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A note about isothermic surfaces in $\mathbb{R}^{n-j,j}$

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Abstract. In this note we survey our results on the description of timelike isothermic surfaces in $\mathbb{R}^{n-j,j}$ using the Grassmannian systems or U/K-systems. We give the natural extensions of the definition of Ribaucour and Darboux transformations for timelike isothermic surfaces and review how those transformations correspond to dressing actions of suitable simple elements.

1. Introduction

An isothermic surface in $\mathbb{R}^3$ is, by definition, a surface which admits, away from its umbilic points, a coordinates system which is a conformal line of curvature system. Several works about this kind of surface and its geometric transformations, such as the Christoffel, Ribaucour and Darboux transformations, are known ([3], [4], [5], [6]). In particular, a new approach using integrable systems was given recently by Bruck-Du-Park-Terng in [2], by Burstall-Hertrich-Pedit-Pinkall in [4] and by Burstall in [3]. In [2] is found a connection between the isothermic surfaces and the integrable system constituted by the U/K-system, while in [3] and [4], they are investigated using curved flats. Turning our attention to the Lorentzian setting, recently the authors in [8], [9] obtained a description of the timelike isothermic surfaces in the pseudo-Riemannian space $\mathbb{R}^{n-j,j}$ with signature $j \geq 1$, through of the U/K-systems. Since in the Lorentzian case, the shape operators can be of different algebraic types, the authors considered the cases when all the shape operators are diagonalizable over $\mathbb{R}$ or over $\mathbb{C}$.

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The purpose of this note is to review isothermic surfaces in $\mathbb{R}^{n-j,j}$ together with explicit examples, and to give a short report of the work done by the authors in [8], [9], for timelike isothermic surfaces.

This note is divided as follows. In Section 2, we review the U/K-systems and some results of Terng-Uhlenbeck ([15]) about Birkhoff-type factorization over certain loop groups, obtaining via dressing action new solutions of the U/K-system. Section 3 contains definitions and explicit examples of isothermic surfaces in $\mathbb{R}^{n-j,j}$. Finally, in Section 4, we make a description of the timelike isothermic surfaces through the U/K-system involving the geometric transformations associated to the dressing actions. We observe that [17] considered the real diagonalizable case.

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2. The Grassmannian Systems

The U/K-system or Grassmannian system for U/K a symmetric space of rank n and Cartan decomposition $\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$, is the first order non-linear system of partial differential equations given by the equation

$$\left[a_i, \frac{\partial v}{\partial x_j}\right] - \left[a_j, \frac{\partial v}{\partial x_i}\right] = [[a_i, v], [a_j, v]], \qquad 1 \leq i \neq j \leq n,$$

for $v : \mathbb{R}^n \to \mathcal{P} \cap \mathcal{A}^{\perp}$, where $\mathcal{A}$ is a maximal abelian subalgebra in the subspace $\mathcal{P}$ and $\{a_i\}_1^n$ is a basis of $\mathcal{A}$.

From of the work done by Terng in [14] one knows that the U/K-system admits a zero curvature formulation given by the flatness of the 1-parameter family of $\mathcal{U} \oplus \mathbb{C}$-connection 1-form

$$\theta_\lambda = \sum_{j=1}^{n} (a_j \lambda + [a_j, v]) dx_j, \tag{1}$$

and in addition, that the flatness of θ_λ for all $\lambda \in \mathbb{C}$ is equivalent to the existence of a map $E : \mathbb{R}^n \times \mathbb{C} \to U$ such that

$$E_{x_j} = E(a_j \lambda + [a_j, v]), \quad 1 \leq j \leq n. \tag{2}$$

Such a solution E of the system (2) is known as a *trivialization* of v.

From the existence of the family θ_λ one can have certain elements belonging to an infinite dimensional loop group acting on the space of solutions of the U/K-system, giving a dressing action. The key point is to identify which geometric transformation between the submanifolds associated to the two solutions of the U/K-system, corresponds to that dressing action.

Next we recall the construction of that action following Terng-Uhlenbeck ([15]), which is based on the Birkhoff Factorization Theorem.

Let G be a complex and simple Lie group and $\mathcal{G}$ its Lie algebra. Let $S^2 = \mathbb{C} \cup \{\infty\}$ and $\mathcal{V}_{1/\epsilon} = \{\lambda \in S^2 \mid |\lambda| > 1/\epsilon\}$. We denote by $L(G)$ the Lie group containing all the holomorphic maps f from $\mathcal{V}_{1/\epsilon} \cap \mathbb{C}$ to G. In addition, we denote by $L_+(G)$ the Lie subgroup containing all the maps $f \in L(G)$ which extend holomorphically to $\mathbb{C}$, and $L_-(G)$ denotes the subgroup containing all the maps $f \in L(G)$ which extend holomorphically to $\mathcal{V}_{1/\epsilon}$ and such that $f(\infty) = I$.

Theorem 2.1 (Birkhoff Factorization Theorem, [13]). *The multiplication map*

$$\beta : L_+(G) \times L_-(G) \to L(G),$$

given by $\beta(h_+, h_-) = h_+ h_-$, is one-to-one and its image is an open and dense subset of $L(G)$.

This theorem assures that an element $h \in L(G)$ can be factored uniquely as $h = h_+ h_-$ where $h_\pm \in L_\pm(G)$.

The Birkhoff factorization theorem allows one to obtain another factorization for the special subgroups of $L(G)$ given by

$$L^{\tau,\sigma}(G) = \{f \in L(G) \mid \tau(f(\bar{\lambda})) = f(\lambda),\ \sigma(f(-\lambda)) = f(\lambda)\}$$

$$L^{\tau,\sigma}_{\pm}(G) = L^{\tau,\sigma}(G) \cap L_\pm(G),$$

where $\tau, \sigma : G \to G$ are the two involutions on the complex Lie group G that define the symmetric space U/K. Namely,

Proposition 2.2 ([15]). *The multiplication map $\beta : L^{\tau,\sigma}_+(G) \times L^{\tau,\sigma}_-(G) \to L^{\tau,\sigma}(G)$ is one-to-one and its image is open and dense in $L^{\tau,\sigma}(G)$.*

We recall that the conditions on f, namely, $\tau(f(\bar{\lambda})) = f(\lambda)$ and $\sigma(f(-\lambda)) = f(\lambda)$, are called the *$U/K$-reality conditions*, and that a trivialization $E(x, \lambda)$ for a solution v of the U/K-system which satisfies those conditions is called a *frame* for the solution v of U/K-system.

Proposition 2.2 allows one to prove the next theorem which establishes how to construct new solutions of the U/K-system from a given initial solution v (see Terng-Uhlenbeck [15]).

Theorem 2.3 ([15]). *Let v be a solution of the U/K-system, E a trivialization of the Lax connection θ_λ of v with $E(0,\lambda) = I$, and $g \in L_-^{\tau,\sigma}(G)$. Then gE can be factored uniquely as*

$$g(\lambda)E(x,\lambda) = \tilde{E}(x,\lambda)\tilde{g}(x,\lambda),$$

with $\tilde{E}(x) \in L_+^{\tau,\sigma}(G)$ and $\tilde{g}(x) \in L_-^{\tau,\sigma}(G)$. Expand $\tilde{g}(x,\lambda)$ at $\lambda = \infty$:

$$\tilde{g}(x,\lambda) = I + \tilde{g}_1(x)\lambda^{-1} + \tilde{g}_2(x)\lambda^{-2} + \cdots$$

Let $\tilde{v} = v - \pi(\tilde{g}_1)$, where π is the projection onto $\mathcal{P} \cap \mathcal{A}^\perp$. Then,

- **(a)** *$\tilde{v}$ is a new solution of the U/K-system and $\tilde{E}$ is a frame of the Lax connection $\tilde{\theta}_\lambda$ of the $\tilde{v}$ such that $\tilde{E}(0,\lambda) = I$.*
- **(b)** *The map $(g, v) \to g * v := \tilde{v}$ is a local action of $L_-^{\tau,\sigma}(G)$ on the space of local solutions of the U/K-system (dressing action).*

We finish this section reviewing the equivalent gauge system to the U/K-system, called the U/K-systems II which are directly associated to isothermic surfaces ([2]).

Take v to be a solution of the U/K-system where $K = K_1 \times K_2$. Since $\theta_0 = \sum_i [a_i, v]dx_i = \sum_i \xi_i + \eta_i \in \mathcal{K}_1 + \mathcal{K}_2$ is a $\mathcal{K}$-valued flat connection, then $\sum_i \eta_i dx_i$ is also flat and there exists a map $g : \mathbb{R}^n \to K$ so that $g^{-1}dg = \theta_0$. Now take $g_2 : \mathbb{R}^n \to K_2$ a trivialization of $\sum_i \eta_i dx_i$. Then the equation given by the flatness of the connection form obtained by the gauge transformation:

$$\theta_\lambda^{II} := g_2 * \theta_\lambda = \sum_{i=1}^{n} (g_2 a_i g_2^{-1}\lambda + \xi_i)dx_i,$$

is the U/K-system II.

We recall that a trivialization for a solution η of the U/K-system II, which satisfies the U/K-reality conditions, is called the *frame* for the solution η of U/K-system II. We denote this frame by E^{II}.

3. Isothermic surfaces in $\mathbb{R}^{n-j,j}$

The goal in this section is to review some facts about isothermic surfaces in $\mathbb{R}^{n-j,j}$ for any signature $j \geq 1$. Let $\mathbb{R}^{n-j,j}$ be $\mathbb{R}^n$ with the metric

$$\langle v, w\rangle = v_1w_1 + v_2w_2 + \cdots + v_{n-j}w_{n-j} - v_{n-j+1}w_{n-j+1} - \cdots - v_nw_n,$$

for $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$.

Classically, a surface in $\mathbb{R}^3$ is called *isothermic* if, away from its umbilic points, it admits conformal coordinates for which the associated coordinate vectors ∂x_1, ∂x_2 form an eigenvector basis for the second fundamental form, i.e, if it admits a conformal line of curvature coordinate system. On the other hand, the 2-dimensional immersions in $\mathbb{R}^{n-j,j}$ can be spacelike or timelike. For the spacelike case the definition of isothermic surfaces is not so different from the Riemannian case in $\mathbb{R}^n$. One states its formal definition as follows.

Definition 3.1. Let $\mathcal{O}$ be a domain in $\mathbb{R}^2$. An immersion $X : \mathcal{O} \to \mathbb{R}^{n-j,j}$ is called a spacelike isothermic surface if it has flat normal bundle and the two fundamental forms are

$$I = e^{2u}(dx_1^2 + dx_2^2), \qquad II = e^u \sum_{i=1}^{n-2}(r_{i,1}dx_1^2 + r_{i,2}dx_2^2)e_{i+2}, \tag{3}$$

with respect to some parallel normal frame $\{e_\alpha\}$. Or equivalently, if $(x_1, x_2) \in \mathcal{O}$ is conformal and line of curvature coordinate system for X.

Spacelike minimal surfaces, spacelike surfaces with constant mean curvature and spacelike Bonnet surface in $\mathbb{R}^{2,1}$, provide examples of spacelike isothermic surfaces in $\mathbb{R}^{2,1}$ (see [10]). Another typical example of spacelike isothermic surface is spacelike surfaces of revolution in $\mathbb{R}^{2,1}$ ([10]).

In terms of its dual characterization (or Christoffel transform), a spacelike surface is isothermic if it admits a dual surface, which means, the surfaces have the same normal vectors and a conformal immersion reversing orientation.

Here we will construct an explicit example of a dual pair of spacelike isothermic surfaces in $\mathbb{R}^{2,1}$. For that, we first recall the following well known result (see [16]).

Proposition 3.2. *Let M be a spacelike surface of $\mathbb{R}^{2,1}$ with constant curvature -1 and free of umbilic points. Then there exists a local coordinate system x_1, x_2 such that the two* *fundamental forms are*

$$I = \cosh^2 u dx_1^2 + \sinh^2 u dx_2^2, \qquad II = \cosh u \sinh u (dx_1^2 + dx_2^2).$$

Moreover, u satisfies the Elliptic Sinh-Gordon equation

$$u_{x_1x_1} + u_{x_2x_2} = \sinh u \cosh u.$$

So, we have the following result.

Example 3.3. *Let M be a spacelike surface in $\mathbb{R}^{2,1}$ with curvature -1 and free of umbilic points. By the proposition above, it admits coordinates such that*

$$I = \cosh^2 u dx_1^2 + \sinh^2 u dx_2^2, \qquad II = \cosh u \sinh u (dx_1^2 + dx_2^2),$$

so its Gauss-Codazzi equation is $u_{x_1x_1} + u_{x_2x_2} = \sinh u \cosh u$. Let $X(x_1, x_2)$ denote the immersion of M and e_3 the unit normal of M. One can see that e_3 is a local parametrization of an open subset of pseudo-hyperbolic space $H^2(1) = \{p \in \mathbb{R}^{2,1} \mid \langle p, p\rangle = -1\}$ and that the fundamental forms for e_3 are given by:

$$I = \sinh^2 u dx_1^2 + \cosh^2 u dx_2^2, \qquad II = -(\sinh^2 u dx_1^2 + \cosh^2 u dx_2^2).$$

Now, let

$$Z_1 = X - e_3, \qquad Z_2 = X + e_3.$$

Since

$$dX = -(\cosh u dx_1 e_1 + \sinh u dx_2 e_2), \qquad de_3 = \sinh u dx_1 e_1 + \cosh u dx_2 e_2,$$

we have

$$dZ_1 = -e^u(dx_1 e_1 + dx_2 e_2), \qquad dZ_2 = -e^{-u}(dx_1 e_1 - dx_2 e_2),$$

and then

$$I_{Z_1} = e^{2u}(dx_1^2 + dx_2^2), \qquad II_{Z_1} = e^u(\sinh u dx_1^2 + \cosh u dx_2^2)$$

$$I_{Z_2} = e^{-2u}(dx_1^2 + dx_2^2), \qquad II_{Z_2} = e^{-u}(\sinh u dx_1^2 - \cosh u dx_2^2).$$

Hence, it follows that the two surfaces Z_1 and Z_2 constitute an example of a dual pair of spacelike isothermic surfaces in $\mathbb{R}^{2,1}$ according to Definition 3.1.

Now we turn our attention to the principal case, the timelike case (or Lorentzian case). We recall that (x_1, x_2) being isothermal coordinate systems, for the induced metric $\langle \cdot, \cdot \rangle$ on M and for the associated coordinates vectors $\partial x_1,\ \partial x_2$, there is a non-zero function λ so that

$$\langle \partial x_1, \partial x_1 \rangle = -\lambda^2, \quad \langle \partial x_2, \partial x_2 \rangle = \lambda^2, \quad \langle \partial x_1, \partial x_2 \rangle = 0.$$

Definition 3.4 ([12]). A Lorentzian immersion is called isothermic if there is some isothermal coordinate system (x_1, x_2) such that each shape operator with respect to the basis $\{\partial x_1, \partial x_2\}$ has one of the following forms:

$$\text{a)} \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad\qquad \text{b)} \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The first case is called *real isothermic* and the second *complex isothermic*. Since the work is local, one defines an *isothermic immersion* to be an immersion for which each point has a neighborhood which is either real isothermic or complex isothermic.

Formally, in term of a smooth map $u : M \to \mathbb{R}$, we have the following definition.

Definition 3.5 ([8]). Let $\mathcal{O}$ be a domain in $\mathbb{R}^{1,1}$. An immersion $X : \mathcal{O} \to \mathbb{R}^{n-j,j}$ is called a real timelike isothermic surface if it has flat normal bundle and the two fundamental forms are

$$I = e^{2u}(-dx_1^2 + dx_2^2), \qquad II = e^u \sum_{i=2}^{n-1} (r_{i-1,2} dx_2^2 - r_{i-1,1} dx_1^2) e_i, \tag{4}$$

with respect to some parallel normal frame $\{e_i\}$. Equivalently, if $(x_1, x_2) \in \mathcal{O}$ is a conformal and line of curvature coordinate system for X.

As typical examples, the second author shows in [12] that Lorentzian surfaces in $\mathbb{R}^{2,1}$ with constant mean curvature (including minimal surfaces) and with second fundamental form diagonalizable over $\mathbb{R}$, are real isothermic surfaces. Similarly, Lorentzian surfaces of revolution in $\mathbb{R}^{2,1}$ are real isothermic surfaces.

For the complex case, we define it formally as follows:

Definition 3.6 ([8]). Let $\mathcal{O}$ be a domain in $\mathbb{R}^{1,1}$. An immersion $X : \mathcal{O} \to \mathbb{R}^{n-j,j}$ is called a complex timelike isothermic surface if it has flat normal bundle and its two fundamental forms are

$$I = \pm e^{2u}(-dx_1^2 + dx_2^2), \qquad II = \sum_{i=2}^{n-1} e^u (r_{i1}(dx_2^2 - dx_1^2) - 2 r_{i2} dx_1 dx_2) e_i, \tag{5}$$

with respect to some parallel normal frame $\{e_i\}$.

Similarly, the Lorenztian surfaces in $\mathbb{R}^{2,1}$ with constant mean curvature (including minimal surfaces) and with second fundamental form diagonalizable over $\mathbb{C}$ are isothermic surfaces.

In [12] the second author also shows that each real or complex isothermic surface possesses a dual surface, which, following the positive definite terminology, is called the Christoffel transform of the original surface. See [12] for the characterization in terms of coordinates system of the dual pair of timelike isothermic surfaces.

The next is an example of a complex dual pair of isothermic surfaces in $\mathbb{R}^{2,1}$.

Example 3.7 ([12]). *Consider a Lorentzian helicoid*

$$X(x_1,x_2) = (x_2, \sinh x_1 \sinh x_2, \cosh x_2 \sinh x_1)$$

with normal vector $N(x_1,x_2) = \dfrac{1}{\cosh x_1}(-\sinh x_1, \cosh x_2, \sinh x_2)$. *The dual surface to the Lorentzian helicoid is given by*

$$\tilde{X}(x_1,x_2) = \frac{1}{\cosh x_1}(\sinh x_1, -\cosh x_2, -\sinh x_2),$$

which is a parametrization of part of a standard immersion of the Lorentzian sphere.

4. Description of timelike isothermic surfaces

Here we review the main results of [8] and [9], which establish the descriptions of timelike isothermic surfaces in $\mathbb{R}^{n-j,j}$ using Grassmannian systems. The key point for doing this, is the identification of two distinct maximal abelian subalgebras $\mathcal{A}_i$, for which the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-systems correspond respectively to the geometry of the real and complex timelike isothermic surfaces in $\mathbb{R}^{n-j,j}$.

In this section, we assume $U/K = O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$, where

$$O(n-j+1,j+1) = \left\{ X \in GL(n+2) \,\middle|\, X^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} X = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} \right\},$$

$I_{n-j,j} = \begin{pmatrix} I_{n-j} & 0 \\ 0 & -I_j \end{pmatrix}$ and $I_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We shall write down the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-systems explicitly.

From a direct computation one identifies the subspaces $\mathcal{K}$ and $\mathcal{P}$ of the Cartan decomposition of $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$. Next we take two maximal abelian subalgebras $\mathcal{A}_i$ in $\mathcal{P}$ spanned by the following two bases, each one corresponding respectively to the real and complex case.

$$\mathcal{A}_1 : \ a_1 = e_{n,n+1} + e_{n,n+2} + e_{n+1,n} + e_{n+2,n}, \quad a_2 = -e_{1,n+1} + e_{1,n+2} - e_{n+1,1} + e_{n+2,1},$$

$$\mathcal{A}_2 : \ a_1 = e_{1,n+1} + e_{n,n+2} + e_{n+1,n} - e_{n+2,1}, \quad a_2 = -e_{1,n+2} + e_{n,n+1} + e_{n+1,1} + e_{n+2,n},$$

where e_{ij} is the elementary $(n+2)\times(n+2)$ matrix, whose only non-zero entry is 1 in the ij-th place.

The real $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system for $\mathcal{A}_1$ is given by

$$\xi = \begin{pmatrix} \xi_1 & \xi_1 \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ \xi_2 & -\xi_2 \end{pmatrix} : \mathbb{R}^2 \to \mathcal{M}_{n\times 2},$$

$$\begin{cases} (r_{i,2})_{x_1} - (r_{i,1})_{x_1} = -2(r_{i,1}+r_{i,2})\xi_2, \quad i=1,\dots,n-2, \\ (r_{i,2})_{x_2} + (r_{i,1})_{x_2} = 2(r_{i,2}-r_{i,1})\xi_1, \quad i=1,\dots,n-2, \\ 2((\xi_1)_{x_2} + (\xi_2)_{x_1}) = \sum_{i=1}^{n-2} \sigma_i (r_{i1}^2 - r_{i2}^2), \\ (\xi_2)_{x_2} + (\xi_1)_{x_1} = 0, \end{cases} \tag{6}$$

where $\sigma_i = 1$, $i=1,\dots,n-j-1$ and $\sigma_i = -1$, $i = n-j,\dots,n-2$.

In particular, taking

$$B = \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix} = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix},$$

one has that the real $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system II is the set of partial differential equations for $(u, r_{1,1}, r_{1,2},\dots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} (r_{i,2})_{x_1} - (r_{i,1})_{x_1} = -2(r_{i,1}+r_{i,2})u_{x_1}, \quad i=1,\dots,n-2, \\ (r_{i,1})_{x_2} + (r_{i,2})_{x_2} = -2(r_{i,2}-r_{i,1})u_{x_2}, \quad i=1,\dots,n-2, \\ 2(u_{x_1x_1} - u_{x_2x_2}) = \sum_{i=1}^{n-2} \sigma_i (r_{i1}^2 - r_{i2}^2), \end{cases} \tag{7}$$

where $\sigma_i = 1$, $i=1,\dots,n-j-1$ and $\sigma_i = -1$, $i = n-j,\dots,n-2$.

The complex $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system for $\mathcal{A}_2$ is given by

$$\xi = \begin{pmatrix} \xi_1 & \xi_2 \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ \xi_2 & -\xi_1 \end{pmatrix} : \mathbb{R}^2 \to \mathcal{M}_{n\times 2},$$

$$\begin{cases} -r_{i,2_{x_2}} - r_{i,1_{x_1}} = 2(r_{i,2}\xi_1 - r_{i,1}\xi_2), \quad i = 1,\ldots,n-2, \\ -r_{i,1_{x_2}} + r_{i,2_{x_1}} = -2(r_{i,1}\xi_1 + r_{i,2}\xi_2), \quad i = 1,\ldots,n-2, \\ (-2\xi_1)_{x_2} + (2\xi_2)_{x_1} = \sum_{i=1}^{n-2} \sigma_i (r_{i,1}^2 + r_{i,2}^2), \\ (2\xi_2)_{x_2} - (2\xi_1)_{x_1} = 0, \end{cases} \tag{8}$$

where $\sigma_i = 1,\ i = 1,\ldots,n-j-1$ and $\sigma_i = -1,\ i = n-j,\ldots,n-2$.

Now taking $B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}$, the complex $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system II is the PDE for $(u, r_{1,1}, r_{1,2},\ldots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} -r_{i,2_{x_2}} - r_{i,1_{x_1}} = 2(r_{i,2}\xi_1 - r_{i,1}\xi_2), \\ -r_{i,1_{x_2}} + r_{i,2_{x_1}} = -2(r_{i,1}\xi_1 + r_{i,2}\xi_2), \\ -2u_{x_2x_2} + 2u_{x_1x_1} = \sum_{i=1}^{n-2} \sigma_i (r_{i,1}^2 + r_{i,2}^2), \end{cases} \tag{9}$$

where $\sigma_i = 1,\ i = 1,\ldots,n-j-1$ and $\sigma_i = -1,\ i = n-j,\ldots,n-2$.

To understand the geometries involved in the two $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-systems, we need to define the dual pairs of timelike isothermic surfaces in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$. We start with the real timelike case.

Definition 4.1. Let $\mathcal{O}$ be a domain in $\mathbb{R}^{1,1}$ and $X_i : \mathcal{O} \to \mathbb{R}^{n-j,j}$ an immersion with flat and non-degenerate normal bundle for $i = 1, 2$. Then (X_1, X_2) is called a real isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ if:

(i) The normal plane of $X_1(x)$ is parallel to the normal plane of $X_2(x)$ and $x \in \mathcal{O}$.

(ii) There exists a common parallel normal frame $\{e_2,\ldots,e_{n-1}\}$, where $\{e_i\}_2^{n-j}$ and $\{e_i\}_{n-j+1}^{n-1}$ are spacelike and timelike vectors, respectively.

(iii) $x \in \mathcal{O}$ is a conformal line of curvature coordinate system with respect to $\{e_2,\ldots,e_{n-1}\}$ for each X_k such that the fundamental forms of X_k are given by

$$\begin{cases} I_1 = b^{-2}(-dx_1^2 + dx_2^2), \\ II_1 = b^{-1}\sum_{i=2}^{n-1} [-(g_{i-1,1} + g_{i-1,2})dx_1^2 + (g_{i-1,2} - g_{i-1,1})dx_2^2] e_i, \\ I_2 = b^2(-dx_1^2 + dx_2^2), \\ II_2 = b\sum_{i=2}^{n-1} [-(g_{i-1,1} + g_{i-1,2})dx_1^2 - (g_{i-1,2} - g_{i-1,1})dx_2^2] e_i, \end{cases} \tag{10}$$

where $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ is in $O(1,1)$ and a $\mathcal{M}_{(n-2)\times 2}$-valued map $G = (g_{ij})$.

In a similar way, one defines the dual pair in the complex case as follows.

Definition 4.2. Let $\mathcal{O}$ be a domain in $\mathbb{R}^{1,1}$ and $X_i : \mathcal{O} \to \mathbb{R}^{n-j,j}$ an immersion with flat and non-degenerate normal bundle for $i = 1, 2$. Then (X_1, X_2) is called a complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ if:

(i) The normal plane of $X_1(x)$ is parallel to the normal plane of $X_2(x)$ and $x \in \mathcal{O}$.

(ii) There exists a common parallel normal frame $\{e_2, \ldots, e_{n-1}\}$, where $\{e_i\}_2^{n-j}$ and $\{e_i\}_{n-j+1}^{n-1}$ are spacelike and timelike vectors, respectively.

(iii) $x \in \mathcal{O}$ is a isothermal coordinate system with respect to $\{e_2, \ldots, e_{n-1}\}$, for each X_k, such that the fundamental forms of X_k are diagonalizable over $\mathbb{C}$. Namely,

$$\begin{cases} I_1 = b^{-2}(dx_1^2 - dx_2^2), \\ II_1 = -b^{-1}\sum_{i=1}^{n-2}[g_{i,2}(dx_2^2 - dx_1^2) + 2g_{i,1}dx_1dx_2]e_{i+1}, \\ I_2 = b^2(-dx_1^2 + dx_2^2), \\ II_2 = b\sum_{i=1}^{n-2}[g_{i,1}(dx_2^2 - dx_1^2) - 2g_{i,2}dx_1dx_2]e_{i+1}, \end{cases} \tag{11}$$

where $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ is in $O(1,1)$ and a $\mathcal{M}_{(n-2)\times 2}$-valued map $G = (g_{ij})$.

The geometries associated to the two $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-systems II were identified by the authors in [8], [9]. In particular, with the definitions of dual pairs above, the authors proved several results which we summarize in the next theorem.

Theorem 4.3 ([8], [9]). *Based on the notation above, for each case, real and complex, there is a one-to-one correspondence between the solutions* $(u, r_{11}, \ldots, r_{n-2,2})$ *of the* $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$*-systems II* (7) ((9), *respectively) and a dual pair of real* (*complex, respectively*) *isothermic surfaces in* $\mathbb{R}^{n-j,j}$ *of type* $O(1,1)$.

Example 4.4. *Because of the existence of the one-to-one correspondence between the solutions of the* $O(3,2)/O(2,1)\times O(1,1)$*-system II* (9) *and a dual pair of complex timelike isothermic surfaces* (*Theorem* 4.3), *the Lorentzian helicoid and the Lorentzian sphere* (*Example* 3.7), *constitutes an explicit solution of the* $O(3,2)/O(2,1) \times O(1,1)$*-system II* (9). *This means in particular, by looking at the first and second fundamental forms of the two surfaces that* $\xi_1 = 0$, $\xi_2 = \dfrac{\tanh x_1}{2}$ *and hence that the triple* $(u, 0, -e^{-2u})$ *is a solution of system* (9).

Following the ideas in Bruck-Du-Park-Terng in [2], we now describe explicit dressing actions of simple elements belonging to certain loop groups on the space of local solutions of the systems (7) and (9) (see [9] for more details).

We note that the study of the geometric transformations associated to the real timelike case was already considered in [17], so in the rest of this paper, we will focus in the authors' results in [9] which involve the identifications for the complex timelike case.

We identify the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-reality conditions, namely,

$$\begin{cases} \overline{g(\overline{\lambda})} = g(\lambda), \\ I_{n,2}\; g(-\lambda)\; I_{n,2} = g(\lambda), \\ g(\lambda)^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} g(\lambda) = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}. \end{cases} \tag{12}$$

Consider

$$G_+ = \{g : \mathbb{C} \to O(n-j+1, j+1; \mathbb{C}) \mid g \text{ is holomorphic and satisfies (12))}\}$$

$$G_- = \{g : S^2 \to O(n-j+1, j+1; \mathbb{C}) \mid g \text{ is meromorphic, } g(\infty) = I \text{ and satisfies (12)}\}.$$

Next we define the rational element

$$g_{s,\pi}(\lambda) = \left(\pi + \frac{\lambda - is}{\lambda + is}(I-\pi)\right)\left(\overline{\pi} + \frac{\lambda + is}{\lambda - is}(I - \overline{\pi})\right) \tag{13}$$

where $0 \neq s \in \mathbb{R}$, π is the orthogonal projection of $\mathbb{C}^{n+2}$ onto the span of $\begin{pmatrix} W \\ iZ \end{pmatrix}$, with respect to the bilinear form $\langle , \rangle_2$ given by

$$\langle U, V\rangle_2 = \sum_{i=1}^{n-j} \overline{u}_i v_i - \sum_{i=n-j+1}^{n} \overline{u}_i v_i + \overline{u}_{n+1} v_{n+2} + \overline{u}_{n+2} v_{n+1},$$

for $W \in \mathbb{R}^{n-j,j}$, $Z \in \mathbb{R}^{1,1}$ unit vectors. It is a simple computation to see that $g_{s,\pi} \in G_-$.

Lemma 4.5 (Main Lemma, [9]). *Let* $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$ *be a solution of the system* (8), *and* $E(x,\lambda)$ *a frame of* ξ *such that* $E(x,\lambda)$ *is holomorphic for* $\lambda \in \mathbb{C}$. *Let* $g_{s,\pi}$ *be the map defined by* (13) *and* $\widetilde{\pi}(x)$ *the orthogonal projection onto* $\mathbb{C}\begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}(x)$ *with respect to* $\langle \cdot, \cdot \rangle_2$, *where*

$$\begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}(x) = E(x, -is)^{-1}\begin{pmatrix} W \\ iZ \end{pmatrix}. \tag{14}$$

Let $\widehat{W} = \frac{\widetilde{W}}{\|\widetilde{W}\|_{n-j,j}}$ *and* $\widehat{Z} = \frac{\widetilde{Z}}{\|\widetilde{Z}\|_{1,1}}$, $\widetilde{E}(x,\lambda) = g_{s,\pi}(\lambda)E(x,\lambda)g_{s,\widetilde{\pi}(x)}(\lambda)^{-1}$,

$$\widetilde{\xi} = \xi - 2s(\widehat{W}\widehat{Z}^t J')_*, \tag{15}$$

where (ϑ_*) *is the projection onto the span of* $\{a_1, a_2\}^{\perp}$*. Then* $\widetilde{\xi}$ *is a solution of system* (8)*,* $\widetilde{E}$ *is a frame for* $\widetilde{\xi}$ *and* $\widetilde{E}(x, \lambda)$ *is holomorphic in* $\lambda \in \mathbb{C}$*.*

For the proof of the Main Lemma see [9].

Writing the new solution given by Lemma 4.5 as $\widetilde{\xi} = \begin{pmatrix} \widetilde{F} \\ \widetilde{G} \end{pmatrix}$, the components of $\widetilde{\xi}$ are:

$$\begin{cases} \widetilde{f}_{11} = -\widetilde{f}_{22} = f_{11} - s(\widehat{w}_1 \widehat{z}_2 - \widehat{w}_n \widehat{z}_1), \\ \widetilde{f}_{12} = \widetilde{f}_{21} = f_{12} - s(\widehat{w}_1 \widehat{z}_1 + \widehat{w}_n \widehat{z}_2), \\ \widetilde{r}_{i1} = r_{i1} - 2s\widehat{w}_{1+i}\widehat{z}_2, \\ \widetilde{r}_{i2} = r_{i2} - 2s\widehat{w}_{1+i}\widehat{z}_1, \end{cases} \tag{16}$$

for $F = (f_{ij})_{2\times 2},\ G = (r_{ij})_{(n-2)\times 2},\ \widetilde{F} = (\widetilde{f}_{ij})_{2\times 2},\ \widetilde{G} = (\widetilde{r}_{ij})_{(n-2)\times 2}$.

After getting the new solution $\widetilde{\xi}$ of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system (8), one obtains the corresponding new solution $(\widetilde{F}, \widetilde{G}, \widetilde{B}^{\sharp})$ or $(\widetilde{u}, \widetilde{r}_{11}, \ldots, \widetilde{r}_{n-2,2})$ of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (9). Next, one identifies which geometric transformations are associated to the dressing actions of the element $g_{s,\pi}$ on the space of local solutions of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (9). For doing that, we need a natural extension of the definition of Ribaucour transformation given in [7], and of the definition of Darboux transformation for surfaces in $\mathbb{R}^m$, for our case of complex timelike surfaces.

For $x \in \mathbb{R}^{n-j,j}$ and $v \in (T\mathbb{R}^{n-j,j})_x$, let $\gamma_{x,v}(t) = x + tv$ denote the geodesic starting at x in the direction of v.

Definition 4.6 ([9])**.** Let M^m and $\widetilde{M}^m$ be Lorentzian submanifolds whose shape operators are all diagonalizable over $\mathbb{R}$ or $\mathbb{C}$ immersed in the pseudo-Riemannian space $\mathbb{R}^{n-j,j}$, $0 < j < n$. A sphere congruence is a vector bundle isomorphism $P : \mathcal{V}(M) \to \mathcal{V}(\widetilde{M})$ that covers a diffeomorphism $\phi : M \to \widetilde{M}$ with the following conditions:

(1) If ξ is a parallel normal vector field of M, then $P \circ \xi \circ \phi^{-1}$ is a parallel normal field of $\widetilde{M}$.

(2) For any nonzero vector $\xi \in \mathcal{V}_x(M)$, the geodesics $\gamma_{x,\xi}$ and $\gamma_{\phi(x),P(\xi)}$ intersect at a point that is the same parameter value t away from x and $\phi(x)$.

For the following definition we assume that each shape operator is diagonalized over the real or complex numbers. We note that there are submanifolds for which this does not hold.

Definition 4.7 ([9]). A sphere congruence $P : \mathcal{V}(M) \to \mathcal{V}(\widetilde{M})$ that covers a diffeomorphism $\phi : M \to \widetilde{M}$ is called a Ribaucour transformation if it satisfies the following additional properties:

(1) If e is an eigenvector of the shape operator A_ξ of M, corresponding to a real eigenvalue then $\phi_*(e)$ is an eigenvector of the shape operator $A_{P(\xi)}$ of $\widetilde{M}$ corresponding to a real eigenvalue.

If $e_1 + ie_2$ is an eigenvector of A_ξ on $(TM)^{\mathbb{C}}$ corresponding to the complex eigenvalue $a + ib$ (so that $e_1 - ie_2$ corresponds to the eigenvalue $a - ib$), then $\phi_*(e_1) + i\phi_*(e_2)$ is an eigenvector corresponding to a complex eigenvalue for $A_{P(\xi)}$.

(2) The geodesics $\gamma_{x,e}$ and $\gamma_{\phi(x),\phi_*(e)}$ intersect at a point that is equidistant to x and $\phi(x)$ for real eigenvectors e, and γ_{x,e_j} and $\gamma_{\phi(x),\phi_*(e_j)}$ meet for the real and imaginary parts of complex eigenvectors $e_1 + ie_2$, i.e., for $j = 1, 2$.

Definition 4.8 ([9]). Let $M, \widetilde{M}$ be two timelike surfaces in $\mathbb{R}^{n-j,j}$ with flat and non-degenerate normal bundle, shape operators that are diagonalizable over $\mathbb{C}$ and $P : \mathcal{V}(M) \to \mathcal{V}(\widetilde{M})$ a Ribaucour transformation that covers the map $\phi : M \to \widetilde{M}$. If, in addition, ϕ is a sign-reversing conformal diffeomorphism then P is called a Darboux transformation.

In Definition 4.8, by a sign-reversing conformal diffeomorphism we mean that the timelike and spacelike vectors are interchanged and the conformal coordinates remain conformal.

We finish with the theorem which describes that the dressing action of the element $g_{s,\pi}$ on the space of local solutions of the system (9), corresponds to Darboux transformations between complex isothermic surfaces.

Theorem 4.9 ([9]). *Let (X_1, X_2) be a complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ corresponding to the solution (u, G) of the system* (9)*, and let $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$ be the corresponding solution of the system* (8)*, where*

$$F = \begin{pmatrix} u_{x_2} & u_{x_1} \\ u_{x_1} & -u_{x_2} \end{pmatrix}, \qquad B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}.$$

Let $g_{s,\pi}$ defined in (13)*, and $\widehat{W}$, $\widehat{Z}$ as in Lemma 4.5, for the solution ξ of the system* (8)*. Let $(\widetilde{E^\sharp}^{II}, \widetilde{A}^\sharp, \widetilde{B}^\sharp) = g_{s,\pi}.(E^{II}, A, B)$ for the action of $g_{s,\pi}$ over (E^{II}, A, B) where $A, B, \widetilde{A}^\sharp, \widetilde{B}^\sharp$ are the entries of*

$$E(x,0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}, \qquad \widetilde{E^\sharp}(x,0) = \begin{pmatrix} \widetilde{A}^\sharp(x) & 0 \\ 0 & \widetilde{B}^\sharp(x) \end{pmatrix}$$

and E^{II} is the frame corresponding to the solution (F, G, B) of the complex system II (9). *Write $A = (e_1, \ldots, e_n)$ and $\widetilde{A}^\sharp = (\widetilde{e}_1, \ldots, \widetilde{e}_n)$. Set*

$$\begin{cases} \widetilde{X}_1 = X_1 + \frac{2}{s}\widehat{z}_2 e^{-2u} \sum_{i=1}^n \widehat{w}_i e_i, \\ \widetilde{X}_2 = X_2 + \frac{2}{s}\widehat{z}_1 e^{2u} \sum_{i=1}^n \widehat{w}_i e_i \end{cases} \tag{17}$$

Then,

(i) $(\widetilde{u}, \widetilde{G})$ *is the solution of system* (9)*, corresponding to $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$, where $e^{4\widetilde{u}} = \frac{4\widehat{z}_2^4}{e^{4u}}$ and $\widetilde{G} = (\widetilde{r}_{ij})$ is defined by Lemma 4.5, for the new solution $\widetilde{\xi}$ of the system* (8)*.*

(ii) *The fundamental forms of the pair $(\widetilde{X}_1, \widetilde{X}_2)$ are, respectively,*

$$\begin{cases} \widetilde{I}_1 = e^{4\widetilde{u}}(-dx_1^2 + dx_2^2), \\ \widetilde{II}_1 = e^{2\widetilde{u}} \sum_{i=1}^{n-2} [\widetilde{r}_{i,1}(dx_2^2 - dx_1^2) - 2\widetilde{r}_{i,2} dx_1 dx_2]\widetilde{e}_{i+1}, \\ \widetilde{I}_2 = e^{-4\widetilde{u}}(dx_1^2 - dx_2^2), \\ \widetilde{II}_2 = -e^{-2\widetilde{u}} \sum_{i=1}^{n-2} \widetilde{r}_{i,2}(dx_2^2 - dx_1^2) + 2\widetilde{r}_{i,1} dx_1 dx_2]\widetilde{e}_{i+1}. \end{cases}$$

(iii) *The bundle morphism $P(e_k(x)) = \widetilde{e}_k(x)$, $k = 2, \ldots, n-1$ is a Ribaucour transformation covering the map $\phi_i : X_i \to \widetilde{X}_i$ for each $i = 1, 2$.*

(iv) *Each $\phi_i : X_i \to \widetilde{X}_i$, $i = 1, 2$ is a sign-reversing conformal diffeomorphism. This means, the bundle morphism $P(e_k(x)) = \widetilde{e}_k(x)$, $k = 2, \ldots, n-1$ covering the map $\phi_i : X_i \to \widetilde{X}_i$ is a Darboux transformation for each $i = 1, 2$.*

For the proofs and explicit examples of dual pair of complex timelike isothermic surfaces obtained by applying the Darboux transformation given in (17), see [8] and [9].

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