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An Exact Homogeneous Stiff Cosmology that Reduces to the Kasner Solution

Fabio D. Lora* & Guillermo A. González*

Abstract. A family of exact simple solutions of Einstein field equations for homogeneous stiff cosmologies is presented. The method to obtain the solution is based on the introduction of auxiliary functions in order to cast the Einstein equations in such a way that can be explicitly integrated. The obtained solution is expressed in terms of simple functions of the used coordinates. The geometrical and kinematical properties of the solution are analyzed and the parameters are restricted in order to have a physically acceptable behavior. The solution is of the Petrov type I and presents a big-bang singularity. Now, for a particular value of one of the parameters, the solution is a vacuum solution of the Bianchi I type that reduces to the Kasner solution.

Resumen. Se presenta una familia de soluciones exactas sencillas de las ecuaciones de Einstein homogéneas sobre el plano para las cosmologías rígidas. El método para obtener la solución se basa en la introducción de funciones auxiliares, a fin de emitir las ecuaciones de Einstein de tal manera que puedan ser integradas explícitamente. La solución obtenida se expresa en términos de funciones simples de las coordenadas utilizadas. Se analizan las propiedades geométricas y cinemáticas de la solución; los parámetros están restringidas a fin de tener un comportamiento aceptable físicamente. Las soluciones son del tipo Petrov I, y presentan una singularidad de big-bang. Ahora bien, para un cierto valor de uno de los parámetros la solución es una solución de vacío de tipo Bianchi I, que se reduce a la solución de Kasner.

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* Escuela de Física, Universidad Industrial de Santander. Grupo de Investigación en Relatividad y Gravitación. *e-mail*: fadu31@gmail.com, guillego@uis.edu.co

1. The Einstein And Matter Evolution Equations

In order to study inhomogeneous or homogeneous stiff cosmologies, we take as the starting point the metric tensor as given by the line element[1]

$$ds^2 = e^{-2U}[e^{2\gamma}(-dt^2 + dr^2) + W^2 dx^2] + e^{2U} dy^2, \tag{1}$$

where U, γ and W are functions of r and t only. We also consider as the matter contents a perfect fluid with the stiff equation of state $p = \rho$, whose energy-momentum tensor can be written as

$$T_{\alpha\beta} = \rho(2u_\alpha u_\beta + g_{\alpha\beta}). \tag{2}$$

With the above choices, the Einsten equations can be cast as

$$R_{\alpha\beta} = 2\rho u_\alpha u_\beta, \tag{3}$$

whereas the matter evolution equations can be obatined, from the conservation law

$$T^{\alpha\beta}{}_{;\beta} = 0, \tag{4}$$

by projecting it along the temporal and spatial directions. In order to obtain the above projections, we contract the equation (4) with the velocity vector u^α and the "spatial projection tensor" $h_{\alpha\beta} = u_\alpha u_\beta + g_{\alpha\beta}$, respectively. So we obtain

$$\rho_{,\beta} u^\beta + 2\rho u^\beta{}_{;\beta} = 0, \qquad 2\rho u^\beta u^\alpha{}_{;\beta} h^\mu_\alpha + \rho_{,\beta} g^{\beta\alpha} h^\mu_\alpha = 0 \tag{5}$$

where we use the condition $u_\alpha h^{\alpha\beta} = 0$.

We now impose the irrotationality condition [2]

$$u_\alpha = \frac{\Phi_{,\alpha}}{(-\Phi_{,\mu}\Phi^{,\mu})^{1/2}}, \tag{6}$$

so that the equation (5) can be cast as

$$\frac{\rho(\Phi_{,\mu}\Phi^{,\mu})_{,\alpha} h^\alpha_\mu}{(\Phi_{,\mu}\Phi^{,\mu})} = \rho_{,\alpha} h^\alpha_\mu, \tag{7}$$

which can be identically satisfied if we choose [3]

$$\rho = -\frac{F}{2}\Phi_{,\mu}\Phi^{,\mu}, \tag{8}$$

where F is an arbitrary function of the scalar potential Φ. Now, by using (8), the energy-momentum tensor can be cast as

$$T_{\alpha\beta} = F\left[\Phi_{,\alpha}\Phi_{,\beta} - \frac{1}{2} g_{\alpha\beta}\Phi_{,\mu}\Phi^{,\mu}\right], \tag{9}$$

in such a way that the Einstein and evolution equations can be written as

$$R_{\alpha\beta} = F\Phi_{,\alpha}\Phi_{,\beta}, \qquad F\Phi^{,\alpha}{}_{;\alpha} = -\frac{F'}{2}\Phi_{,\mu}\Phi^{,\mu}, \tag{10}$$

where $F' = \frac{\partial F}{\partial \Phi}$. Now, it is easy to see that, for any arbitrary function $F(\Phi)$, the evolution and Einstein system of equations can be cast as

$$W_{,rr} - W_{,tt} = 0, \tag{11}$$

$$(W\psi_{,r})_{,r} - (W\psi_{,t})_{,t} = 0, \tag{12}$$

$$(WU_{,r})_{,r} - (WU_{,t})_{,t} = 0, \tag{13}$$

$$\gamma_{,t}W_{,r} + \gamma_{,r}W_{,t} = 2WU_{,t}U_{,r} + k^2W\psi_{,t}\psi_{,r} + w_{,tr}, \tag{14}$$

$$\gamma_{,t}W_{,t} + \gamma_{,r}W_{,r} = W(U_{,t}^2 + U_{,r}^2) + \frac{1}{2}\left[k^2W(\psi_{,t}^2 + \psi_{,r}^2) + (W_{,tt} + W_{,rr})\right], \tag{15}$$

where k is an arbitrary positive constant and ψ is a new scalar potencial, which is given by the functional dependece $\Phi = \Phi(\psi)$. As we can see, equation (11) is the classical one-dimensional wave equation. On the other hand, equations (12) and (13) are equivalents, so that solutions from (13) are also solutions from (12). According with this, we can suppose that $U(t,r) = \psi(t,r)$. Finally, the integrability conditions of the overdetermined system (14)–(15) are equivalent to the equations (11)–(13), guaranting so the existence of solutions. On the other hand, we can see that by taking the stiff equation of state $p = \rho$, the stiff fluids equations are easy to integrate because the metric functions U, W decouple from the pressure [4].

2. Homogeneous Stiff Solution

In order to solve the system (11) – (15), we consider solutions of the equation (11) of the general form

$$W(r,t) = \Psi(r+t) + \Omega(r-t), \tag{16}$$

where Ψ and Ω are arbitrary functions. Now, by taking $\Psi = \dfrac{r+t}{2}$ and $\Omega = \dfrac{t-r}{2}$, we obtain for the metric functions the expresions

$$\begin{aligned} W(r,t) &= t, & (17)\\ \gamma(r,t) &= q\left(a_1^2r^2 + a_3^2 + 2a_1a_3r + a_1a_2\right)\frac{t^2}{2} \\ &\quad + q\frac{a_1^2}{16}t^4 + qa_2^2\ln t + qa_1a_2r^2 + 2qa_2a_3r, & (18)\\ U(r,t) &= \frac{a_1}{4}\left(t^2 + 2r^2\right) + a_2\ln t + a_3r, & (19) \end{aligned}$$

in such a way that the fluid density is given by

$$\rho = \frac{k^2}{2}\left[\frac{a_1^2 t^2}{4} + \frac{a_2^2}{t^2} + a_1 a_2 - a_1\left(a_1 r^2 + 2a_3 r\right) - a_3^2\right] e^{2(U-\gamma)}, \tag{20}$$

whereas the velocity components are given by

$$u_r = \frac{k}{\sqrt{2\rho}}\left(a_1 r + a_3\right), \qquad u_t = \frac{k}{\sqrt{2\rho}}\left(\frac{a_1 t}{2} + \frac{a_2}{t}\right). \tag{21}$$

Now, we require that $\rho \geq 0$ for any value of r and t in order to obtain a physically acceptable distribution of matter. From expression (20) it is easy to see that ρ will be no negative everywhere only if we take $a_1 = a_3 = 0$, so that the expression for the density reduces to

$$\rho = \frac{k^2 a_2^2}{2} t^{-2(q a_2^2 - a_2 + 1)}. \tag{22}$$

Now, as $(a_2 - 1)/(a_2^2) < 1 < q$, we have an initial singularity and then the density decreases to zero as $t \to \infty$. On the other hand, the velocity vector is given by

$$u^\alpha = t^{a_2(1-q a_2)}(1, 0, 0, 0), \tag{23}$$

where, in order to have a future oriented timelike vector, we have taken $a_2 < 0$. Also, we can see that the spatial velocity is zero and thus we again have a comoving reference frame.

The line element can be written as

$$ds^2 = t^{-2a_2}[t^{2q a_2^2}(-dt^2 + dr^2) + t^2 dx^2] + t^{2a_2} dy^2, \tag{24}$$

so that, when $q = 1$ (or $k = 0$) we have a vacuum solution of the Bianchi I type that reduces to the Kasner solution [5, 6], which can be written as [7, 1]

$$ds^2 = t^{(d^2-1)/2}(-dt^2 + dr^2) + t^{1+d} dx^2 + t^{1-d} dy^2, \tag{25}$$

with the Kasner parameter given by $d = 1 - 2a_2$. Now, it is worth to mention that another kind o inhomogeneous stiff cosmologies were obtained by Patel and Dadich [8], which also reduce to the Kasner solution. However, in contrast with the solution here presented, the solutions of Patel and Dadich are singularity free.

Now, in order to see if the solution has a real initial singularity, we computed the Weyl tensor in the natural null tetrad of the metric[9, 10] and obtain

$$\Psi_0(t, r) = \frac{1}{2} a_2(2a_2 - 1)(a_2 q - 1) t^{-2q a_2^2 + 2a_2 - 2}, \tag{26}$$

$$\Psi_2(t, r) = -\frac{1}{2}(a_2 - 1) a_2 t^{-2q a_2^2 + 2a_2 - 2}, \tag{27}$$

$$\Psi_4(t, r) = \frac{1}{2} a_2(2a_2 - 1)(a_2 q - 1) t^{-2q a_2^2 + 2a_2 - 2}. \tag{28}$$

The scalars constructed from the Ricci and Weyl tensors diverge as $t \longrightarrow 0$, which corresponds to a big-bang singularity. Also, it is easy to see that in the algebraic classification of the Reimann tensor, the metric is of Petrov type I.

On the other hand, the kinematical quantities for this model can also be easily computed and so, by taking $a_2 < 0$, we obtain for the acceleration the expression

$$a_\alpha = (0, 0, 0, 0), \tag{29}$$

where all the components have been computed in the natural orthonormal tetrad of the metric. It is interesting to see that, as the pressure gradient is zero, the acceleration is equal to zero and thus the fluid is geodesic.

3. Discussion

We present a simple family of exact homogeneous stiff cosmologies. The solution was obtained by introducing an auxiliary function that permit to cast the Einstein and matter evolution equations as a complete integrable system. The general solution is expressed in terms of simple functions of the used coordinates. The simple family presented reduces to the Kasner solution, with Kasner parameter $d = 1 - 2a_2$, when $q = 1$. This solution is a vacuum solution of Bianchi I type. The Weyl's scalars diverge as $t \to 0$, which can be interpreted as a big-bang singularity. In the algebraic classification of the Riemann tensor, the metric is of Petrov type I. On the other hand is interesting to see that the acceleration is equal to zero, so that the fluid is geodesic.

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Fabio D. Lora & Guillermo A. González
Escuela de Física, Universidad Industrial de Santander.
Grupo de Investigación en Relatividad y Gravitación.
e-mail: fadu31@gmail.com, gonzalez@gag-girg-uis.net