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The Classical Isotropic bi-Dimensional Oscilator in the Eisenhart Formulation of Classical Mechanics

U. Percoco*, L.A. Núñez** & M. Zambrano*

Abstract. Accordingly with the general theory of relativity, the motion of a particle by the only action of inertia and gravity is described by a space-time geodesic. We use the Eisenhart geometric formulation of classical mechanics to establish a correspondence between geodesics and paths in phase space of the classical bi-dimensional isotropic oscillator. The Killing vectors and its associated constants of motion are presented and compared with non-Noetherian motion constant calculated by S. Hojman and collaborators.

Resumen. De acuerdo con la Teoría de la Relatividad General, el movimiento de partículas por acción de su inercia y la gravedad es descrito por geodésicas en el espacio-tiempo. Utilizamos la formulación Geométrica de Eisenhart de la Mecánica Clásica para establecer una correspondencia entre geodésicas y trayectorias en el espacio de fases del oscilador clásico isótropo. Se presentan los vectores de Killing y las constantes de movimiento asociadas, se comparan con las constantes de movimiento no noetheriano calculadas por S. Hojman y colaboradores.

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* Grupo de Física Teórica, Departamento de Física, Fac. de Ciencias Universidad de los Andes, Mérida Venezuela *e-mails*: upercoco@ula.ve, mzambrano@ula.ve

** Centro de Física Fundamental, Departamento de Física, Facultad de Ciencias, Universidad de Los Andes, Mérida 5101, Venezuela y Centro Nacional de Cálculo Científico, Universidad de Los Andes, (CeCalCULA), Corporación Parque Tecnológico de Mérida, Mérida 5101, Venezuela.
e-mail: nunez@ula.ve

1. Introduction

The paths in the configuration space of mechanical systems are constituted by points representing states of the system. To every set of initial conditions ($t = 0$) there corresponds (at least locally) a unique orbit classified by its initial values and parameterized by the time t which in Newtonian Physics is common to all dynamical systems whereas in Special and General Relativity depends on the proper time. Those paths are the solutions of Lagrange equations of motion for the mechanical system. Then we have a new manifold called the tangent phase space describing the evolution of the dynamical system for given initial conditions.

L. Eisenhart's idea (pionner in this approach) geometrizes the motion of a dynamical system schematically as follows:

Hamilton's principle states that the natural motions of a Hamiltonian system are the extremal curves of the functional:

$$S = \int L dt,$$

where L is the lagrangian function of the system. On the other hand, the geodesics of a riemannian manifold are the extremal curves of the length functional

$$l = \int ds,$$

where s is the arc-length parameter.

If by choosing a suitable metric, it is possible to establish a relationship between length and action, then it becomes possible to identify geodesics with physical trajectories and to search symmetries from geometric objects.

Among the many choices of a metric to achieve a geometric formulation of dynamics, the one originally introduced by Eisenhart on an enlarged configuration space-time, has proven to be useful due to the fact that, besides having simple curvature properties, the dynamics can be geometrized with an affine parameterization of the arc-length, i.e. $ds = adt$.

The article is organized as follows: In Sec 2, we consider a particular form of Eisenhart's metric and find its Killing vectors with the associated constants of motion. The results obtained here will be useful in Sec. 3 where we compared these constants of motion with a non-Noetherean conserved quantity (based on Hamiltonian approach) calculated by S. Hojman [1]. Finally in Sec. 4 we present some concluding remarks.

2. *Einsernhart's Metric for the Classical Isotropic 2-D Oscillator. Killing Equations. Associated Constants of Motion*

The description of the motion of mechanical system can be done by their paths in the phase space and in the extended Eisenhart manifold, where the Eisenhart formulation is equivalent to T and satisfies that geodesics equations are Lagrange equations of the mechanical system for the first n dimensions:

$$T \Rightarrow \left[\frac{d^2}{ds^2}q^i + \Gamma^i_{jk}\frac{dq^i}{ds}\frac{dq^k}{ds}\right],\ (i,j,k) \in [1,2,\ldots,n],$$

equations related with the two leaned dimentions *(t,u)*

$$\begin{cases} n+2 \Rightarrow \left[u(q^i,t) = \dfrac{1}{2}\dfrac{t}{a^2} - \int Ldt + b\right], \\ n+1 \Rightarrow [t = as]. \end{cases}$$

By establishing the Eisenhart's metric for the oscillator here considered, these equations (and the related energies and its line element) take the form

$$\text{Potencial energy} \longrightarrow V = \frac{1}{2}k(q^1)^2 + \frac{1}{2}k(q^2)^2.$$

Kinetic energy:

$$T_E = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + g_{(n+1)i}\dot{q}^i + \frac{1}{2}g_{(n+1)(n+1)} = \frac{1}{2}(\dot{q}^1)^2 + \frac{1}{2}(\dot{q}^2)^2, n = 2, i = (q^1,q^2) = (1,2).$$

Line element:

$$\begin{aligned}(ds)^2 = G_{\mu\nu}dx^\mu dx^\nu &= g_{ij}dq^i dq^j + 2g_{(n+1)i}dq^i dt + \bar{A}(dt)^2 + 2\bar{B}dtdu \\ &= (dq^1)^2 + (dq^2)^2 - 2V(dt)^2 + 2dtdu,\end{aligned}$$

$$u = \frac{t}{a^2} - \int Ldt \rightarrow \dot{u} = 1 - \frac{1}{2}\left[(\dot{q}^1)^2 + (\dot{q}^2)^2\right] + \frac{k}{2}\left[(q^1)^2 + (q^2)^2\right],$$

where $\bar{B} = ctte \equiv 1$, $j = (1,2)$, $(1,2,3,4) \rightarrow (q^1,q^2,t,u)$, $\bar{A} = g_{(n+1)(n+1)} - 2V = -2V$, with the correspondent equation of motion

$$\ddot{q}^\mu = kq^\mu, \mu = (q^1,q^2,t,u) = (1,2,3,4)$$

of general solution

$$q^1 = \cos(\sqrt{k}t + \alpha), q^2 = \cos(\sqrt{k}t + \beta), q^t = t, q^\mu = \mu;$$

for the metric tensor we note that

$g_{(n+1)i}\dot{q}^i = g_{(t)1}\dot{q}^1 = g_{(t)2}\dot{q}^2 = 0 \rightarrow g_{31} = g_{32} = 0$	$\frac{1}{2}g_{tt} = \frac{1}{2}g_{33} = 0$
$g_{11} = g_{22} = 1$	$n+1 = 3 = t$

$$(G)_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2V_{(q_1+q_2)} & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

o

$$(G)_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -k((q^1)^2 + (q^2)^2) & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- Killing vectors:

Having calculated the metric and the connections (first Christoffel of and second type) and established the vectorial field on which Lie-it was derived, we determined the Killing vectors:

$$\begin{aligned}
\xi_1 =& C_3 q^2 + C_1 \sin(\sqrt{k}t) + C_2 \cos(\sqrt{k}t),\\
\xi_2 =& - C_3 q^1 + C_4 \sin(\sqrt{k}t) + C_5 \cos(\sqrt{k}t),\\
\xi_3 =& \sqrt{k}(C_2 q^1 + C_5 q^2)\sin(\sqrt{k}t) - \sqrt{k}(C_1 q^1 + C_4 q^2)\cos(\sqrt{k}t) - C_6 k\left((q^1)^2 - (q^2)^2\right) + C_1,\\
\xi_4 =& C_6,\\
\xi_\mu =& \xi_\mu(q^1, q^1, t, u).
\end{aligned}$$

For the calculus of the conserved quantity on the geodesic that it describes to the oscillator, we determined the four-moment (remembering that $\dot{x}_\nu = G_{\nu\mu}\dot{x}^\mu$):

$$\begin{aligned}
P^1 =& P^{q^1} = \dot{q}^1 = \dot{q}_1,\\
P^2 =& P^{q^2} = \dot{q}^2 = \dot{q}_2,\\
P^3 =& P^t = \dot{t} = \frac{dt}{ds} = 1, \quad (ds = a(dt), a \equiv 1),\\
P^4 =& P^u = \dot{x}^3 + 2V\dot{x}^4 = (-2V + \dot{u}) + 2V = \dot{u} = \frac{du}{dt} = \frac{du}{ds},
\end{aligned}$$

where,

$$u = \frac{t}{a^2} - \int L dt \rightarrow \dot{u} = 1 - \frac{1}{2}\left[(\dot{q}^1)^2 + (\dot{q}^2)^2\right] + \frac{k}{2}\left[(q^1)^2 + (q^2)^2\right].$$

Now we have all the ingredients to calculate the conserved quantity in the extended space and that we will denominate Eisenhart's conserved quantity of movement (J_E):

$$
\begin{aligned}
J_E = \xi_a P^a = &- (C_3 \cos(\sqrt{k}t + \beta) + C_1 \sin(\sqrt{k}t) + \\
&+ C_2 \cos(\sqrt{k}t) \sin(\sqrt{k}t + \alpha)\sqrt{k} \cos(\sqrt{k}t + \beta)\sqrt{k} \sin(\sqrt{k}t)) + \\
&+ C_4 \sin(\sqrt{k}t) + C_5((-\cos(\sqrt{k}t + \beta) \sin(\sqrt{k}t + \beta))) + \\
&+ \sqrt{k}(C_2 \cos(\sqrt{k}t + \alpha) + C_5 \cos(\sqrt{k}t + \beta)\sqrt{k} \sin(\sqrt{k}t)) + \\
&+ C_5(\sqrt{k}t + \beta)\sqrt{k} \sin(\sqrt{k}t - C_4)\sqrt{k} \cos(\sqrt{k}t + \beta) + \\
&+ C_5(-\cos(\sqrt{k}t + \beta)) \sin(\sqrt{k}t + \beta)\sqrt{k} + C_2 \cos(\sqrt{k}t + \beta) + \\
&+ C_2 \cos(\sqrt{k}t + \beta) + (C_6(1 - \frac{1}{2} \sin(\sqrt{k}t + \alpha)^2)k - \sqrt{k}C_7) + \\
&+ \frac{1}{2} \sin(\sqrt{k}t + \beta)^2 - \frac{1}{2}k(\cos(\sqrt{k}t + \alpha)^2) + \sin(\sqrt{k}t + \beta)^2.
\end{aligned}
$$

3. Hojman's non-Noetherian Conserved Quantity Versus Eisenhart's Conserved Quantity

From now on we consider $k = 1$ and "on shell calculations" with the corresponding equations of motions.

The conserved Hojman's quantity of movement is [2]:

$$J_H = AB \left[\cos(\alpha) \cos(\beta) + \sin(\beta) \sin(\alpha)\right].$$

The conserved Eisenhart's constant of motion is:

$$J_E = C_7 - C_2 \sin(\alpha) - C_3 \sin(\alpha) \cos(\beta) + C_3 \sin(\beta) \cos(\alpha) - C_4 \cos(\beta) - C_1 \cos(\alpha) + C_5 \sin(\beta).$$

We recover Hojman's constant as a particular case of Eisenhart formalism for the following set of conditions $(C_1, ..., C_6, A, B, k, \alpha, \beta)$:

$$
\begin{aligned}
C_7 = &C_2 \sin(\alpha) + C_3 \sin(\alpha) \cos(\beta) - C_3 \sin(\beta) \cos(\alpha) + C_4 \cos(\beta) + \\
&+ C_1 \cos(\alpha) + C_5 \sin(\beta) + AB \cos(\beta) \cos(\alpha) + AB \sin(\beta) \sin(\alpha).
\end{aligned}
$$

4. Final Comments

The results obtained in this contribution, show that Hojman's constant of the motion has been reobtained as a particular case in Eisenhart's formulation. Therefore, we could consider this geometric approach to be more general in the following sense: it may provide constants of the motion from which conserved quantities, obtained following completely different approaches, can be constructed. In a following contribution we will apply Eisenhart's approach to the study of the anisotropic two-dimensional oscillator.

References

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U. Percoco & M. Zambrano
Grupo de Física Teórica,
Departamento de Física, Fac. de Ciencias
Universidad de los Andes, Mérida Venezuela
e-mails: upercoco@ula.ve, mzambrano@ula.ve

L. A. Núñez
Centro de Física Fundamental,
Departamento de Física,
Facultad de Ciencias,
Universidad de Los Andes,
Mérida 5101, Venezuela y
Centro Nacional de Cálculo Científico,
Universidad de Los Andes, (CeCalCULA),
Corporación Parque Tecnológico de Mérida,
Mérida 5101, Venezuela. *e-mail*: nunez@ula.ve