Revista INTEGRACIÓN
Universidad Industrial de Santander
Escuela de Matemáticas
Vol. 15, No 2, p. 107–116, julio–diciembre de 1997

Generalized Riemann Boundary Value Problem for Generalized Analityc Functions

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Key words and phrases: Boundary value problemsm, generalized analytic functions, Cauchy integral tye, singular integral operator. 1991 Mathematics.
Subjet classification. Primary 45E05, 30E20, 30E25.

1. Introduction

The classical Riemann boundary value problem for analytic functions and solutions of more general elliptic systems in the plane modeled many problems of Mathematical Physics, but closed form solutions of such problems are known only in a few cases (see Gakhov [4]; Vekua [9]; and Begehr-Gilbert [2]).

Using a successive reduction method, generalized Riemann boundary value problems and its conjugates are solved in closed form in [1, 3, 6, 8].

We are investigating classes of generalized Riemann boundary value problems for generalized analytic function w, which means, in this context, that w satisfies the equation

$$\frac{\partial w}{\partial \bar{z}} + Aw + B\bar{w} = 0,$$

where coefficients are in $L_{p,2}(C)$ $(2 < p)$ and also have compact support.

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2. Smootheness of generalized integral of Cauchy type

First, we describe some notation. Let L be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$ bounding a domain D^+, 0 D^+ and satisfying the condition $\theta(\delta) = 0(\delta)$, $\delta \to 0+$, which is defined using a metrical characteristic of a curve: $\theta(\delta) = \sup_{t\in L} \text{mes}\{\tau \in L : |t-\tau| \leq \delta\}$, $\delta > 0$.

Let $D^- = \bar{C}/\bar{D}^+$, $\varphi \in H_\omega(L)$ (i.e., φ is a uniformly Hölder continuous functions on L with characteristic nonnegative, nondecreasing function ω defined on $(0,d]$, $d =$ diameter L, and such that $\delta^{-1}\omega(\delta)$ is noncreasing, $\omega(\delta) \to 0$, $\delta \to 0+$, moreover,

$$\int_0^\delta \omega(\tau)/\tau d\tau + \delta \int_\delta^d \frac{\omega(\tau)}{\tau^2} d\tau = O(\omega(\delta)).$$

On the other hand, let D_0 be a regular domain, A and B complex functions in $L_{p,2}(C)$ $(2 < p)$ which vanish identically out side $\bar{D}_0$; and let $\Omega_1(z,t)$ and $\Omega_2(z,t)$ be the fundamental Kernels of the class $U_{p,2}(A,B,C/t)$ (see [9]).

Throughout the paper, L, ω , D_0 , A and B will be as before, with the following additional condition: $\delta^{\frac{p-2}{p}}\omega^{-1}(\delta) \to 0$, $\delta \to 0^+$.

Lemma 1 *Let $\varphi \in H_\omega(L)$, $K(\varphi,z)$ be the generalized integral of Cauchy type of φ in L; namely,*

$$K(\varphi,z) = \frac{1}{2\pi i}\int_L \{\Omega_1(z,\tau)\,\varphi(\tau)d\tau - \Omega_2(z,\tau)\bar{\varphi}(z,\tau)\,\overline{d\tau}\}. \tag{1}$$

Then $K(\varphi,z)$ is a sectionally generalized analytic function in $C\setminus L$ and

$$\begin{aligned} &P(A,B)[\varphi](t) = lim_{z\to t,\, z\in D^+}\, K(\varphi,z) = \\ &\tfrac{1}{2\pi i}\int_L [\Omega_1(t,\tau)\,\varphi(\tau) - \tfrac{\varphi(t)}{\tau-t}] - \Omega_2(t,\tau)\bar{\varphi}(\tau)\overline{d\tau} + \varphi(t), \\ &Q(A,B)[\varphi](t) = -lim_{z\to t,\, z\in D^-}\, K(\varphi,z) = \\ &-\tfrac{1}{2\pi i}\int_L [\Omega_1(t,\tau)\,\varphi(\tau) - \tfrac{\varphi(t)}{\tau-t}] - \Omega_2(t,\tau)\bar{\varphi}(\tau)\overline{d\tau}, \end{aligned} \tag{2}$$

are continuous projectors on $H_\omega(L)$. The first integrals have to be understood in the Cauchy principal value sence.

The proof of the last lemma and the following assertions involve only simple calculations and are omitted. We define formally an operator S(A,B), acting on $H_\omega(L)$ by the formula:

$$S(A,B) = P(A,B) - Q(A,B), \tag{3}$$

and we note that if A $\equiv$ B $\equiv 0$, then $\Omega_2(z,t) \equiv 0$, $\Omega_1(z,t) \equiv \frac{1}{t-z}$ and $S(A,B)$ reduces to the singular integral operator S (see [5]). Hence, formula (2) are the classical Plemelj-Sokholski formula related with the limiting values $\Phi^{\pm}(\varphi, t)$ on L of the integral of Cauchy type

$$\Phi(\varphi, z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} \, d\tau. \tag{4}$$

Remark 1 *If one introduces the spaces $H_\omega^{\pm}(L)$ of complex Hölder continuous in L functions, with characterist ω , and generalized analytic in $D^{\pm}$, vanishing at infinity, using the the fact that $S(A,B)$ maps the space $H_\omega(L)$ boundedly into itself, we are able to set the equality $H_\omega(L) = H_\omega^+(L) \oplus H_\omega^-(L)$.*

Lemma 2 *If $M(t) \in H_\omega(L)$ and the operator M is defined by means of the equality*

$$M[\varphi](t) = M(t)\varphi(t),$$

then the commutator operator $S(A,B)M - MS(A,B)$ is compact on $H_\omega(L)$.

The conjugate operator to the operator $S^*(A,B)$ that is defined on the total space of all functional Ξ of the form

$$\Xi(\varphi) = \mathrm{Re}\left(\frac{1}{2i}\int_L \varphi(\tau)\xi(\tau)\, d\tau\right), \quad \xi \in H_\omega(L)$$

(this space is identified with the space $H_\omega(L)$), is given by $-S(-A, -\bar{B})$ (see [7]). Hence,

$$P^*(A,B) = Q(-A, -\bar{B})$$

and

$$Q^*(A,B) = P(-A, -\bar{B}).$$

Riemann Boundary value problem

Let a, b and f be Hölder continuous functions on L and a and b nonvanishing on L. We consider the Riemann boundary value problem for generalized analytic functions in the following operational interpretation:

$$\begin{array}{l}\text{"To find a function } \varphi \in H_\omega(L) \text{ such that} \\ aP(A,B)[\varphi] + bQ(A,B)[\varphi] = f \text{ on } L\text{".}\end{array} \tag{5}$$

Expressing $b(t)/a(t)$, as a ratio of boundary values of the canonical function $X(z) = \exp\, \Phi(\ln[\tau^{-\chi}\frac{b(\tau)}{a(\tau)}],\ z)$ (see [4]), we have $\frac{b(t)}{a(t)} = \frac{X^+(t)}{t^{-\chi}X^-(t)}$, where

$$X^{\pm}(t) = \exp\, \Phi^{\pm}(\ln[\tau^{-\chi}\frac{b(\tau)}{a(\tau)}],\, t) \quad \text{and} \quad \chi = \frac{1}{2\pi}\left[\arg\frac{b(t)}{a(t)}\right]_L.$$

Now, introducing the notation

$$\hat{A} = A,$$
$$\hat{B} = \begin{cases} B\exp\left(2i\,\mathrm{Im}[\ln X(z)]\right), & z \in D^+, \\ z^{\chi}\bar{z}^{-\chi}B\exp(2i\,\mathrm{Im}[\ln X(z)]), & z \in D^-, \end{cases}$$

the following theorem holds:

Theorem 1 *For $\chi \geq 0$, the problem (5) is solvable for every function $f \in H_\omega(L)$. The solution for $t \in L$ is representable in the form $\varphi(t) = P(A,B)[\varphi](t) + Q(A,B)[\varphi](t)$, where*

$$P(A,B)[\varphi](t) = X^+(t)\left\{P(\hat{A},\hat{B})[\frac{f}{aX^+}](t) + \hat{R}_{\chi-1}(t)\right\}, \tag{6}$$

$$Q(A,B)[\varphi](t) = t^{-\chi}\, X^-(t)\left\{Q(\hat{A},\hat{B})[\frac{f}{aX^+}](t) - \hat{R}_{\chi-1}(t)\right\},$$

and $\hat{R}_{\chi-1}(t)$ is an arbitrary generalized polynomial, of degree not bigger than $\chi - 1$ (for $\chi = 0$, $\hat{R}_{\chi-1} \equiv 0$), of the class $U_{p,2}(\hat{A},\hat{B},C)$.

For $\chi < 0$, the inhomogeneous problem is solvable only under the assumptions

$$\operatorname{Re}\left(\frac{1}{2i}\int_L \hat{K}\prime(z^k,\tau)\,\frac{f(\tau)}{a(\tau)X^+(\tau)}\,d\tau\right) = 0,$$

$$\operatorname{Re}\left(\frac{1}{2i}\int_L \hat{K}\prime(iz^k,\tau)\,\frac{f(\tau)}{a(\tau)X^+(\tau)}\,d\tau\right) = 0, \tag{7}$$

$$k = 0,1,\ldots,\chi-7$$

where $\hat{K}'$ is the generalized Cauchy type integal on the circle $|z| = \rho$ (ρ has to be chosen in such a way that $\bar{D}_0 \cup \bar{D}^+$ is contained on it) associated to the adjoint equation

$$\frac{\partial w}{\partial \bar{z}} - \hat{A}w - \hat{B}\bar{w} = 0.$$

Remark 2 The corresponding adjoint problem, equations, operator, system, coefficient and so on, is denoted briefly by adding the suscript$'$ (for example, $\ll$ *problem* $(8)' \gg$.

The additional conditions (7) as well as the analytic case, can be obtained for the representation of generalized analytic integral of Cauchy type as power series in a neighourhood of infinity (see [4]).

3. The generalized boundary value problem for generalized analytic functions

In the space $H_\omega(L)$ we consider the problem

$$O_n[\varphi] = a_1P(A,B)a_2P(A,B).....a_nP(A,B)[\varphi]+$$

$$+b_1Q(A,B)b_2Q(A,B).....b_nQ(A,B)[\varphi] = f, \text{ on } L. \tag{8}$$

The functions $a_1, a_2, ...a_n$, b_1 , b_2 , ...b_n are asumed to be in $H_\omega(L)$ and nonvanishing; furthermore, the function f is also asumed to be in $H_\omega(L)$.

For analytic functions (i.e., A $\equiv$ B $\equiv$ 0), the problem (8) is discused in the papers [1,6] under certain conditions for the contour, coefficients and free term.

For the sake of simplicity, we present the solution of (8) for n = 2. The reasoning is similar in the general case. The basic scheme of the solution considered here is the same as in the latter references.

Problem (8) can be written as a Riemann boundary value problem:

$$a_1P(A,B)[u] + b_1Q(A,B)[v] = f, \tag{9}$$

where the functions u and v belong to $H_\omega(L)$ and are given by the equalities

$$u(t) = a_2P(A,B)[\varphi](t)\ ; \quad v(t) = b_2Q(A,B)[\varphi](t).$$

The index of this problem is $\alpha = \frac{1}{2\pi}\left(\arg \frac{b_1(t)}{a_1(t)}\right)_L$ and its general solution is following according to

$$P(A,B)[u](t) = X^+(t)\left\{P(\hat{A},\hat{B})[\frac{f}{a_1X^+}](t) + \hat{R}_{\alpha-1}(t)\right\},$$

$$Q(A,B)[v](t) = t^{-\alpha}X^-(t)\left\{Q(\hat{A},\hat{B})[\frac{f}{a_1X^+}](t) - \hat{R}_{\alpha-1}(t)\right\},$$

where $\hat{R}_{\alpha-1}$ is a generalized polynomial of degree not higher than $\alpha - 1$ with arbitrary coefficients and $\hat{R}_{\alpha-1} \equiv 0$, for $\alpha \leq 0$. Problem (9) has a solution for $\alpha < 0$ if and only if the corresponding condition (7) holds.

The determination of φ , using $P(A,B)[u]$ and $Q(A,B)[v]$ already determined, leads to the problems

$$a_2P(A,B)[\varphi] - Q(A,B)[u] = P(A,B)[u], \tag{10}$$

$$P(A,B)[v] - b_2Q(A,B)[\varphi] = Q(A,B)[v]. \tag{11}$$

We factorize the functions $\frac{1}{a_2}$ and b_2 as follows:

$$a_2(t)Y^+(t) = t^{-\beta}Y^-(t) \quad \text{and} \quad Z^+(t) = b_2(t)t^{-\delta}Z^-(t),$$

where

$$Y^\pm(t) = \exp \Phi^\pm(\ln[\tau^{-\beta}\frac{1}{a_2(\tau)}], t) \quad \text{and} \quad Z^\pm(t) = \exp \Phi^\pm(\ln[\tau^{-\delta}b_2(\tau)], t);$$

$$\beta := \frac{1}{2\pi}\left(\arg \frac{1}{a_2(t)}\right)_L \quad \text{and} \quad \delta := \frac{1}{2\pi}(\arg b_2(t))_L .$$

If $\beta < 0$ or $\delta < 0$ the conditions

$$\begin{cases} \operatorname{Re}\left(\frac{1}{2i}\int\limits_L \frac{P(A,B)[u](\tau)}{a_2(\tau)Y^+(\tau)}\hat{K}\prime(z^k,\tau)\,d\tau\right) = 0, \\ \operatorname{Re}\left(\frac{1}{2i}\int\limits_L \frac{P(A,B)[u](\tau)}{a_2(\tau)Y^+(\tau)}\hat{K}\prime(iz^k,\tau)\,d\tau\right) = 0, \end{cases} \qquad k = 0,1,\ldots,-\beta-1, \tag{12}$$

or

$$\begin{cases} \operatorname{Re}\left(\frac{1}{2i}\int\limits_L \frac{Q(A,B)[v](\tau)}{Z^+(\tau)}\hat{K}\prime(z^m,\tau)\,d\tau\right) = 0, \\ \operatorname{Re}\left(\frac{1}{2i}\int\limits_L \frac{Q(A,B)[v](\tau)}{Z^+(\tau)}\hat{K}\prime(iz^m,\tau)\,d\tau\right) = 0, \end{cases} \qquad m = 0,1,\ldots,-\delta-1, \tag{13}$$

are necessary and sufficient for the solvability of the problems (10) and (11) respectively.

For $\alpha > 0$, and taken into account that $\hat{K}(z^k,t)$ and $\hat{K}(iz^k,t)$ is a complete system (see [9]) in $U_{p,2}(\hat{A},\hat{B},\,|z| < \rho)$, one gets the expansion

$$\hat{R}_{\alpha-1}(t) = \sum_{n=0}^{\alpha-1}\lambda_n\hat{K}(z^n,t) + \gamma_n\hat{K}(iz^n,t)\ ,$$

where λ_n, γ_n are arbitrary real constants. Then condition (12) can be expressed as the system

$$\begin{cases} \sum\limits_{n=0}^{\alpha-1} f_{n,k}\lambda_n + g_{n,k}\gamma_n = h_k, \\ \sum\limits_{n=0}^{\alpha-1} p_{n,k}\lambda_n + q_{n,k}\gamma_n = r_k, \end{cases} \qquad k = 0,1.\ldots,-\beta-1, \tag{14}$$

of algebraic equations, where

$$f_{n,k} = \operatorname{Re}\left(\frac{1}{2i}\int\limits_L A_2(\tau)\hat{K}(z^n,\tau)\hat{K}\prime(z^k,\tau)\,d\tau\right),$$

$$p_{n,k} = \operatorname{Re}\left(\frac{1}{2i}\int\limits_L A_2(\tau)\hat{K}(z^n,\tau)\hat{K}\prime(iz^k,\tau)\,d\tau\right),$$

$$g_{n,k} = \operatorname{Re}\left(\frac{1}{2i}\int\limits_L A_2(\tau)\hat{K}(iz^n,\tau)\hat{K}\prime(z^k,\tau)\,d\tau\right),$$

$$q_{n,k} = \operatorname{Re}\left(\frac{1}{2i}\int\limits_L A_2(\tau)\hat{K}(iz^n,\tau)\hat{K}\prime(iz^k,\tau)\,d\tau\right),$$

$$h_k = -\operatorname{Re}\left(\frac{1}{2i}\int_L A_2(\tau)P(\hat{A},\hat{B})[\frac{f(\tau)}{a_1(\tau)X^+(\tau)}]\hat{K}\prime(z^k,\tau)\,d\tau\right),$$

$$r_k = -\operatorname{Re}\left(\frac{1}{2i}\int_L A_2(\tau)P(\hat{A},\hat{B})[\frac{f(\tau)}{a_1(\tau)X^+(\tau)}]\hat{K}\prime(iz^k,\tau)\,d\tau\right),$$

$$A_2(\tau) = \frac{X^+(\tau)}{a_2(\tau)Y^+(\tau)}.$$

The analogous algebraic system

$$\begin{cases} \sum\limits_{n=0}^{\alpha-1} \tilde{f}_{n,m}\lambda_n + \tilde{g}_{n,m}\gamma_n = \tilde{h}_m, \\ \sum\limits_{n=0}^{\alpha-1} \tilde{p}_{n,m}\lambda_n + \tilde{q}_{n,m}\gamma_n = \tilde{r}_m, \end{cases} \qquad m = 0,1....,-\delta-1, \tag{15}$$

where

$$\tilde{f}_{n,m} = \operatorname{Re}\left(\frac{1}{2i}\int_L B_2(\tau)\hat{K}(z^n,\tau)\hat{K}'(z^m,\tau)\,d\tau\right),$$

$$\tilde{p}_{n,m} = \operatorname{Re}\left(\frac{1}{2i}\int_L B_2(\tau)\hat{K}(z^n,\tau)\hat{K}'(iz^m,\tau)\,d\tau\right),$$

$$\tilde{g}_{n,m} = \operatorname{Re}\left(\frac{1}{2i}\int_L B_2(\tau)\hat{K}(iz^n,\tau)\hat{K}'(z^m,\tau)\,d\tau\right),$$

$$\tilde{q}_{n,m} = \operatorname{Re}\left(\frac{1}{2i}\int_L B_2(\tau)\hat{K}(iz^n,\tau)\hat{K}'(iz^m,\tau)\,d\tau\right),$$

$$\tilde{h}_m = \operatorname{Re}\left(\frac{1}{2i}\int_L B_2(\tau)\,Q(\hat{A},\hat{B})[\frac{f(\tau)}{a_1(\tau)X^+(\tau)}]\hat{K}'(z^m,\tau)\,d\tau\right),$$

$$\tilde{r}_m = \operatorname{Re}\left(\frac{1}{2i}\int_L B_2(\tau)Q(\hat{A},\hat{B})[\frac{f(\tau)}{a_1(\tau)X^+(\tau)}]\hat{K}'(iz^m,\tau)\,d\tau\right),$$

$$B_2(\tau) = \frac{\tau^{-\alpha}X^-(\tau)}{b_2(\tau)\tau^{-\delta}Z^-(\tau)},$$

can be obtained as a representation of condition (13).

We are going to state our results in form of a theorem that deals with the dimensionality, over the algebra R of the real complex numbers, of the Kernel of the operator

O_2. This dimension is determined by α, β, δ and also depends on the rank of a certain algebaic system.

Theorem 2 *If $\alpha \geq 0$, $\alpha \geq 0$ and $\delta \geq 0$, then dim Ker $O_2 = 2(\alpha+\beta+\delta)$ and the inhomogeneous problem (8) is inconditionally solvable; moreover, its solution is determined by making use of the above reduction method.*

If at least one of the index α, β or δ is negative, the condition (7), (12) or (13), depending on the negative index, is necessary and sufficient to ensure the solvability of (8).

If $\alpha > 0$, $\beta < 0$, $\delta \geq 0$, then a necessary and sufficient condition for the solvability of the problem (8) is the compatibility of the algebraic system (14); if r is the rank of this system, then dim Ker $O_2 = 2(\alpha+\delta) - r$.

If $\alpha \geq 0$, $\beta \geq 0$, $\delta < 0$, then a necessary and sufficient condition for the solvability of the problem (8) is the compatibility of the algebraic system (15); if r is the rank of this system, then dim Ker $O_2 = 2(\alpha+\beta) - r$.

If $\alpha > 0$, $\beta < 0$, $\delta < 0$, then a necessary and sufficient condition for the solvability of the problem (8) is the compatibility of the combined algebraic system formed by the systems (14) and (15); if r is the rank of this system, then dim Ker $O_2 = 2 - r$.

4. The adjoint homogeneous problem; Orthogonality relations

Let us now consider the homogeneous adjoint problem to problem (8). Using the scheme for the solution in the preceding chapter, we write the general solution of homogeneous problem (8)$'$ as :

$$\psi(t) = \frac{1}{a_1(t)X^+(t)} \left\{ \begin{array}{c} \hat{R}'_{-\alpha-1}(t) + P(\hat{A}',\hat{B}')[\frac{X^+(t)a_1(t)}{Z^+(t)b_1(t)}\hat{R}'_{-\delta-1}(t)] - \\ -Q(\hat{A}',\hat{B}')\left[\frac{X^+(t)}{t^{-\beta}Y^-(t)}\hat{R}'_{-\beta-1}(t)\right] \end{array} \right\}, \quad (16)$$

where $\hat{A}' = -\hat{A}$, $\hat{B}' = -\hat{B}$.

If $\alpha > 0$, $\beta < 0$, $\delta \geq 0$, then the vector of the real coefficients of the polynomial $\hat{R}'_{-\beta-1}(t)$ is a solution of system (14)$'$.

If $\alpha > 0$, $\beta \geq 0$, $\delta < 0$, then the vector of the real coefficients of the polynomial $\hat{R}'_{-\delta-1}(t)$ is a solution of system (15)$'$.

If $\alpha > 0$, $\beta < 0$, $\delta < 0$, then the vector of the real coefficients of the polynomials $\hat{R}'_{-\beta-1}(t)$, $\hat{R}'_{-\delta-1}(t)$ is a solution of the combined system composed by the systems (14)$'$ and (15)$'$.

The following theorem, concerning the dimensionality over R of the operator O_2^* , is an analogous to the corresponding result for the operator O_2 .

Theorem 3 *The number of linearly independent, over R , solutions of the homogeneous adjoint problem $O_2^*\psi = 0$* is

$$\max\{0,-2\alpha\}+\max\{0,-2\beta\}+\max\{0,-2\delta\}-r',$$

where r' is the rank of the system $(14)'$ when $\alpha > 0$, $\beta \geq 0$, $\delta < 0$; r' is the rank of the system $(15)'$ when $\alpha > 0$, $\beta \geq 0$, $\delta < 0$; it is the rank of the combined system formed of the systems $(14)'$ and $(15)'$ when $\alpha > 0$, $\beta < 0$, $\delta < 0$; finally, $r' = 0$ in any other cases.

In what follows we shall quote without proof a result of Noether theorem type.

"The problem (8) is solvable if and only if

$$\operatorname{Re}\left(\frac{1}{2i}\int_L f(\tau)\psi(\tau)\,d\tau\right)=0, \tag{17}$$

where ψ is a solution of the homogeneous adjoint problem $(8)'$".

Under the assumption $\alpha > 0$, $\beta < 0$, $\delta \geq 0$, the relation (17) takes the form:

$$\begin{aligned}&\sum_{k=0}^{-\beta-1}\lambda'_k \operatorname{Re}\left(\frac{1}{2i}\int_L Q(\hat{A}',\hat{B}')\left[\frac{X^+(\tau)\hat{K}'(z^k,\tau)}{\tau^{-\beta}Y^-(\tau)}\right]\frac{f(\tau)}{a_1(\tau)X^+(\tau)}\,d\tau\right) + \\ &+\sum_{k=0}^{-\beta-1}\gamma'_k \operatorname{Re}\left(\frac{1}{2i}\int_L Q(\hat{A}',\hat{B}')\left[\frac{X^+(\tau)\hat{K}'(iz^k,\tau)}{\tau^{-\beta}Y^-(\tau)}\right]\frac{f(\tau)}{a_1(\tau)X^+(\tau)}\,d\tau\right)=0,\end{aligned} \tag{18}$$

where $(\lambda'_0,\ldots,\lambda'_{-\beta-1},\gamma'_0,\ldots,\gamma'_{-\beta-1})$ is the general solution of the system $(14)'$.

According to the identity $Q^*(\hat{A}',\hat{B}') = P(\hat{A},\hat{B})$, it follows that (18) can be represented in the form

$$\begin{aligned}&\sum_{k=0}^{-\beta-1}\lambda'_k \operatorname{Re}\left(\frac{1}{2i}\int_L \frac{X^+(\tau)\hat{K}'(z^k,\tau)}{\tau^{-\beta}Y^-(\tau)}P(\hat{A},\hat{B})\left[\frac{f(\tau)}{a_1(\tau)X^+(\tau)}\right]d\tau\right) + \\ &+\sum_{k=0}^{-\beta-1}\gamma'_k \operatorname{Re}\left(\frac{1}{2i}\int_L \frac{X^+(\tau)\hat{K}'(iz^k,\tau)}{\tau^{-\beta}Y^-(\tau)}P(\hat{A},\hat{B})\left[\frac{f(\tau)}{a_1(\tau)X^+(\tau)}\right]d\tau\right)=0.\end{aligned} \tag{19}$$

One can verify that (19) is the necessary and sufficient condition for the compatibility of the system (14).

We remark also that if $\alpha > 0$, $\beta \geq 0$, $\delta < 0$ ($\alpha > 0$, $\beta < 0$, $\delta < 0$) using the same technique as before, (17) is the necessary and sufficient condition for the compatibility of the system (15) (the combined system of (14) and (15)).

Let us observe finally that (17) can be reduced directly to the conditions (7), (12) and (13), when $\alpha > 0$, $\beta \geq 0$, and $\delta \geq 0$.

5. Acknowledgement

The authors would like to thank Professor Dr. M. Borges Trenard for his valuable suggestions about the final version of the manuscript.

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