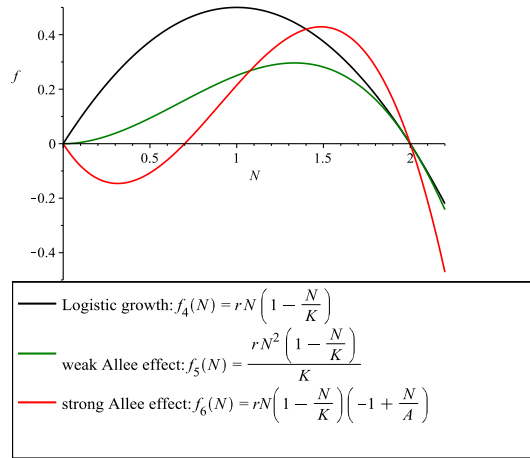


Figure 2. Population growth rate: in the case of logistic growth (black curve), we note that there is no critical point in low densities below which population size is affected, unlike to the weak and strong Allee effects; in the case of weak Allee effect (green curve), we note that the populational growth rate is negatively affected by low population sizes, but where the per capita population growth rate cannot go below zero. Therefore populations will grow at low population sizes; in the case of strong Allee effect (red curve), we note that there is a critical point in population size below which the population will tend towards extinction since for lower population densities, the growth rate is negative; the following parameter values $r = 0.7$, $K = 50$ and $A = 10$, were considered for simulations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

environment and m is the Allee effect constant. If $-K < m < 0$ it shows the weak Allee effect, while if $0 < m < K$, it shows the strong Allee effect.

Other mathematical forms have been proposed to describe this phenomenon. Dennis [15] was the first who introduced the equation that modeled the *additive Allee effect*, represented by the following ordinary differential equation

$$\frac{dN}{dt} = N \left[r \left(1 - \frac{N}{K} \right) - \frac{m}{N+b} \right]. \tag{3}$$

In model (3), $\frac{mN}{N+b}$ represents the additive term of the Allee effect, m and b are the Allee effect constant with $K > b$. In absence of predator the prey N has a populational growth given by

$$f(N) = N \left[r \left(1 - \frac{N}{K} \right) - \frac{m}{N+b} \right].$$

It is clear that $f(N)$ satisfies $f(0) = 0$ and $f'(0) = r - \frac{m}{b}$. In the sense of [33], we have the weak Allee effect when $m < br$, i.e., $f'(0) = r - \frac{m}{b} > 0$; the Allee effect will be strong when $m > br$, i.e., $f'(0) = r - \frac{m}{b} < 0$. With the previous notation, we conclude that the population presents a weak Allee effect or pure depensation if there exists a value $z \in (0, K)$, such that $f''(z) = 0$ and $f(N) > 0$ for all $N \in (0, K)$ (see Figure 3 green curve). On the other hand, the population presents a strong Allee effect or a *critical compensation* if for lower population densities, the growth rate is negative, that is, for small values of N near to zero, we have $f(N) < 0$ (see Figure 3 red curve). When the

species is submitted to a strong Allee effect, it may have a bigger tendency to be less able to overcome these additional mortality causes, to have a slower recovery, and to be prone to extinction than other species [28]. These curves are called decompensation curves, and there exist a value of $z \in (0, K)$, such that $f(z) = 0$. The value of $N = z$ is an unstable equilibrium and coincides with m in the equation for the multiplicative Allee effect, which is called the minimal level of viable population or threshold level. If the initial population N_0 at time $t = 0$ is less than m , i.e., $N(0) = N_0 < m$, then for this initial size of population less than m the population tends to extinction.

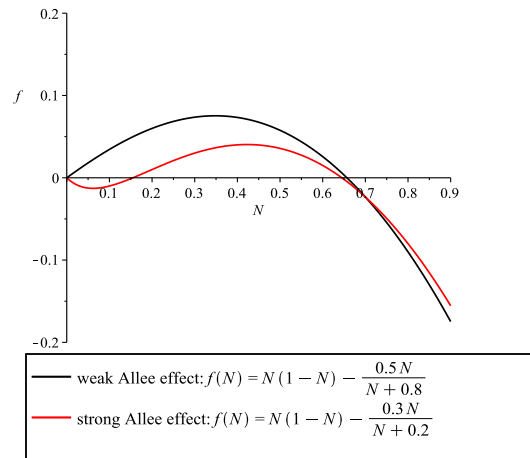


Figure 3. Graphics for the function $f(N) = N(1 - N) - \frac{mN}{N+b}$ in the cases: Weak Allee effect $m < br$ (green curve): $r = 1$, $K = 1$, $m = 0.5$, $b = 0.8$; Strong Allee effect $m > br$ (red curve): $r = 1$, $K = 1$, $m = 0.3$, $b = 0.2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In recent studies [23], [15], [11], [20], [13], the authors studied the spatio-temporal dynamics for a predator-prey model with diffusion and Allee effect. In [4], [3] the existence and stability of limit cycles for a Leslie-Gower predator prey model with additive Allee effect was studied. In [13] the authors analyzed a predator-prey model with diffusion and Holling type II functional response subject to additive Allee effect in the prey equation.

Following [13], we focus our attention on the following system of equations:

$$\begin{aligned} \frac{dN}{dt} &= N \left(r \left(1 - \frac{N}{K} \right) - \frac{m}{N+b} \right) - \frac{NP}{N+a} \\ \frac{dP}{dt} &= sP \left(-\beta + \frac{N}{N+a} \right). \end{aligned} \quad (4)$$

The parameters composing the model (4) are positive, and r , K , m , b have the same meaning as in Equations (2) and (3); a is a saturation constant; β is the death rate of the predator; and s is the feed concentration; Dennis [15] showed that the probability of mating encounters among individual of a population with increasing population density, could be modeled using a rectangular hyperbola function $\frac{N}{N+b}$, where b is the population size at which mating fitness is half of its maximum value that is, b is the population density at which the probability of mating is $\frac{1}{2}$. Therefore, $1 - \frac{N}{N+b} = \frac{b}{N+b}$ is the probability of not mating. Consequently, the term $-\frac{m}{N+b}$ considered in the per capita

growth rate of prey population in model (4) represents the reduction due to mating shortage. We then have the logistic model adjusted for mating encounters. $-\frac{NP}{N+a}$ is the predation rate where the saturation constant a reflects the prey density level at which the predation rate begins to saturate, meaning that predators have a limited capacity to consume prey and this capacity stabilizes at high prey densities; $\frac{N}{N+a}$ is the fraction of the maximum consumption rate of prey by predators as a function of prey density.

In [13] the authors investigated the dynamics of model (4) and its diffusive version. They analysed the local and global asymptotic stability behavior of the positive equilibrium for both the ODE system and the reaction diffusion system. Furthermore, they obtained conditions over the parameters for the occurrence of the Hopf bifurcation in model (4) in the cases of weak Allee effect as well as strong Allee effect. However, to the best of our knowledge there is no results related to the direction and stability of the bifurcating periodic orbits for the system (4). In this paper, we determine the direction and stability of the bifurcating periodic orbits for the system, in the case of weak Allee effect as well as strong Allee effect by calculating the value of the first Lyapunov coefficient.

This paper is organized as follows: In Section 2 we present a review of the method found in [26] for the calculation of the first Lyapunov coefficient for a general ordinary differential equation system. In Section 3, positivity, dissipation, boundedness and permanence of the solutions for model (4) are studied. In Section 4, equilibrium points of the system (4) are identified, and a Hopf bifurcation analysis for the positive equilibrium is carried out. We determine the direction and stability of the bifurcating periodic orbits for the system (4), in the case of weak Allee effect as well as strong Allee effect by calculating the value of the first Lyapunov coefficient; simulations are presented to corroborate the analytical results. Also, in the case of strong Allee effect it is observed by a numerical experiment the presence of an interesting global bifurcation not discussed earlier for this mode. A discussion follows in the final section.

2. Outline of the Hopf Bifurcation Methods

In this section we present a very brief summary of the projection method described in [26] for the calculation of the first Lyapunov coefficient associated with the Hopf bifurcation, denoted by l_1 . Other equivalent definitions and algorithmic procedures to write the expressions of the Lyapunov coefficients for two-dimensional systems can be found in [7] and [27], among others.

Definition 2.1. Consider the differential equation

$$X' = \mathbf{f}(X, m), \quad (5)$$

where $X \in \mathbb{R}^2$ is a vector representing phase variables and $m \in \mathbb{R}$ is a parameter representing control parameter. Assume that f is of class \mathbb{C}^∞ in $\mathbb{R}^2 \times \mathbb{R}$. Suppose that system (5) has an equilibrium point $X = X_0$, which may depend on m . Let the eigenvalues of the linearised system about this equilibrium point be given by $\lambda_{1,2}(m) = \alpha(m) \pm i\omega(m)$, and suppose that this pair of simple complex eigenvalues reaches the imaginary axis as the parameter m varies, say at a critical value $m = m^*$, and we have $\lambda_{1,2}(m^*) = \pm i\omega_0$, $\omega_0 > 0$. The bifurcation corresponding to the presence of $\lambda_{1,2}(m^*) = \pm i\omega_0$, $\omega_0 > 0$, is called a **Hopf (or Andronov-Hopf) bifurcation**. Specifically, at the

bifurcation point, a complex conjugate pair of eigenvalues crosses the imaginary axis of the complex plane, leading to the creation or destruction of a periodic orbit.

Now, consider the system (5) and suppose that it has an equilibrium point $X = X_0$ at $m = m^*$ and, denoting the variable $X - X_0$ also by X , we write

$$X' = JX + F(X)$$

where

$$F(X) = \frac{1}{2}\mathbf{B}(X, X) + \frac{1}{6}\mathbf{C}(X, X, X) + O(\|X\|^4), \quad (6)$$

with $J = \mathbf{f}_X(\mathbf{0}, m^*)$ and $\mathbf{B}(X, Y)$, $\mathbf{C}(X, Y, Z)$ are *multilinear functions*. In coordinates, we have

$$\mathbf{B}_i(X, Y) = \sum_{j,k=1}^2 \frac{\partial^2 F_i(\eta)}{\partial \eta_j \partial \eta_k} \Big|_{\eta=0} x_j y_k \quad \text{and} \quad \mathbf{C}_i(X, Y, Z) = \sum_{j,k,l=1}^2 \frac{\partial^3 F_i(\eta)}{\partial \eta_j \partial \eta_k \partial \eta_l} \Big|_{\eta=0} x_j y_k z_l, \quad (7)$$

where $i = 1, 2$.

Suppose that $(\mathbf{0}, m^*)$ is an equilibrium point of (5) where the Jacobian matrix J has a pair of pure imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, and these eigenvalues are the only pure imaginary eigenvalues. Let T^c be the generalized eigenspace of J corresponding to $\lambda_{1,2}$. By this it is meant the largest subspace invariant by J on which the eigenvalues are $\lambda_{1,2}$.

Let $p, q \in \mathbb{C}^2$ be *complex* eigenvectors corresponding to $-i\omega_0$ and $+i\omega_0$ respectively, such that

$$Jq = i\omega_0 q, \quad J^T p = -i\omega_0 p, \quad \langle p, q \rangle = 1, \quad \langle p, \bar{q} \rangle = 0,$$

where J^T is the transpose of the matrix J and $\langle p, q \rangle = \sum_{i=1}^2 \bar{p}_i q_i$ is the standard scalar product in \mathbb{C}^2 (linear with respect to the second argument). Thus, the critical real eigenspace T^c corresponding to $\pm i\omega_0$ is now two-dimensional, and is spanned by $\{\mathbf{Re}q, \mathbf{Im}q\}$ and any vector $Y \in T^c$ can be represented as $Y = wq + \bar{w}\bar{q}$, where $w = \langle p, Y \rangle \in \mathbb{C}$.

By introducing a complex variable z , system (5) can be written for sufficiently small $|m|$ as a single equation:

$$\dot{z} = \lambda(m)z + G(z, \bar{z}, m)$$

where $G = O(|z^4|)$ is a smooth function of (z, \bar{z}, m) and

$$G(z, \bar{z}, m) = \langle p(m), F(zq(m) + \bar{z}\bar{q}(m), m) \rangle.$$

Write G as a formal Taylor series in two complex variables (z and \bar{z}):

$$G(z, \bar{z}, m) = \sum_{k+n \geq 2} \frac{1}{k!n!} G_{kn}(m) z^k \bar{z}^n \quad (8)$$

where

$$G_{kn}(m) = \frac{\partial^{k+n}}{\partial z^k \partial \bar{z}^n} \langle p(m), F(zq(m) + \bar{z}\bar{q}(m), m) \rangle_{z=0}$$

for $k + n \geq 2$.

On the other hand

$$\begin{aligned} G(z, \bar{z}, m) &= \langle p(m), F[zq(m) + \bar{z}\bar{q}(m), m] \rangle \\ &= \frac{1}{2}z^2 \langle p, B(q, q) \rangle + z\bar{z} \langle p, B(q, \bar{q}) \rangle + \frac{1}{2}\bar{z}^2 \langle p, B(\bar{q}, \bar{q}) \rangle \\ &\quad + \frac{1}{6}z^3 \langle p, C(q, q, q) \rangle + \frac{1}{2}z^2\bar{z} \langle p, C(q, q, \bar{q}) \rangle + \frac{1}{2}z\bar{z}^2 \langle p, C(q, \bar{q}, \bar{q}) \rangle \\ &\quad + \frac{1}{6}\bar{z}^3 \langle p, C(\bar{q}, \bar{q}, \bar{q}) \rangle + \dots \end{aligned} \tag{9}$$

comparing (8) with (9), we have

$$\begin{aligned} G_{20} &= \langle p, B(q, q) \rangle, \quad G_{11} = \langle p, B(q, \bar{q}) \rangle, \quad G_{02} = \langle p, B(\bar{q}, \bar{q}) \rangle, \quad G_{30} = \langle p, C(q, q, q) \rangle, \\ G_{21} &= \langle p, C(q, q, \bar{q}) \rangle, \quad G_{12} = \langle p, C(q, \bar{q}, \bar{q}) \rangle, \quad \text{and} \quad G_{03} = \langle p, C(\bar{q}, \bar{q}, \bar{q}) \rangle. \end{aligned}$$

Also, the equations

$$\dot{z} = \lambda z + \sum_{k+n \geq 2} \frac{1}{k!n!} G_{kn}(m) z^k \bar{z}^n + O(|z|^4)$$

where $\lambda = \lambda(m) = \alpha(m) \pm i\omega(m)$, $\alpha(m^*) = 0$, $\omega(m^*) = \omega_0 > 0$ can be transformed by an invertible parameter- dependent change of complex coordinate, for all sufficiently small $|m|$, into an equation with only the resonant cubic term:

$$\dot{v} = \lambda v + c_1 v^2 \bar{v} + O(|v|^4)$$

where $c_1 = c_1(m)$ and

$$c_1(m) = \frac{G_{20}G_{11}(2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|G_{11}|^2}{\lambda} + \frac{|G_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{G_{21}}{2}.$$

At the bifurcation parameter value $m = m^*$, the previous equation reduces to

$$c_1(m^*) = \frac{i}{2\omega_0} \left[G_{20}G_{11} - 2|G_{11}|^2 - \frac{1}{3}|G_{02}|^2 \right] + \frac{G_{21}}{2}.$$

From Lemma 3.7 pp. 98 of [26], the equation

$$\dot{v} = (\alpha(m) + i\omega(m))v + c_1(m)v^2\bar{v} + O(|v|^4)$$

where $\alpha(m^*) = 0$, $\omega(m^*) > 0$ and supposing that $\alpha'(m^*) \neq 0$, and $Re(c_1(m^*)) \neq 0$, then, the equation can be transformed by a parameter-dependent linear coordinate

transformation, a time rescaling, and a nonlinear time reparametrization into an equation of the form

$$\frac{du}{d\theta} = (\hat{\beta} + i)u + \frac{l_1(\hat{\beta})}{|l_1(\hat{\beta})|}u|u|^2 + O(|u|^4)$$

where u is a new complex coordinate, and θ, β are the new time and parameter, respectively,

$$\frac{l_1(\hat{\beta})}{|l_1(\hat{\beta})|} = \text{signRe}(c_1(m^*)) = \pm 1 \quad \text{and} \quad l_1(m^*) = \frac{\text{Re}(c_1(m^*))}{\omega(m^*)}. \quad (10)$$

Definition 2.2. The real function $l_1(\hat{\beta})$ is called the first Lyapunov coefficient. It follows from (19) that the first Lyapunov coefficient at $\hat{\beta} = 0$ can be computed by the formula

$$l_1(m^*) = \frac{1}{2\omega_0^2} \text{Re} \left(iG_{20}G_{11} + \omega_0 \frac{G_{21}}{2} \right). \quad (11)$$

Proposition 2.3. Consider system (5) where $X \in \mathbb{R}^2$ is a vector representing phase variables and $m \in \mathbb{R}$ is a parameter representing control parameter. Assume that f is of class C^∞ in $\mathbb{R}^2 \times \mathbb{R}$. Suppose that (5) has an equilibrium point $X = X_0$ at $m = m^*$. Then the following holds:

i) Suppose that at (X_0, m^*) the Jacobian matrix \mathbf{A} of (5) has a pair of pure imaginary eigenvalues $\lambda_{2,3}(m^*) = \pm i\omega_0$, $\omega > 0$, and these eigenvalues are the only eigenvalues with null real part. Then (X_0, m^*) is a Hopf point of (5).

ii) At (X_0, m^*) a two-dimensional center manifold is well-defined, it is invariant under the flow generated by (5) and can be continued with arbitrary high class of differentiability to nearby parameter values.

iii) If $\frac{d}{dm} \text{Re}\lambda(m)|_{m=m^*} \neq 0$ then (X_0, m^*) is a transversal Hopf point. In this case, if $l_1 \neq 0$ then the system (5) undergoes a Andronov-Hopf bifurcation, that is, in a neighborhood of (X_0, m^*) the dynamic behavior of the system (5), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the following complex normal form

$$w' = (\eta + i\omega)w + l_1 w|w|^2,$$

where $w \in \mathbb{C}$, η, ω and l_1 are real functions having derivatives of arbitrarily higher order, which are continuations of $0, \omega_0$ and the first Lyapunov coefficient at the Hopf point. When $l_1(m^*) < 0$ (resp. $l_1(m^*) > 0$) a family of stable (resp. unstable) periodic orbits can be found in these manifolds, shrinking to an equilibrium point at the Hopf point.

iv) The Hopf bifurcation is supercritical (resp. subcritical) if $(\frac{d}{dm} \text{Re}\lambda(m)|_{m=m^*})^{-1} l_1(m^*) < 0$ (resp. > 0). In addition, the bifurcating periodic solutions are stable (resp. unstable) if $l_1(m^*) < 0$ (resp. $l_1(m^*) > 0$).

3. Positivity, dissipation, boundedness and permanence of the solutions

In this section, we are mainly concerned with some simple properties of the solution to model (4). In theoretical ecology, positivity and boundedness of the system establishes

the biological well behaved nature of the system. On the other hand, persistence and permanence are important in the sense that they describe long term behavior of the system. Due to many reasons, for example, pollution, over predation, over exploitation, mismanagement of natural resources, among others, several species become extinct and many others are at the bound of extinction. Accordingly, the concept of persistence, that means the survival of species for a longer time, has drawn lot of attention (see [9], [21], [1], [2]).

Analytically, a system is said to be persistent if it persist for each population, that is, if $\liminf_{t \rightarrow \infty} x(t) > 0$ (stronger case) or $\limsup_{t \rightarrow \infty} x(t) > 0$ (weaker case) for each population $x(t)$ of the system. Geometrically, persistence means that trajectories that initiate in a positive cone are eventually bounded away from coordinate planes. On the other hand permanently coexistence (uniform persistence) implies the existence of a region in the phase space at a non zero distance from boundary, in which all the population vectors must ultimately lie. The later one assures the survival of species in biological sense.

It is worth observing that in many significant examples of periodic differential systems modeling infectious diseases transmission, competition of species among others ecological models, it is of interest to prove the existence of a nontrivial fixed point for the operator associated with the differential system, which establishes the global existence of nontrivial periodic solution that may implies the coexistence of the species. To this end, some well known theorems or techniques can be applied such as Browder theorem [12], Brouwer degree techniques, Leray-Schauder degree techniques if the space of initial values is infinite dimensional, among others.

Although the phenomenons such as periodicity of a differential systems, uniform persistence, existence of a compact global attractor with some more conditions, often implies the global existence of a positive fixed point for the system, the focus of our paper is to study the existence of periodic orbits for the non-periodic system (4) by Hopf bifurcation arguments, which are local existence results. Nevertheless, for sake of completeness, results concerning the persistence and permanence of system (4) are presented in this section, since those results guarantee the the survival of the total population which is desirable on the study of population dynamics. Mathematically, persistence of a system means that strictly positive solutions do not have any omega limit points on the boundary of the positive cone.

We highlight that the study of system (4) involving periodic cases to obtain some sufficient conditions on the boundedness, permanence, extinction and global existence of a positive fixed point will be considered in a next exposition, since our focus here is carried out a local analysis by Hopf bifurcation arguments.

Definition 3.1. The system system (4) is said to be *weakly persistent* if every solution $(N(t), P(t))$ satisfies two conditions:

- i. $N(t) \geq 0, P(t) \geq 0$, for all $t \geq 0$.
- ii. $\limsup_{t \rightarrow \infty} N(t) > 0, \limsup_{t \rightarrow \infty} P(t) > 0$.

System (4) is said to be *strongly persistent* if every solution $(N(t), P(t))$ satisfies the following condition along with the first condition of the weak persistence:

$\liminf_{t \rightarrow \infty} N(t) > 0$, $\liminf_{t \rightarrow \infty} P(t) > 0$.

Remark 3.2. Weak persistence allows that $\liminf_{t \rightarrow \infty} N(t) = 0$ or $\liminf_{t \rightarrow \infty} P(t) = 0$ so that arbitrarily small statistical fluctuations may annihilate a species [24].

Definition 3.3. The system (4) is said to be permanent if there exist positive constants s and S , with $0 < s \leq S$ such that

$$\min\{\liminf_{t \rightarrow \infty} N(t), \liminf_{t \rightarrow \infty} P(t)\} \geq s,$$

$$\max\{\limsup_{t \rightarrow \infty} N(t), \limsup_{t \rightarrow \infty} P(t)\} \leq S,$$

for all solutions $(N(t), P(t))$ of system (4) with positive initial values. System (4) is said to be non-permanent if it has a positive solution $(N(t), P(t))$ such that $\min\{\liminf_{t \rightarrow \infty} N(t), \liminf_{t \rightarrow \infty} P(t)\} = 0$.

The following results ensure the positivity, boundedness, dissipativeness and permanence of solutions of system (4).

Lemma 3.4. *The positive quadrant of \mathbb{R}_+^2 is invariant for the system (4).*

Proof. The first quadrant is an invariant region for (4) since $\{(N, P) \in \mathbb{R}_+^2 : N = 0\}$ and $\{(N, P) \in \mathbb{R}_+^2 : P = 0\}$ are invariant manifolds of (4). Therefore, the solutions of (4) with the initial values $N(0) > 0$ and $P(0) > 0$ are positive. The basic existence and uniqueness theorem for differential equations ensures that positive solutions and the axis cannot intersect. \square

Theorem 3.5. *The system (4) is dissipative when $t \geq 0$ and $N(t_0), P(t_0) > 0$.*

Proof. From the first equation of system (4), we have $\frac{dN}{dt} \leq \frac{rN}{K}(K - N)$. By integrating the previous inequality from t_0 to t , it follows that

$$N(t) \leq \frac{N(t_0)Ke^{-rt_0}}{N(t_0)e^{-rt_0} + (K - N(t_0))e^{-rt}} = \frac{K}{1 + \left(\frac{K}{N(t_0)} - 1\right)e^{-r(t-t_0)}}.$$

Letting $t \rightarrow \infty$ in the previous inequality, it follows that $\limsup_{t \rightarrow \infty} N(t) \leq K$. As a consequence, for any $\epsilon_1 > 0$, there exists a $T_1 > 0$ such that $N \leq K + \epsilon_1$ for $t > T_1$.

On the other hand, multiplying the first equation in (4) by s and then adding with the second one, we obtain $(sN + P)' \leq srN - s\beta P$. Let $\xi = \max_{t \geq 0}\{srN(t) + s^2\beta N(t)\}$. Then we have $(sN + P)' \leq \xi - s\beta(sN + P)$. From Gronwall's inequality, we obtain that

$$sN + P \leq (N(t_0) + P(t_0))e^{-s\beta t} + \frac{\xi}{s\beta}(1 - e^{-s\beta t})$$

as long as the solution exists. Letting $t \rightarrow \infty$ in the previous inequality, it follows that $\limsup_{t \rightarrow \infty}(sN + P) \leq \frac{\xi}{s\beta}$. Hence $P(t)$ is also bounded. As a consequence, for any $\epsilon_2 > 0$, there exists a $T_2 > 0$ such that $sN + P \leq \frac{\xi}{s\beta} + \epsilon_2$ for $t > T_2$. This completes the proof of the boundedness of solutions and the system under consideration is weakly

persistent (dissipative). Hence, any solution is defined for $t \geq 0$ and enter the attracting compact set

$$J = \left\{ (N, P) : N \geq 0, P \geq 0, \text{ and } N + P \leq K + \frac{\xi}{s\beta} \right\}$$

for sufficiently large values of t , which implies that J is positive invariant. Consequently, the system is dissipative and its global attractor is contained in J . \square

Remark 3.6. We observe that in the case of $m = 0$, i.e., in model (4) without Allee effect, the dissipation and boundedness of the solution are similar to the results in Theorem 3.5. In other words, the solutions of the system without Allee effect are always dissipative and uniformly bounded.

Theorem 3.7. *If $m > rb - \frac{b\xi}{sa\beta}$ holds, then any solution of system (4) starting from the interior of the first quadrant satisfies $\liminf_{t \rightarrow \infty} N(t) \geq 0$, $\liminf_{t \rightarrow \infty} P(t) \geq 0$.*

Proof. From Theorem 3.5 it is clear that $N(t) \leq K + \epsilon_1$ and $P(t) \leq \frac{\xi}{s\beta} + \epsilon_2$ for $t \geq T = \max\{T_1, T_2\}$. Therefore, from the first equation of system (4) we have

$$N' \geq rN \left[\left(1 - \frac{N}{K}\right) - \frac{m}{br} - \frac{P}{ar} \right] \geq \frac{r}{K} N \left[\left(K - \frac{Km}{br} - \frac{K\xi}{ra(s\beta + \epsilon_2)}\right) - N \right]$$

for $t \geq T$. Thus, using a comparison theorem (see [8] pp. 1071) and the arbitrariness of $\epsilon_2 > 0$, we obtain $\liminf_{t \rightarrow \infty} N(t) \geq 0$ provided $K - \frac{Km}{br} - \frac{K\xi}{ras\beta} < 0$ or equivalently $m > rb - \frac{b\xi}{sa\beta}$. On the other hand, observe that $P' \geq -\beta P$, hence $\liminf_{t \rightarrow \infty} P(t) \geq 0$. \square

Remark 3.8. According with Theorem 3.7, we may have $\liminf_{t \rightarrow \infty} N(t) = \liminf_{t \rightarrow \infty} P(t) = 0$, provided $m > rb - \frac{b\xi}{sa\beta}$. This condition implies that the system is non-permanent. Such a scenario occur when the system (4) is weakly persistent, allowing arbitrarily small statistical fluctuations to lead to the extinction of a species. This annihilating phenomenon may take place when the system (4) is subject to strong Allee effect, i.e., $m > rb$. Clearly this inequality highlights the contribution of the strong Allee effect.

4. Equilibria, stability and Hopf bifurcation

In this section, equilibrium points of the system (4) are identified and a Hopf bifurcation analysis for the positive equilibrium is carried out. It is worth noting that in [13], assuming $a < b$ in case of weak Allee effect and $a \neq b$ in case of strong Allee effect, the authors proved that the model (4) undergoes a Hopf bifurcation around its positive equilibrium at

$$m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a - b)(1 - \beta)^3}. \tag{12}$$

Even though they have focused on the study of Hopf bifurcation for the system (4), there is no result concerning the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions. Our main goal in this section is to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions for this system, when additive Allee effect is considered in the prey equation. First, we focuses

on the case when the Allee effect in (4) is the weak one, i.e., $m < br$. Next, we analyze the system (4) when the Allee effect is the strong one, i.e., $m > br$.

It was observed in [25] that because of difficulties in finding mates when prey population density becomes low due to increasing emigration rate, Allee effect may occur in prey species. These two mechanisms are related, because high rates of emigration further decrease density, thereby reducing the probability of locating mates at low densities. Consequently, these two mechanism may increase the risk of extinction of small local population. As we will see in this section, the predator-prey model with weak Allee effect may not increase the extinction risk of both predators and prey. However, when the model is subject to strong Allee effect it may increase the extinction risk of both species.

4.1. The case of weak Allee effect

Let us assume $m < br$. As a starting point of this subsection, we discuss the equilibria of system (4) that is, the solutions of the system

$$\begin{aligned} N \left(r \left(1 - \frac{N}{K} \right) - \frac{m}{N+b} \right) - \frac{NP}{N+a} &= 0 \\ sP \left(-\beta + \frac{N}{N+a} \right) &= 0. \end{aligned} \quad (13)$$

Following the ideas in [13], one can verify that model (4) has always two boundary equilibria

$$E_0 = (0, 0), \quad \text{and} \quad E_1 = (N_{11}, P_{11}) = \left(\frac{r(K-b) + \sqrt{r^2(K-b)^2 + 4Kr(br-m)}}{2r}, 0 \right).$$

On the other hand, system (4) has a unique positive equilibrium $E^* = (N^*, P^*)$ provided

$$0 < \beta < 1, \quad m < br + \frac{a\beta r[(K-b)(1-\beta) - a\beta]}{K(1-\beta)^2} \quad \text{and} \quad (14)$$

$$\max \left\{ b, \frac{a\beta}{1-\beta} \right\} < K < b + \frac{a\beta}{1-\beta}, \quad (15)$$

where

$$N^* = \frac{a\beta}{1-\beta} \quad ; \quad P^* = \frac{ar(K - K\beta - a\beta)}{K(1-\beta)^2} - \frac{am}{b(1-\beta) + a\beta}. \quad (16)$$

In this subsection, we will assume the conditions given in (14) from now on.

The Jacobian matrix associated with the system (4) is given by

$$J(N, P) = \begin{pmatrix} r - \frac{2rN}{K} - \frac{mb}{(N+b)^2} - \frac{Pa}{(N+a)^2} & -\frac{N}{N+a} \\ \frac{asP}{(N+a)^2} & -s\beta + \frac{sN}{N+a} \end{pmatrix}. \quad (17)$$

The authors in [13] observed that the equilibria E_0 and E_1 are saddle point for model (4).

Remark 4.1. When the system (4) is compared with the predator prey model without Allee effect, whose boundary equilibria $E_0 = (0, 0)$ and $E_1 = (N_1, 0)$ are also saddle points, we can see that if the Allee effect of the prey population in system (4) is very weak, it may not increase the extinction risk of both predators and prey. However, when model (4) is subject to a strong Allee effect it may not be true as we will see in the next subsection. Furthermore, since the positive equilibrium of model (4) without Allee effect is given by $(N^*, P^*) = (\frac{a\beta}{1-\beta}, \frac{ar(K-K\beta-a\beta)}{K(1-\beta)^2})$, provided $K > \frac{a\beta}{1-\beta}$ and $\beta < 1$, this is obvious from (16) that the equilibrium density of prey population is the same, however, the equilibrium density of predator population is smaller. That is, a decrease of predator population density at the equilibrium is caused by the Allee effect of prey population. This highlights the extinction risk caused by the Allee effect.

In the following, we collect the main results given in [13] related to the stability of the equilibrium E^* , with few modifications which guarantee the positivity of m^* .

Theorem 4.2. (i) If $K(1-\beta)^3(a-b)m < r(K\beta-K+a+a\beta)(a\beta+b-b\beta)^2$ holds, then the equilibrium point E^* is locally asymptotically stable for the system (4).

(ii) If $K(1-\beta)^3(a-b)m > r(K\beta-K+a+a\beta)(a\beta+b-b\beta)^2$ holds, then the equilibrium point E^* is unstable for the system (4).

(iii) Assume that $a < b$ and $a\beta + a + K\beta - K < 0$. Then the system (4) undergoes a Hopf bifurcation around the positive equilibrium E^* at

$$m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a-b)(1-\beta)^3}.$$

(iv) Assume that $a < b$, $rb^2(K-a) < K(b-a)m$ and $\max\{b, \frac{a\beta}{1-\beta}\} < K < \min\{a+b, b + \frac{a\beta}{1-\beta}\}$ holds. Then E^* is globally asymptotically stable for the system (4).

The Jacobian matrix of model (4) evaluated at $E^* = (N^*, P^*)$ takes the form

$$J = J(E^*, m) = \begin{pmatrix} \frac{mN^*}{(N^*+b)^2} + \frac{P^*N^*}{(N^*+a)^2} - \frac{rN^*}{K} & -\frac{N^*}{N^*+a} \\ \frac{asP^*}{(N^*+a)^2} & 0 \end{pmatrix}. \tag{18}$$

It is easy to see that $\det(J) = \frac{asN^*P^*}{(N^*+a)^3} > 0$ and $tr(J) = \frac{mN^*}{(N^*+b)^2} + \frac{P^*N^*}{(N^*+a)^2} - \frac{rN^*}{K}$.

In the next theorem we determine the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions for the system (4). To this purpose, we follow the frame work of Section 2 (see also [35], [19] and [26]). We highlight that these results were not established in [13].

Theorem 4.3. Assume that one of the following two conditions is true:

i) $a < b$ and $a\beta + a + K\beta - K < 0$;

ii) $a > b$ and $a\beta + a + K\beta - K > 0$.

Then the system (4) undergoes a Hopf bifurcation around the positive equilibrium E^* at $m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a-b)(1-\beta)^3}$. Furthermore, the first Lyapunov coefficient associated with the equilibrium E^* is given by

$$l_1(m^*) = \frac{\operatorname{Re}(c_1(m^*))}{\omega_0(m^*)} \quad (19)$$

or equivalently

$$l_1(m^*) = \frac{1}{2\omega_0^2} \operatorname{Re} \left(iG_{20}G_{11} + \omega_0 \frac{G_{21}}{2} \right) \quad (20)$$

with

$$\omega_0(m^*) = \sqrt{\frac{\operatorname{ars}\beta((K-b)(1-\beta) - 2a\beta)}{K(a-b)(1-\beta)}}$$

where

$$\operatorname{Re}(c_1(m^*)) = - \left(\frac{m^*b(2N^{*2} - (b-4a)N^* + ab)}{2(N^*+b)^4N^*(N^*+a)} + \frac{aP^*(2N^*+a)}{2(N^*+a)^4N^*} + \frac{r(N^*-a)}{2kN^*(N^*+a)} \right). \quad (21)$$

If $(\frac{d}{dm} \operatorname{tr}(J)|_{m=m^*})^{-1} l_1(m^*) < 0$ (resp. > 0) the Hopf bifurcation is supercritical (resp. subcritical) for the system (4). On the other hand, if $l_1(m^*) < 0$ (resp. > 0) the bifurcating periodic solutions are stable (resp. unstable).

Proof. To determine the direction of the Hopf bifurcation, we will calculate the first Lyapunov coefficient for the system (4) at $m = m^*$. Note that $E^* = (N^*, P^*)$ is the unique positive equilibrium of system (4) and the Jacobian matrix of system (4) evaluated at (E^*, m^*) is given by

$$J^* = J(N^*, P^*, m^*) = \begin{pmatrix} 0 & -\frac{N^*}{N^*+a} \\ \frac{asP^*}{(N^*+a)^2} & 0 \end{pmatrix}. \quad (22)$$

The Jacobian matrix (22) has the purely imaginary eigenvalues

$$\lambda(m^*) = \pm i\omega_0(m^*) = \pm i \sqrt{\frac{\operatorname{ars}\beta((K-b)(1-\beta) - 2a\beta)}{K(a-b)(1-\beta)}}.$$

From [13] we have

$$\frac{d}{dm} \operatorname{tr}(J)|_{m=m^*} = \frac{d}{dm} \operatorname{Re}\lambda(m)|_{m=m^*} = \frac{\beta(a-b)(1-\beta)^2}{(b\beta - a\beta - b)^2}. \quad (23)$$

Note that the characteristic equation associated with the matrix given in (22) is

$$\lambda^2 + \frac{\operatorname{ars}\beta[(K-b)(1-\beta) - 2a\beta]}{K(a-b)(1-\beta)} = 0 \quad \text{with} \quad \frac{\operatorname{ars}\beta[(K-b)(1-\beta) - 2a\beta]}{K(a-b)(1-\beta)} > 0.$$

The eigenvectors associated with the eigenvalues $\lambda(m^*) = i\sqrt{\frac{ars\beta[(K-b)(1-\beta)-2a\beta]}{K(a-b)(1-\beta)}}$ and $\bar{\lambda}(m^*) = -i\sqrt{\frac{ars\beta[(K-b)(1-\beta)-2a\beta]}{K(a-b)(1-\beta)}}$ are, respectively,

$$q(m^*) = \begin{pmatrix} 1 \\ -\frac{i\omega_0(N^*+a)}{N^*} \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \quad \text{and} \quad \bar{q}(m^*) = \begin{pmatrix} 1 \\ \frac{i\omega_0(N^*+a)}{N^*} \end{pmatrix} = \begin{pmatrix} \bar{a}_0 \\ \bar{b}_0 \end{pmatrix}. \quad (24)$$

On the other hand, the transposed matrix of J is given by

$$J^T = J^T(N^*, P^*, m^*) = \begin{pmatrix} 0 & \frac{asP^*}{(N^*+a)^2} \\ -\frac{N^*}{N^*+a} & 0 \end{pmatrix}$$

where

$$p(m^*) = \frac{1}{2} \begin{pmatrix} 1 \\ -\frac{iN^*}{\omega_0(N^*+a)} \end{pmatrix} = \begin{pmatrix} a_0^* \\ b_0^* \end{pmatrix} \quad \text{and} \quad \bar{p}(m^*) = \frac{1}{2} \begin{pmatrix} 1 \\ \frac{iN^*}{\omega_0(N^*+a)} \end{pmatrix} = \begin{pmatrix} \bar{a}_0^* \\ \bar{b}_0^* \end{pmatrix} \quad (25)$$

are the eigenvectors of J^T associated with the eigenvalues $\lambda(m^*) = i\sqrt{\frac{ars\beta[(K-b)(1-\beta)-2a\beta]}{K(a-b)(1-\beta)}}$ and $\bar{\lambda}(m^*) = -i\sqrt{\frac{ars\beta[(K-b)(1-\beta)-2a\beta]}{K(a-b)(1-\beta)}}$, respectively.

We calculate the terms $\mathbf{B}(q, q)$, $\mathbf{B}(q, \bar{q})$, $\mathbf{B}(\bar{q}, \bar{q})$, $\mathbf{C}(q, q, q)$, $\mathbf{C}(q, q, \bar{q})$, $\mathbf{C}(q, \bar{q}, \bar{q})$ and $\mathbf{C}(\bar{q}, \bar{q}, \bar{q})$ given in Section 2 which are evaluated at (E^*, m^*) . Given that

$$\begin{aligned} f_{NN}(m_0^*, 0, 0) &= -\frac{2r}{K} + \frac{2m_0^*b}{(N^*+b)^3} + \frac{2aP^*}{(N^*+a)^3} & ; g_{NN}(m_0^*, 0, 0) &= -\frac{asP^*}{(N^*+a)^3} \\ f_{NP}(m_0^*, 0, 0) &= -\frac{a}{(N^*+a)^2} & ; g_{NP}(m_0^*, 0, 0) &= \frac{as}{(N^*+a)^2} \\ f_{PN}(m_0^*, 0, 0) &= -\frac{a}{(N^*+a)^2} & ; g_{PN}(m_0^*, 0, 0) &= \frac{as}{(N^*+a)^2} \\ f_{PP}(m_0^*, 0, 0) &= 0 & ; g_{PP}(m_0^*, 0, 0) &= 0 \end{aligned}$$

we have

$$\begin{aligned} \mathbf{B}(q, q) &= \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} f_{NN}(m_0^*, \mathbf{0})a_0^2 + f_{PP}(m_0^*, \mathbf{0})b_0^2 + 2f_{NP}(m_0^*, \mathbf{0})a_0b_0 \\ g_{NN}(m_0^*, \mathbf{0})a_0^2 + g_{PP}(m_0^*, \mathbf{0})b_0^2 + 2g_{NP}(m_0^*, \mathbf{0})a_0b_0 \end{pmatrix} \\ &= \begin{pmatrix} \left(-\frac{2r}{K} + \frac{2m_0^*b}{(N^*+b)^3} + \frac{2aP^*}{(N^*+a)^3} \right) + \frac{2a}{(N^*+a)^2} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \\ -\frac{asP^*}{(N^*+a)^3} - \frac{2as}{(N^*+a)^2} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{B}(q, \bar{q}) &= \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \\ &= \begin{pmatrix} f_{NN}(m_0^*, \mathbf{0})|a_0|^2 + f_{NP}(m_0^*, \mathbf{0})(a_0\bar{b}_0 + \bar{a}_0b_0) + f_{PP}(m_0^*, \mathbf{0})|b_0|^2 \\ g_{NN}(m_0^*, \mathbf{0})|a_0|^2 + g_{NP}(a_0\bar{b}_0 + \bar{a}_0b_0) + g_{PP}|b_0|^2 \end{pmatrix} \\ &= \begin{pmatrix} \left(-\frac{2r}{K} + \frac{2m_0^*b}{(N^*+b)^3} + \frac{2aP^*}{(N^*+a)^3} \right) \\ -\frac{asP^*}{(N^*+a)^3} \end{pmatrix}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} f_{NNN}(m_0^*, 0, 0) &= -\frac{6m_0^*b}{(N^*+b)^4} - \frac{6aP^*}{(N^*+a)^4} & ; & \quad f_{PPP}(m_0^*, 0, 0) = 0 \\ f_{NNP}(m_0^*, 0, 0) &= \frac{2a}{(N^*+a)^3} & ; & \quad f_{PNP}(m_0^*, 0, 0) = 0 \\ f_{NPN}(m_0^*, 0, 0) &= \frac{2a}{(N^*+a)^3} & ; & \quad f_{PNN}(m_0^*, 0, 0) = \frac{2a}{(N^*+a)^3} \\ f_{NPP}(m_0^*, 0, 0) &= 0 & ; & \quad f_{PPN}(m_0^*, 0, 0) = 0, \end{aligned}$$

$$\begin{aligned} g_{NNN}(m_0^*, 0, 0) &= \frac{6asP^*}{(N^*+a)^4} & ; & \quad g_{PNN}(m_0^*, 0, 0) = -\frac{2as}{(N^*+a)^3} \\ g_{NNP}(m_0^*, 0, 0) &= -\frac{2asP^*}{(N^*+a)^3} & ; & \quad g_{PNP}(m_0^*, 0, 0) = 0 \\ g_{NPN}(m_0^*, 0, 0) &= -\frac{2as}{(N^*+a)^3} & ; & \quad g_{PPN}(m_0^*, 0, 0) = 0 \\ g_{NPP}(m_0^*, 0, 0) &= 0 & ; & \quad g_{PPP}(m_0^*, 0, 0) = 0, \end{aligned}$$

we have

$$\begin{aligned} \mathbf{C}(q, q, \bar{q}) &= \begin{pmatrix} g_0 \\ h_0 \end{pmatrix} \\ &= \begin{pmatrix} f_{NNN}|a_0|^2 a_0 + f_{NNP}(2|a_0|^2 b_0 + a_0^2 \bar{b}_0) + f_{NPP}(2|b_0|^2 a_0 + b_0^2 \bar{a}_0) + f_{PPP}|b_0|^2 b_0 \\ g_{NNN}|a_0|^2 a_0 + g_{NNP}(2|a_0|^2 b_0 + a_0^2 \bar{b}_0) + g_{NPP}(2|b_0|^2 a_0 + b_0^2 \bar{a}_0) + g_{PPP}|b_0|^2 b_0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{6m_0^*b}{(N^*+b)^4} - \frac{6aP^*}{(N^*+a)^4} - \frac{2a}{(N^*+a)^3} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \\ \frac{6asP^*}{(N^*+a)^4} + \frac{2as}{(N^*+a)^3} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \end{pmatrix}. \end{aligned}$$

For the sake of convenience we denote $q(m^*)$ as q and $p(m^*)$ as p , as well as its conjugates. On the other hand, we get

$$\begin{aligned} G_{20} &= \langle p, \mathbf{B}(q, q) \rangle, \quad G_{11} = \langle p, \mathbf{B}(q, \bar{q}) \rangle, \quad G_{02} = \langle p, \mathbf{B}(\bar{q}, \bar{q}) \rangle, \quad G_{30} = \langle p, \mathbf{C}(q, q, q) \rangle \\ G_{21} &= \langle p, \mathbf{C}(q, q, \bar{q}) \rangle, \quad G_{12} = \langle p, \mathbf{C}(q, \bar{q}, \bar{q}) \rangle, \quad G_{03} = \langle p, \mathbf{C}(\bar{q}, \bar{q}, \bar{q}) \rangle \end{aligned} \tag{26}$$

where

$$\begin{aligned} \langle p, \mathbf{B}(q, q) \rangle &= \left\langle \frac{1}{2} \begin{pmatrix} 1 \\ -\frac{iN^*}{\omega_0(N^*+a)} \end{pmatrix}, \begin{pmatrix} \left(-\frac{2r}{K} + \frac{2m_0^*b}{(N^*+b)^3} + \frac{2aP^*}{(N^*+a)^3} \right) + \frac{2a}{(N^*+a)^2} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \\ -\frac{asP^*}{(N^*+a)^3} - \frac{2as}{(N^*+a)^2} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \end{pmatrix} \right\rangle \\ &= \left(-\frac{r}{K} + \frac{m^*b}{(N^*+b)^3} + \frac{aP^*}{(N^*+a)^3} + \frac{as}{(N^*+a)^2} \right) + \left(\frac{2a-N^*}{2(N^*+a)^2} \sqrt{\frac{asP^*}{N^*(N^*+a)}} \right) i \\ \langle p, \mathbf{B}(q, \bar{q}) \rangle &= \left\langle \frac{1}{2} \begin{pmatrix} 1 \\ -\frac{iN^*}{\omega_0(N^*+a)} \end{pmatrix}, \begin{pmatrix} \left(-\frac{2r}{K} + \frac{2m_0^*b}{(N^*+b)^3} + \frac{2aP^*}{(N^*+a)^3} \right) \\ -\frac{asP^*}{(N^*+a)^3} \end{pmatrix} \right\rangle \\ &= -\frac{r}{K} + \frac{m_0^*b}{(N^*+b)^3} + \frac{aP^*}{(N^*+a)^3} - \frac{N^*}{2(N^*+a)^2} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \langle p, \mathbf{C}(q, q, \bar{q}) \rangle = \\ & \frac{1}{2} \left\langle \frac{1}{2} \begin{pmatrix} 1 \\ -\frac{iN^*}{\omega_0(N^*+a)} \end{pmatrix}, \begin{pmatrix} -\frac{6m_0^*b}{(N^*+b)^4} - \frac{6aP^*}{(N^*+a)^4} - \frac{2a}{(N^*+a)^3} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \\ \frac{6asP^*}{(N^*+a)^4} + \frac{2as}{(N^*+a)^3} \sqrt{\frac{asP^*}{N^*(N^*+a)}} i \end{pmatrix} \right\rangle \\ & = \frac{1}{4} \left(-\frac{6m_0^*b}{(N^*+b)^4} - \frac{6aP^*}{(N^*+a)^4} - \frac{2as}{(N^*+a)^3} \right. \\ & \quad \left. + \left(\frac{6asP^*}{(N^*+a)^4} \sqrt{\frac{N^*(N^*+a)}{asP^*}} - \frac{2a}{(N^*+a)^3} \sqrt{\frac{asP^*}{N^*(N^*+a)}} \right) i \right) \end{aligned}$$

therefore,

$$\begin{aligned} & Re \left(\frac{i}{2\omega_0} \langle p, \mathbf{B}(q, q) \rangle \langle p, \mathbf{B}(q, \bar{q}) \rangle \right) = \\ & \frac{(N^*+a)}{2N^*(N^*+a)^2} \left((N^* - a) \left(-\frac{r}{K} + \frac{m_0^*b}{(N^*+b)^3} + \frac{aP^*}{(N^*+a)^3} \right) + \frac{asN^*}{(N^*+a)^2} \right), \end{aligned}$$

$$\begin{aligned} Re \left(\frac{1}{2} \langle p, \mathbf{C}(q, q, \bar{q}) \rangle \right) & = -\frac{1}{2} \left[\left(\frac{6mb}{(N^*+b)^4} + \frac{6aP^*}{(N^*+a)^4} \right) \sqrt{\frac{asP^*}{N^*[N^*+a]}} \left(\frac{N^*[N^*+a]}{asP^*} \right)^{\frac{3}{2}} \right. \\ & \quad \left. + \frac{2N^*}{P^*(N^*+a)^2} \right] \\ & \quad - \frac{1}{2} \left[\left(\frac{6mb}{(N^*+b)^4} + \frac{6aP^*}{(N^*+a)^4} \right) \sqrt{\frac{asP^*}{N^*[N^*+a]}} \left(\sqrt{\frac{N^*[N^*+a]}{asP^*}} \right)^3 \right. \\ & \quad \left. + \frac{2N^*}{P^*(N^*+a)^2} \right] \\ & \quad - \frac{N^*}{P^*} \left[3 \left(\frac{mb}{(N^*+b)^4} + \frac{aP^*}{(N^*+a)^4} \right) \frac{[N^*+a]}{as} + \frac{1}{(N^*+a)^2} \right]. \end{aligned}$$

and

$$Re \left(\frac{1}{2} \langle p, \mathbf{C}(q, q, \bar{q}) \rangle \right) = \frac{1}{4} \left(-\frac{6m_0^*b}{(N^*+b)^4} - \frac{6aP^*}{(N^*+a)^4} - \frac{2as}{(N^*+a)^3} \right)$$

finally,

$$Re(c_1(m^*)) = - \left(\frac{m^*b(2N^{*2} - (b-4a)N^* + ab)}{2(N^*+b)^4 N^*(N^*+a)} + \frac{aP^*(2N^*+a)}{2(N^*+a)^4 N^*} + \frac{r(N^*-a)}{2KN^*(N^*+a)} \right). \quad (27)$$

Consequently, according with [26] and taking into account (23) if $a < b$, $a\beta + a + K\beta - K < 0$, and $l_1(m^*) > 0$ (resp. $l_1(m^*) < 0$), the Hopf bifurcation is supercritical (resp. subcritical) for the system (4), and the bifurcating periodic solutions are unstable (resp. stable). On the other hand, if $a > b$, $a\beta + a + K\beta - K > 0$ and $l_1(m^*) < 0$ (resp. $l_1(m^*) > 0$), the Hopf bifurcation is subcritical (resp. supercritical) for the system (4) and the bifurcating periodic solutions are stable (resp. unstable). \square

Remark 4.4. It is worth highlighting that the results obtained in Theorems 4 and 5 concerning the existence of periodic orbits for the non-periodic system (4), are local

results which were obtained by Hopf bifurcation arguments. As observed in Section 3, the analysis of system (4) involving periodic cases, will be considered in a next exposition since our main purpose in this paper is to carry out a local stability and bifurcation analysis for the proposed system.

In the following propositions we give conditions on the parameters such that the expression (21) becomes negative, which implies that $l_1(m^*) < 0$ and consequently, the bifurcating periodic orbits of system (4) around the equilibrium E^* are stable.

Remark 4.5. It is clear that $\varphi(N^*) = 2N^{*2} - (b - 4a)N^* + ab$ is a quadratic function, which geometrically represents a parabola that opens upwards and whose vertex is given by the point $\left(\frac{b-4a}{4}, \frac{16ab - (b^2 + 16a^2)}{8}\right)$, which is positive as long as the discriminant of the equation $\varphi(N^*) = 0$ is negative or equivalent when $16ab > b^2 + 16a^2$. Consequently $\varphi(N^*) > 0$, if $16ab > b^2 + 16a^2$.

Taking into account Remark 4.5, we can establish the following propositions.

Proposition 4.6. *If $m < br$, $a < b$, $\max\left\{b + \frac{a\beta}{1-\beta}, \frac{a(\beta+1)}{1-\beta}\right\} < K < b + \frac{a\beta}{1-\beta}$, $16ab > b^2 + 16a^2$ and*

$$\frac{m^*b(2N^{*2} - (b-4a)N^* + ab)}{(N^*+b)^4} + \frac{aP^*(2N^*+a)}{(N^*+a)^3} > \frac{r(a-N^*)}{K} \quad (28)$$

then $l_1(m^*) < 0$. Moreover, if $a < b$ we have $\frac{d}{dm}tr(J)|_{m=m^*} < 0$. Consequently, the system (4) undergoes a subcritical Hopf bifurcation around the equilibrium point E^* at

$$m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a-b)(1-\beta)^3}.$$

Furthermore, according with Theorem 3.3 page 100 in [35], the bifurcating periodic solutions are stable.

Proof. It is clear that the conditions $m < br$, $a < b$, $\max\left\{b, \frac{a(\beta+1)}{1-\beta}\right\} < K < b + \frac{a\beta}{1-\beta}$, $16ab > b^2 + 16a^2$ and (28) imply that $m^* > 0$, $P^* > 0$, $\omega_0(m^*) > 0$ and hence $l_1(m^*) = \frac{Re(c_1(m^*))}{\omega(m^*)} < 0$. Furthermore, by (23) we have $\frac{d}{dm}tr(J)|_{m=m^*} < 0$ if $a < b$. \square

The Figure 4 illustrates the result obtained in Proposition 4.6.

Proposition 4.7. *If $m < br$, $a > b$, $b + \frac{2a\beta}{1-\beta} < K < \frac{a(1+\beta)}{1-\beta}$, $16ab > b^2 + 16a^2$ and*

$$\frac{m^*b(2N^{*2} - (b-4a)N^* + ab)}{(N^*+b)^4} + \frac{aP^*(2N^*+a)}{(N^*+a)^3} > \frac{r(a-N^*)}{k}, \quad (29)$$

then $l_1(m^*) < 0$. Moreover, if $a > b$ we have $\frac{d}{dm}tr(J)|_{m=m^*} > 0$. Consequently, the system (4) undergoes a supercritical Hopf bifurcation around the equilibrium point E^* at

$$m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a-b)(1-\beta)^3}.$$

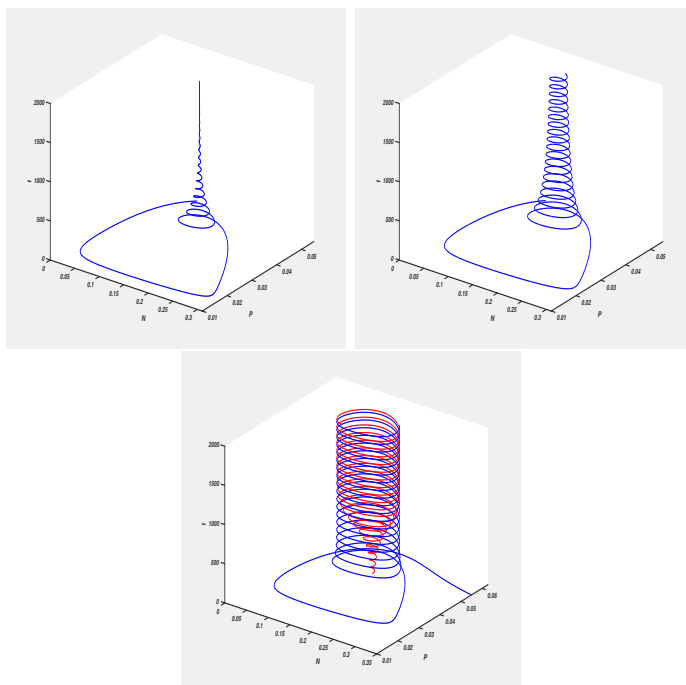


Figure 4. Subcritical Hopf bifurcation around the positive equilibrium $(0.157, 0.085)$. The simulations were carried using MATCONT [16] to illustrate the result obtained in Proposition 4.6, by choosing the values of parameters: (a) $a = 0.1, b = 0.7, \beta = 0.6, K = 0.71, r = 1, s = 0.1, m = 0.52, (N^*, P^*) = (0.15, 0.043)$. (b) $a = 0.1, b = 0.7, \beta = 0.6, K = 0.71, r = 1, s = 0.1, m^* = 0.516, (N^*, P^*) = (0.15, 0.0446)$. (c) $a = 0.1, b = 0.7, \beta = 0.6, K = 0.71, r = 1, s = 0.1, m = 0.5, (N^*, P^*) = (0.15, 0.049)$.

Furthermore, according with Theorem 3.3 page 100 in [26], the bifurcating periodic solutions are stable.

Proof. It is clear that the conditions $m < br, a > b b + \frac{2a\beta}{1-\beta} < K < \frac{a(1+\beta)}{1-\beta}$, $16ab > b^2 + 16a^2$ and (29) imply that $m^* > 0, P^* > 0, \omega_0(m^*) > 0$ and hence $l_1(m^*) = \frac{Re(c_1(m^*))}{\omega(m^*)} < 0$. Furthermore, by (23) we have $\frac{d}{dm}tr(J)|_{m=m^*} > 0$ if $a > b$. \square

The Figure 5 illustrates the result obtained in Proposition 4.7.

4.2. The case of strong Allee effect

Following the ideas in [13], assume that $m > br$, that is, the Allee effect in model (4) is strong. One can verify that model (4) has one boundary equilibrium $E_0 = (0, 0)$. If $m < \frac{r(K+b)^2}{4K}$ the system (4) has other two boundary equilib-

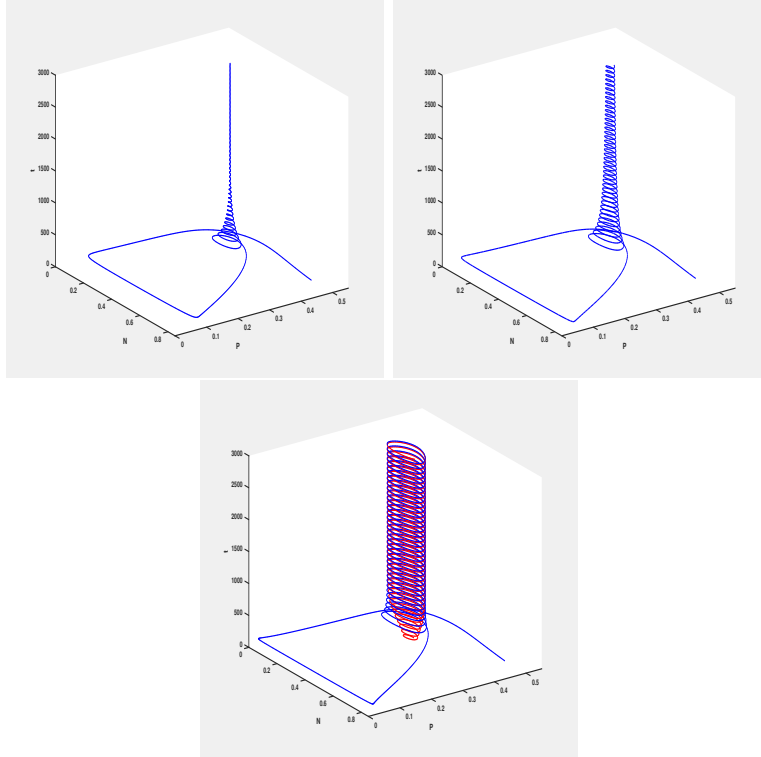


Figure 5. Supercritical Hopf bifurcation around the positive equilibrium $(0.2, 0.5)$. The simulations were carried using MATCONT [16] to illustrate the result obtained in Proposition 4.7, by choosing the values of parameters: (a) $a = 0.7$, $b = 0.2$, $\beta = \frac{1}{3}$, $K = 1$, $r = 1$, $s = 0.2$, $m = 0, 14$, $(N^*, P^*) = (0.3, 0.42)$. (b) $a = 0.7$, $b = 0.2$, $\beta = \frac{1}{3}$, $K = 1$, $r = 1$, $s = 0.2$, $m^* = 0, 15$, $(N^*, P^*) = (0.3, 0.4)$. (c) $a = 0.7$, $b = 0.2$, $\beta = \frac{1}{3}$, $K = 1$, $r = 1$, $s = 0.2$, $m = 0, 16$, $(N^*, P^*) = (0.3, 0.38)$.

$$\text{ria } E_1 = (N_1, P_1) = \left(\frac{r(K-b) - \sqrt{r^2(K-b)^2 + 4Kr(br-m)}}{2r}, 0 \right) \quad \text{and} \quad E_2 = (N_2, P_2) = \left(\frac{r(K-b) + \sqrt{r^2(K-b)^2 + 4Kr(br-m)}}{2r}, 0 \right).$$

It is not difficult to determine that the equilibria E_0 is always locally stable, E_1 and E_2 are saddle points.

Finally, if

$$K > b + \frac{a\beta}{1-\beta} \quad \text{and} \quad br < m < br + \frac{a\beta r}{K(1-\beta)^2} [(K-b)(1-\beta) - a\beta] \quad (30)$$

holds, then the system (4) has a unique positive equilibrium $E^* = (N^*, P^*)$ where $N^* = \frac{a\beta}{1-\beta}$; $P^* = \frac{ar(K-K\beta-a\beta)}{K(1-\beta)^2} - \frac{am}{b(1-\beta)+a\beta}$. In this subsection, we will assume the condition (30) from now on.

In the following theorem, we summarize the main result from [13] regarding the stability

of the equilibrium point E^* under the influence of a strong Allee effect, with a few adjustments ensuring the positivity of m^* .

Theorem 4.8. Assume that one of the following conditions are true:

- i) $a > b$ and $b + \frac{2a\beta}{1-\beta} < K < \frac{a}{1-\beta} + \frac{a\beta}{1-\beta}$
- ii) $a < b$ and $\max\{b + \frac{a\beta}{1-\beta}, \frac{a}{1-\beta} + \frac{a\beta}{1-\beta}\} < K < b + \frac{2a\beta}{1-\beta}$.

Then:

(1) If $K(1 - \beta)^3(a - b)m < r(K\beta - K + a + a\beta)(a\beta + b - b\beta)^2$ holds, the equilibrium point E^* is locally asymptotically stable for system (4).

(2) If $K(1 - \beta)^3(a - b)m > r(K\beta - K + a + a\beta)(a\beta + b - b\beta)^2$ holds, the equilibrium point E^* is unstable for system (4).

(3) The system (4) undergoes a Andronov-Hopf bifurcation at

$$m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a - b)(1 - \beta)^3}. \tag{31}$$

We prove in the next theorem that if model (4) does not have neither boundary equilibria nor interior equilibrium, i.e., if the Allee effect of prey population is very strong, then E_0 is globally asymptotically stable. Any positive orbit converges to E_0 as t tends to infinity, meaning prey and predators cannot coexist even if the initial population density of prey is abundant. As observed in Remark 3.8, this annihilating phenomenon of the species may occur due to the weakly persistence of system (4) subject to strong Allee effect. A typical phase portrait is shown in Figure 6.

Theorem 4.9. Let c_1 and c_2 be positive constants. The equilibrium point E_0 of system (4) is globally asymptotically stable provided $c_1 > c_2s$ and $m > r(b + K)$. In this case, the only equilibrium point of system (4) is E_0 .

Proof. Consider the following Lyapunov function $V(N, P) : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ defined by $V(N(t), P(t)) = c_1N(t) + c_2P(t)$. The time derivative of V along the solutions of system (4) is

$$\frac{dV}{dt} = c_1rN - \frac{c_1r}{K}N^2 - \frac{c_1m}{N+b}N + (sc_2 - c_1)\frac{NP}{N+a} - s\beta c_2P$$

Taking into account that $\lim_{t \rightarrow \infty} N(t) \leq K$, we have, for $t \rightarrow \infty$, that

$$\begin{aligned} \frac{dV}{dt} &\leq -\frac{c_1r}{K}N^2 + c_1rN - \frac{c_1m}{K+b}N + (sc_2 - c_1)\frac{NP}{N+a} - s\beta c_2P \\ &= N\left(-\frac{c_1r}{K}N + c_1r - \frac{c_1m}{K+b}\right) + (sc_2 - c_1)\frac{NP}{N+a} - s\beta c_2P. \end{aligned} \tag{32}$$

Now, $\frac{dV}{dt} \leq 0$ if $N\left(-\frac{c_1r}{K}N + c_1r - \frac{c_1m}{K+b}\right) \leq 0$ and $sc_2 - c_1 \leq 0$. The last inequality is equivalent to $c_1 > sc_2$. On the other hand, define $\Phi(N) = N\left(-\frac{c_1r}{K}N + c_1r - \frac{c_1m}{K+b}\right)$.

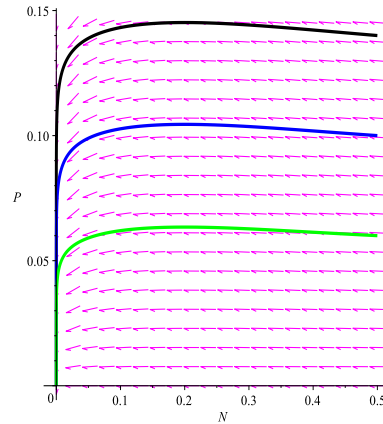


Figure 6. We made the phase portrait using MAPLE 13 to illustrate the global stability of $E_0 = (0, 0)$ by choosing: $a = 0.6$; $b = 0.6$; $\beta = 0.25$; $K = 1$; $r = 1$; $m = 0.7$; $s = 0.1$. Here, $m < \frac{r(K+b)^2}{4K}$, $K > b + \frac{a\beta}{1-\beta}$ and $m > br + \frac{a\beta r((K-b)(1-\beta) - a\beta)}{K(1-\beta)^2}$.

$\frac{c_1 m}{K+b}$) which is a parabola with concavity towards down, whose zeros are $N_1^* = 0$ and $N_2^* = \frac{[r(K+b) - m]K}{r(K+b)}$. Thus, if $N_2 < 0$ we have $\Phi(N) \leq 0$ for all $N \geq 0$. This is the case if $m \geq r(K+b)$. Finally, observe that:

- i) $m > r(K+b) > \frac{r(K+b)^2}{4K} \iff K > \frac{b}{3}$, which is true since $K > b$. This implies that system (4) has no boundary equilibria other than E_0 .
- ii) $m > r(K+b) > br + \frac{a\beta r}{K(1-\beta)^2} [(K-b)(1-\beta) - a\beta] \iff K^2 - \frac{a\beta}{1-\beta}K + (\frac{a\beta b}{1-\beta} + \frac{a^2\beta^2}{(1-\beta)^2}) > 0$. If we define $\Psi(K) = K^2 - \frac{a\beta}{1-\beta}K + (\frac{a\beta b}{1-\beta} + \frac{a^2\beta^2}{(1-\beta)^2})$, we easily conclude that $\Psi(K) > 0$ for all $K > 0$ since the the parabola has concavity towards up and its zeros are complex conjugate. Consequently, this implies that system (4) has no positive equilibria within \mathbb{R}_+^2 . □

Remark 4.10. In model (4) without a strong Allee effect, the boundary equilibria are $E_0 = (0, 0)$ and $E_1 = (N_1, 0)$ both of which are saddle points. There exists a positive equilibrium E^* , feasible under conditions $K > \frac{a\beta}{1-\beta}$ and $\beta < 1$. The equilibrium E_0 of system (4) is always a locally stable node, implying that both prey and predators will become extinct if their population densities enter the attraction region of E_0 . In particular, if the population density of prey becomes less than $N_1 = \frac{r(K-b) - \sqrt{r^2(K-b)^2 + 4Kr(br-m)}}{2r}$, then both predators and prey will go extinct (A typical phase portrait is shown in Figure 7). It is important to observe that the unstable manifold of the equilibrium E_2 goes toward the interior equilibrium (blue curve in Figure 7), and the stable manifold of E_1 (black curve in Figure 7) is above the unstable manifold of E_2 . The model (4) when compared with the predator-prey model without Allee effect, increases the extinction risk of both predators and prey due to the presence of strong Allee effect in the prey population.

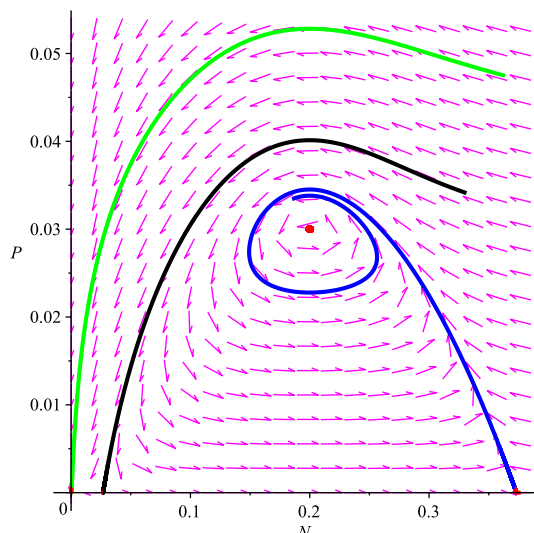


Figure 7. We made the phase portrait using MAPLE 13 to illustrate that prey and predators will become extinct when their population densities lie in the attraction region of E_0 by choosing: $a = 0.6$; $b = 0.6$; $\beta = 0.25$; $K = 1$; $r = 1$; $m = 0.61$; $s = 0.1$. The critical points on the plane $N P$ are points: $E_0 = (0, 0)$, $E_1 = (0.02679491925, 0)$, $E_2 = (0.3732050808, 0)$ and $E^* = (0.2, 0.03)$.

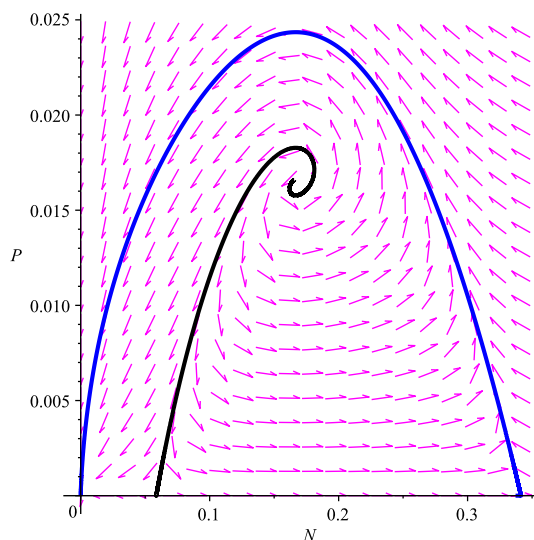


Figure 8. We made the phase portrait using MAPLE 13 to illustrate that prey and predators will become extinct when their population densities lie in the attraction region of E_0 . However, if we take initial conditions on the stable manifold of E_1 (black curve) it means the coexistence of the prey population. We choose the parameter values: $a = 0.5$; $b = 0.6$; $\beta = 0.25$; $K = 1$; $r = 1$; $m = 0.62$; $s = 0.1$. The critical points on the plane $N P$ are points: $E_0 = (0, 0)$, $E_1 = (0.02679491925, 0)$, $E_2 = (0.3732050808, 0)$ and $E^* = (0.2, 0.03)$.

Remark 4.11. It is worth observing that in Figure 7 the stable manifold of the equilibrium point E_1 is above the unstable manifold of the equilibrium point E_2 . On the other hand,

in Figure 8 the stable manifold of the equilibrium point E_1 is below the unstable manifold of the equilibrium point E_2 . This suggests that heteroclinic cycles occurs in model (4) for certain values of the parameters. This numerical experiment confirms the presence of an interesting global bifurcation not discussed earlier for this model. Accordingly we have the following:

Conjecture In view of remark 4.11, we conjecture that heteroclinic cycles occurs in model (4) for certain values of the parameters.

In the following, we determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions in the case of strong Allee effect for the system (4).

Theorem 4.12. *Assume that one of the following two conditions is true:*

- i) $a > b$ and $b + \frac{2a\beta}{1-\beta} < K < \frac{a}{1-\beta} + \frac{a\beta}{1-\beta}$,
- ii) $a < b$ and $\max\left\{b + \frac{a\beta}{1-\beta}, \frac{a}{1-\beta} + \frac{a\beta}{1-\beta}\right\} < K < b + \frac{2a\beta}{1-\beta}$.

Then, the system (4) undergoes a Hopf Bifurcation around the positive equilibrium $E^* = (N^*, P^*)$ at $m = m^*$ given by (31). Furthermore, the first Lyapunov coefficient associated with the equilibrium E^* is given by (19), where

$$\operatorname{Re}(c_1(m^*)) = - \left(\frac{b(2N^{*2} - (b-4a)N^* + ab)}{2(N^*+b)^4 N^*(N^*+a)} m^* + \frac{aP^*(2N^*+a)}{2(N^*+a)^4 N^*} + \frac{r(N^*-a)}{2kN^*(N^*+a)} \right). \quad (33)$$

If $(\frac{d}{dm} \operatorname{tr}(J)|_{m=m^*})^{-1} l_1(m^*) < 0$ (resp. > 0) the Hopf bifurcation is supercritical (resp. subcritical) for the system (4). On the other hand, if $l_1(m^*) < 0$ (resp. > 0) the bifurcating periodic solutions are stable (resp. unstable).

Proof. The proof is analogous to that given in Theorem 4.3 and therefore will be omitted. □

In the following propositions, taking into account Remark 4.5, we give conditions on the parameters such that the expression (33) becomes negative, which implies that $l_1(m^*) < 0$ and consequently the bifurcating periodic orbits of system (4) around the equilibrium E^* are stable.

Proposition 4.13. *If $br < m < br + \frac{a\beta r}{K(1-\beta)^2} [(K-b)(1-\beta) - a\beta]$, $\max\left\{b + \frac{a\beta}{1-\beta}, \frac{a(\beta+1)}{1-\beta}\right\} < K < b + \frac{2a\beta}{1-\beta}$, $a < b$, $16ab > b^2 + 16a^2$, and*

$$\frac{m^*b(2N^{*2} - (b-4a)N^* + ab)}{2(N^*+b)^4 N^*(N^*+a)} + \frac{aP^*(2N^*+a)}{2(N^*+a)^4 N^*} > \frac{r(a-N^*)}{2kN^*(N^*+a)}, \quad (34)$$

then $l_1(m^*) < 0$. Moreover, if $a < b$ we have $(\frac{d}{dm} \operatorname{tr}(J)|_{m=m^*}) < 0$. Consequently, the system (4) undergoes a subcritical Hopf bifurcation around the equilibrium point E^* at

$$m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a-b)(1-\beta)^3}.$$

Furthermore, the bifurcating periodic solutions are stable.

Proof. The conditions $br < m < br + \frac{a\beta r}{K(1-\beta)^2} [(K-b)(1-\beta) - a\beta]$, $\max\{b + \frac{a\beta}{1-\beta}, \frac{a(\beta+1)}{1-\beta}\} < K < b + \frac{2a\beta}{1-\beta}$, $a < b$, $16ab > b^2 + 16a^2$ and (34) imply that $N^* > 0$, $P^* > 0$, $m^* > 0$ and hence $l_1(m^*) < 0$. Furthermore, by (23) we have $\frac{d}{dm} (tr J(m))_{m=m^*} < 0$ if $a < b$. \square

The Figure 9 illustrates the result obtained in Proposition 4.13.

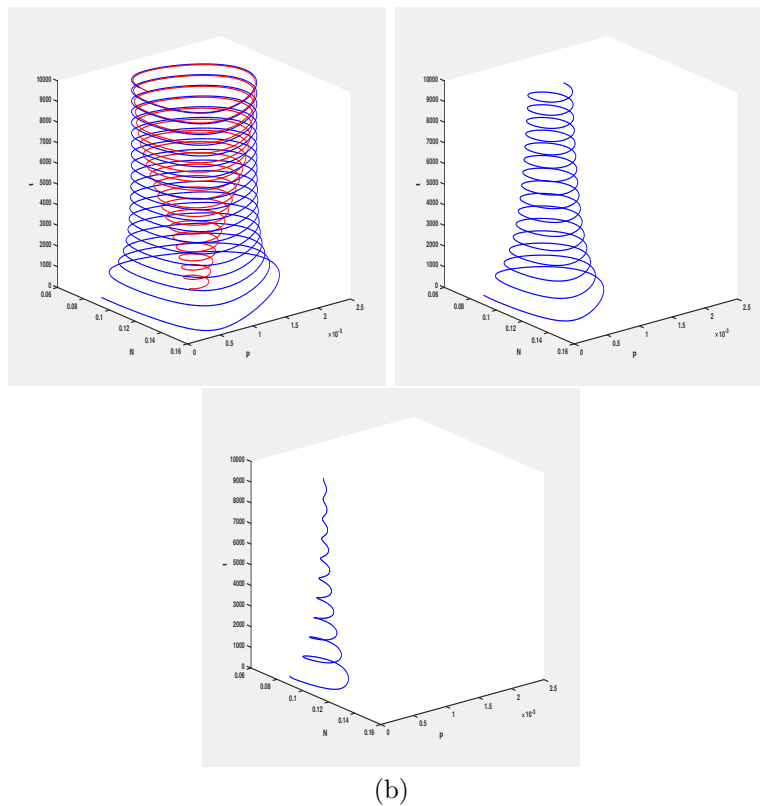


Figure 9. Subcritical Hopf bifurcation around the positive equilibrium (0.3, 0.22). The simulations were carried using MATCONT [16] to illustrate the result given by Proposition 4.13, by choosing the values of parameters: (a) $a = 0.1$, $b = 0.3$, $\beta = 0.5$, $K = 0.49$, $r = 0.2$, $s = 0.1$, $m = 0.061$, $(N^*, P^*) = (0.1, 0.0013)$. (b) $a = 0.1$, $b = 0.3$, $\beta = 0.5$, $K = 0.49$, $r = 0.2$, $s = 0.1$, $m^* = 0.062$, $(N^*, P^*) = (0.1, 0.00083)$. (c) $a = 0.1$, $b = 0.3$, $\beta = 0.5$, $K = 0.49$, $r = 0.2$, $s = 0.1$, $m = 0.063$, $(N^*, P^*) = (0.1, 0.00033)$.

Proposition 4.14. If $br < m < br + \frac{a\beta r}{K(1-\beta)^2} [(K-b)(1-\beta) - a\beta]$, $b + \frac{2a\beta}{1-\beta} < K$, $a > b$, $16ab > b^2 + 16a^2$, and

$$\frac{m^* b (2N^{*2} - (b - 4a) N^* + ab)}{2(N^* + b)^4 N^* (N^* + a)} + \frac{aP^* (2N^* + a)}{2(N^* + a)^4 N^*} > \frac{r(a - N^*)}{2KN^* (N^* + a)}, \quad (35)$$

then $l_1(m^*) < 0$. Moreover, if $a > b$ we have $\frac{d}{dm} \text{tr}(J)|_{m=m^*} > 0$. Consequently, the system (4) undergoes a supercritical Hopf bifurcation around the equilibrium point E^* at $m = m^* = \frac{r(a\beta + a + K\beta - K)(a\beta - b\beta + b)^2}{K(a-b)(1-\beta)^3}$. Furthermore, the bifurcating periodic solutions are stable.

Proof. The conditions $br < m < br + \frac{a\beta r}{K(1-\beta)^2} [(K-b)(1-\beta) - a\beta]$, $b + \frac{2a\beta}{1-\beta} < K$, $a > b$, $16ab > b^2 + 16a^2$, and (35), imply that $N^* > 0$, $P^* > 0$, $m^* > 0$, and $l_1(m^*) < 0$. Furthermore, by (23) we have $\frac{d}{dr} (\text{tr}J(m))|_{m=m^*} > 0$ if $a > b$. \square

The Figure 10 illustrates the result obtained in Proposition 4.

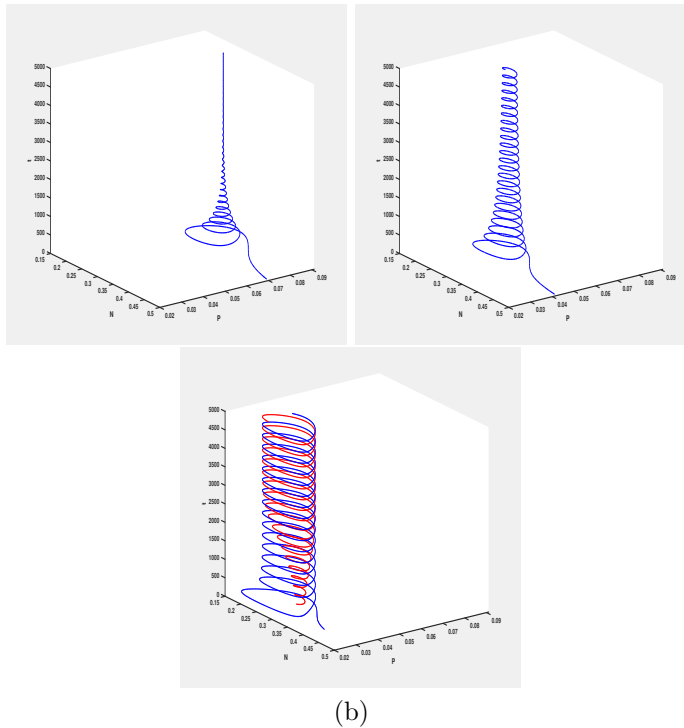


Figure 10. Supercritical Hopf bifurcation around the positive equilibrium $(0.3, 0.22)$. The simulations were carried using MATCONT [16] to illustrate the result given by Proposition 4.14, by choosing the values of parameters: (a) $a = 0.8$, $b = 0.45$, $\beta = 0.25$, $K = 1$, $r = 1$, $s = 0.1$, $m = 0.047$, $(N^*, P^*) = (0.26, 0.082)$. (b) $a = 0.8$, $b = 0.45$, $\beta = 0.25$, $K = 1$, $r = 1$, $s = 0.1$, $m^* = 0.04891$, $(N^*, P^*) = (0.26, 0.054259)$. (c) $a = 0.8$, $b = 0.45$, $\beta = 0.25$, $K = 1$, $r = 1$, $s = 0.1$, $m = 0.5$, $(N^*, P^*) = (0.26, 0.038)$.

5. Discussion

It is well known that in the past decades, the Allee effect has been drawn remarkable attention in many aspects of populational dynamics, and its consequences become more and more significant. In this paper, a model describing the interaction of a predator-prey

system with Holling type II functional response subject to additive Allee effect in the prey equation has been analyzed. It is observed that this model exhibits interesting dynamics of the interacting populations due to the presence of the Allee effect. We investigate the impact of the Allee effect on the stability of the equilibrium states of model (4) when the prey population is subject to both, weak and strong Allee effect. When we compare model (4) with the predator-prey model without Allee effect it is observed that, although the extinction risk of both predators and prey may not occur in the case of weak Allee effect, the equilibrium density of predator population is smaller. In other words, a decrease in the population density at a equilibrium is caused by the presence of weak Allee effect. It is also observed that, the model (4) subject to strong Allee effect when compared with the predator-prey model without Allee effect, increases the extinction risk of both predators and prey due to the presence of the strong Allee effect. Conditions are derived for the occurrence of Hopf bifurcation of the equilibria of model (4) when subject to weak and strong Allee effect. Compared with the model that does not include Allee effect, it is observed that the Allee effect of the prey population may be a destabilizing force in the predator-prey system when the Allee effect constant m is shifted.

Finally, through computer simulations carried in MATLAB, we conjecture that this model may admit heteroclinic bifurcation which has important biological implications.

Note that the model (4) does not take into account the fact neither that the populations naturally develop life strategies nor that they are not spatially dependent. Both of these considerations involve diffusion processes that can be quite intricate as different concentration levels of individuals cause different population movements. Accordingly, it would be interesting to introduce diffusion in appropriate terms of the model since it is natural to assume that members of the same or different populations may meet any other member with different probability. Although the authors in [13] has considered the spatial version of model (4), to the best of our knowledge there is no results related to the complete analyzes of codimension 1 Hopf bifurcation for this system. We defer our work in this direction to a subsequent exposition.

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