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Continuous dependence of very weak solutions for the stationary Navier-Stokes equations

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Abstract. In this work we show the continuous dependence of the very weak solutions for the stationary Navier-Stokes system with respect to boundary data belonging to space $L^2(\Gamma)$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$, be a bounded domain with boundary Γ of class $C^{1,1}$. Let $\mathbf{g} = (g_1, g_2, g_3)$ and $\mathbf{f} = (f_1, f_2, f_3)$ two given fields defined on the boundary Γ and Ω, respectively. The stationary Navier-Stokes system can be written as follows:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \boldsymbol{\Omega}, \\ \operatorname{div}\mathbf{u} &= \mathbf{0}, \quad \text{in } \boldsymbol{\Omega}, \\ \mathbf{u} &= \mathbf{g}, \quad \text{on } \boldsymbol{\Gamma}, \end{cases} \tag{1}$$

where $\mathbf{u}$ denotes the velocity field, p the pressure, $\mathbf{f}$ the density of he body force, $\mathbf{g}$ the prescribed velocity on the boundary of Ω and ν the viscosity of fluid. Without loss of generality we consider the density equals one.

As usual $(L^p(\Omega), |\cdot|_p)$, with $1 \le p \le +\infty$ and $(W^{k,p}, \|\cdot\|_{k,p})$ are the usual Sobolev spaces. In particular $H^k(\Omega) = W^{k,2}(\Omega)$ with the norm $\|\cdot\|_k = \|\cdot\|_{k,2}$. By $(\cdot,\cdot)$, we represent the inner product in $L^2(\Omega)$. We denote

$$H^1_\Gamma(\Omega) := \{u \in H^1(\Omega) : u \equiv 0 \ \text{ on } \Gamma\}$$

and $\mathbf{V}$ the closure of $\{\mathbf{u} \in C_0^\infty : \ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\}$ in the norm of $H^1(\Omega)$, being $((\cdot,\cdot))$ and $\|\cdot\|$ the corresponding inner product and norm.

If the boundary data $\mathbf{g}$ is regular enough we can extend the boundary condition on the whole domain in order to obtain a new system with homogeneous boundary conditions, which can be solved in a standard way obtaining the existence of a weak solution for (1) (see for instance [5]). However, if a boundary condition $\mathbf{g} \in L^2(\Gamma)^3$,

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which is not the trace of a function in the Sobolev space $W^{1,2}(\Omega)$, a new notion of solution for (1) is necessary. In fact, Marusic-Paloka in [4] introduced the following definition of solution for the stationary Navier-Stokes system (1) when $\mathbf{g} \in L^2(\Gamma)^3$.

Definition 1.1. [4] Let $\mathbf{g} \in L^2(\Gamma)^3$ and $\mathbf{f} \in W^{-1,3}(\Omega)^3$. A function $\mathbf{u} \in L^3(\Omega)^3$ is called a very weak solution of Problem (1) if, for any

$$\mathbf{w} \in V = \left\{\phi \in W^{2,3/2}(\Omega)^3 \cap W^{1,3/2}(\Omega)^3 :\ \operatorname{div}\phi = 0\right\}$$

and any $\pi \in W = \left\{\varphi \in W^{1,3/2}(\Omega) : \int_\Gamma \varphi = 0\right\}$, the following equalities hold:

$$-\int_\Omega \nu \mathbf{u}\Delta\mathbf{w} - \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{w}\mathbf{u} + \int_\Gamma \mathbf{g}\cdot\frac{\partial\mathbf{w}}{\partial\mathbf{n}} = \langle \mathbf{f}, \mathbf{w}\rangle_{W^{-1,3},W^{1,3/2}}, \tag{2}$$

$$\int_\Omega \nabla\pi - \int_\Gamma \pi(\mathbf{g}\cdot\mathbf{n}) = 0. \tag{3}$$

The notion of very weak solution for the Navier-Stokes system is a natural extension of the definition of weak solution. In [1], Conca introduced the notion of very weak solution for the case of Stokes equations, that is, for the problem

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } \Omega,\\ \operatorname{div}\mathbf{u} &=& \mathbf{0}, & \text{in } \Omega,\\ \mathbf{u} &=& \mathbf{g}, & \text{on } \Gamma, \end{array}\right. \tag{4}$$

where $\mathbf{g} \in L^2(\Gamma)^3$. This problem arise in homogenization problems, and the solvability was showed using the transposition method which was introduced by Lions and Magenes [3].

Later, Marusič-Paloka [4] extended the Conca's results for the case of Navier-Stokes equations. He proved the existence of very weak solution of the System (1), in the sense of Definition 1.1, with L^2 boundary data. Under the small data assumption he also proved the uniqueness. The proof was obtained using the penalization method to study the linearized problem and then, applying Banach's fixed point theorem for the nonlinear problem with small data. In the case with no small data assumption the proof is obtained by splitting the data on a large regular and small irregular ones.

In this paper we use the method used by Mausic-Paloka in [4] in order to show the continuous dependence of the very weak solutions with respect to the density of the body force $\mathbf{f}$ and the prescribed velocity $\mathbf{g}$ on the boundary Γ.

2. Continuous dependence

Let $\mathbf{v}_1^\epsilon, \mathbf{v}_2^\epsilon$ be two very weak solutions of (1) with data $\mathbf{f}_1$, $\mathbf{g}_1$, and $\mathbf{f}_2$, $\mathbf{g}_2$, respectively. We will study the problem which is satisfied by the difference $\mathbf{w} = \mathbf{v}_1^\epsilon - \mathbf{v}_2^\epsilon$, that is:

$$(P)\quad \left\{\begin{array}{rcll} -\nu\Delta\mathbf{w} + (\mathbf{v}_1^\epsilon\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}_2^\epsilon + \nabla(p_1^\epsilon - p_2^\epsilon) &=& \mathbf{f}_1 - \mathbf{f}_2, & \text{in } \Omega,\\ \operatorname{div}\mathbf{w} &=& 0, & \text{in } \Omega,\\ \mathbf{w} &=& \mathbf{g}_1 - \mathbf{g}_2, & \text{on } \Gamma. \end{array}\right.$$

Firstly, we consider the Problem (P) with $\mathbf{f}_1 - \mathbf{f}_2 \equiv \mathbf{0}$, and for this case we consider the sequence of penalized problems:

$$(P_p) \quad \begin{cases} -\nu\Delta\mathbf{w}^m + (\mathbf{v}_1^\epsilon\cdot\nabla)\mathbf{w}^m + (\mathbf{w}^m\cdot\nabla)\mathbf{v}_2^\epsilon + \nabla(p_1^{m,\epsilon} - p_2^{m,\epsilon}) & = \mathbf{0}, \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{w}^m & = 0, \quad \text{in } \Omega, \\ \mathbf{w}^m\Big|_\Gamma + \dfrac{1}{m}\Big\{\nu\,\partial_{\mathbf{n}}\mathbf{w}^m - (p_1^{m,\epsilon} - p_2^{m\epsilon})\,\mathbf{n} - & \\ \qquad \dfrac{1}{2}(\mathbf{v}_1^\epsilon\cdot\mathbf{n})\mathbf{w}^m - (\mathbf{w}^m\cdot\mathbf{n})\mathbf{v}_2^\epsilon\Big\}\Big|_\Gamma & = \mathbf{g}_1 - \mathbf{g}_2. \end{cases}$$

Formally speaking ,when $m \to \infty$, we obtain the Problem (P). Problem (P_p) can be written in variational form as:

$$\nu\int_\Omega \nabla\mathbf{w}^m : \nabla\boldsymbol{\Psi} - \int_\Omega (\mathbf{v}_1^\epsilon\cdot\nabla)\boldsymbol{\Psi}\cdot\mathbf{w}^m - \int_\Omega (\mathbf{w}^m\cdot\nabla)\boldsymbol{\Psi}\cdot\mathbf{v}_2^\epsilon \\ + m\int_\Gamma \mathbf{w}^m\cdot\boldsymbol{\Psi} = m\int_\Gamma (\mathbf{g}_1 - \mathbf{g}_2)\cdot\boldsymbol{\Psi} - \frac{1}{2}\int_\Gamma (\mathbf{v}_1^\epsilon\cdot\mathbf{n})\,\mathbf{w}^m\cdot\boldsymbol{\Psi}, \quad (5)$$

for all $\boldsymbol{\Psi} \in \mathbf{H}^1_{\text{div}} \equiv \{\boldsymbol{\Psi} \in \mathbf{H}^1(\Omega) : \ \operatorname{div}\boldsymbol{\Psi} = 0\}$. Using the arguments of [4], we can obtain the existence of a unique solution $\mathbf{w}^m$ in $\mathbf{H}^1_{\text{div}}(\Omega)$ of the equation (5) such that:

$$|\mathbf{w}^m|_{L^2(\Gamma)} \le C(\mathbf{v}_1^\epsilon, \mathbf{v}_2^\epsilon)|(\mathbf{g}_1 - \mathbf{g}_2)|_{L^2(\Gamma)}. \quad (6)$$

In fact, we consider the bilinear form $a : \mathbf{H}^1_{\text{div}} \times \mathbf{H}^1_{\text{div}} \to \mathbb{R}$ defined by

$$a(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = \nu\int_\Omega \nabla\boldsymbol{\Phi} : \nabla\boldsymbol{\Psi} - \int_\Omega (\mathbf{v}_1^\epsilon\cdot\nabla)\boldsymbol{\Psi}\cdot\boldsymbol{\Phi} + m\int_\Gamma \boldsymbol{\Phi}\cdot\boldsymbol{\Psi} + \frac{1}{2}\int_\Gamma (\mathbf{v}_1^\epsilon\cdot\mathbf{n})\,\boldsymbol{\Phi}\cdot\boldsymbol{\Psi},$$

which satisfies: $a(\boldsymbol{\Phi}, \boldsymbol{\Phi}) = \nu\displaystyle\int_\Omega |\nabla\boldsymbol{\Phi}|^2 + m\int_\Gamma |\boldsymbol{\Phi}|^2$.

Due the Lax-Milgram theorem we obtain the existence of a unique solution $\mathbf{w}^m \in \mathbf{H}^1_{\text{div}}$ of (5). In order to obtain the ellipticity of the bilinear form a, we use the following inequality of the Poincaré-type:

$$|\phi|_{L^2(\Omega)} \le C(|\nabla\phi|_{L^2(\Omega)} + |\phi|_{L^2(\Gamma)}), \ \forall\phi \in H^1(\Omega). \quad (7)$$

Note that $\mathbf{w}^m$ satisfies

$$\begin{aligned} a(\mathbf{w}^m, \mathbf{w}^m) &= m\int_\Gamma (\mathbf{g}_1 - \mathbf{g}_2)\cdot\mathbf{w}^m + \int_\Omega (\mathbf{w}^m\cdot\nabla)\mathbf{w}^m\cdot\mathbf{v}_2^\epsilon \\ &\le m|\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)}\,|\mathbf{w}^m|_{L^2(\Gamma)} + \|\mathbf{w}^m\|_1^2\,|\mathbf{v}_2^\epsilon|_3. \end{aligned} \quad (8)$$

We remark that using the inequality (7) and noting that $|\mathbf{v}_2^\epsilon|_3 \le \dfrac{\nu}{2}$, the former expression implies that

$$\frac{\nu}{2}\int_\Omega |\nabla\mathbf{w}^m|^2 + m\int_\Gamma |\mathbf{w}^m|^2 \le m|\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)}\,|\mathbf{w}^m|_{L^2(\Gamma)},$$

and consequently, $|\mathbf{w}^m|_{L^2(\Gamma)} \le |\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)}$.

In order to obtain the very weak solution $\mathbf{w} = \mathbf{v}_1^\epsilon - \mathbf{v}_2^\epsilon$, we study the dual problem, that is,

$$\begin{cases} -\nu\Delta\mathbf{\Phi} - (\mathbf{v}_1^\epsilon \cdot \nabla)\mathbf{\Phi} - (\mathbf{v}_2^\epsilon \cdot \nabla^t)\mathbf{\Phi} + \nabla\pi & = \mathbf{F}, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{\Phi} & = 0, \quad \text{in } \Omega, \\ \mathbf{\Phi}| & = \mathbf{0}, \quad \text{on } \Gamma, \end{cases}$$

where $\mathbf{F} \in \mathbf{L}^{3/2}(\Omega)$. Therefore we obtain

$$\|\mathbf{\Phi}\|_{W^{2,3/2}(\Omega)} + \|\pi\|_{W^{1,3/2}(\Omega)} \leq c\,|\mathbf{F}|_{3/2}\,\left(1 + |\mathbf{v}_1^\epsilon|_3 + |\mathbf{v}_2^\epsilon|_3\right).$$

This proof is an adaptation of the proof of Lemma VIII.5.1 from [2]. It is based on a similar estimate for the Stokes system. Now, taking $\mathbf{\Psi}$ as test function in (5), with $\mathbf{F} = |\mathbf{w}^m|\mathbf{w}^m$, we obtain that

$$|\mathbf{w}^m|_3^3 + \int_\Gamma (\nu\,\partial_{\mathbf{n}}\mathbf{\Psi} - \pi\,\mathbf{n}) \cdot \mathbf{w}^m\, d\sigma = 0.$$

Hence, using (6) we have:

$$\begin{aligned} \int_\Gamma (\nu\,\partial_{\mathbf{n}}\mathbf{\Phi} - \pi\,\mathbf{n}) \cdot \mathbf{w}^m\, d\sigma &\leq \left(\nu\,|\partial_{\mathbf{n}}\mathbf{\Phi})|_{L^2(\Gamma)} + |\pi|_{L^2(\Gamma)}\right)|\mathbf{w}^m|_{L^2(\Gamma)} \\ &\leq C\,\left(\|\mathbf{\Psi}\|_{W^{2,3/2}(\Omega)} + \|\pi\|_{W^{1,3/2}(\Omega)}\right)|\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)} \\ &\leq C\,|\mathbf{w}^m|_3^2\,\left(1 + |\mathbf{v}_1^\epsilon|_3 + |\mathbf{v}_2^\epsilon|_3\right)\,|\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)}. \end{aligned}$$

Consequently we obtain

$$|\mathbf{w}^m|_3 \leq C\left(1 + |\mathbf{v}_1^\epsilon|_3 + |\mathbf{v}_2^\epsilon|_3\right)|\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)}. \tag{9}$$

The former remarks allow to show the following theorem:

Theorem 2.1. *Let* $\mathbf{v}_1^\epsilon, \mathbf{v}_2^\epsilon$ *two very weak solutions of Navier-Stokes system with external forces* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{H}^{-1}(\mathbf{\Omega})$ *and boundary data* $\mathbf{g}_1, \mathbf{g}_2 \in \mathbf{L}^2(\Gamma)$*, respectively. Then there exists a unique very weak solution* $\mathbf{w}$ *of Problem* (P) *satisfying*

$$|\mathbf{w}|_3 \leq C\left(1 + |\mathbf{v}_1^\epsilon|_3 + |\mathbf{v}_2^\epsilon|_3\right)|\mathbf{g}_1 - \mathbf{g}_2|_{L^2(\Gamma)} + C|\mathbf{f}_1 - \mathbf{f}_2|_{H^{-1}(\Omega)}. \tag{10}$$

Proof. Firstly we consider the case $\mathbf{f}_1 - \mathbf{f}_2 = 0$. By the former remarks we have that there exists $\mathbf{w}^0 \in \mathbf{L}^3(\Omega)$ and $\xi \in \mathbf{L}^2(\Gamma)$ such that, passing eventually to a subsequence,

$$\mathbf{w}^m \rightharpoonup \mathbf{w}^0, \text{ weakly in } \mathbf{L}^2(\Omega), \qquad \mathbf{w}^m\big|_\Gamma \rightharpoonup \xi, \text{ weakly in } \mathbf{L}^2(\Gamma).$$

First we prove that $\xi = \mathbf{g}_1 - \mathbf{g}_2$. Taking $\mathbf{\Psi} \in C^2(\bar\Omega)$ as test function in (5), we obtain

$$\begin{aligned} m\int_\Gamma (\mathbf{w}^m - (\mathbf{g}_1 - \mathbf{g}_2))\mathbf{\Psi} &= \int_\Omega \mathbf{w}^m\Delta\mathbf{\Psi} - \int_\Gamma \mathbf{w}^m\partial_{\mathbf{n}}\mathbf{\Psi} + \int_\Omega (\mathbf{v}_1^\epsilon \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{w}^m + \\ &\quad + \int_\Omega (\mathbf{w}^m \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{v}_2^\epsilon + \frac{1}{2}\int_\Gamma (\mathbf{v}_1^\epsilon \cdot \mathbf{n})\,\mathbf{w}^m \cdot \mathbf{\Psi} \\ &\leq |\mathbf{w}^m|_3|\Delta\mathbf{\Psi}|_{3/2} + |\mathbf{w}^m|_{L^2(\Gamma)}|\partial_{\mathbf{n}}\mathbf{\Psi}|_{L^2(\Gamma)} \\ &\qquad + |\mathbf{v}_1^\epsilon|_3|\mathbf{w}^m|_3|\nabla\mathbf{\Psi}|_3 + |\mathbf{v}_2^\epsilon|_3|\mathbf{w}^m|_3|\nabla\mathbf{\Psi}|_3 \\ &\qquad\quad + |\mathbf{v}_1^\epsilon \cdot \mathbf{n}|_{L^2(\Gamma)}|\mathbf{w}^m|_{L^4(\Gamma)}|\mathbf{\Psi}|_{L^4(\Gamma)} \;\leq\; C. \end{aligned}$$

Therefore, taking to the limit when $m \to \infty$, we get

$$\int_\Gamma (\xi - (\mathbf{g}_1 - \mathbf{g}_2))\mathbf{\Psi} = 0. \tag{11}$$

Hence, (11) holds for $\mathbf{\Psi} \in \mathbf{L}^2$; in particular for $\mathbf{\Psi} = \xi - (\mathbf{g}_1 - \mathbf{g}_2)$. This implies that $\xi = (\mathbf{g}_1 - \mathbf{g}_2)$. On the other hand, for $\mathbf{\Psi} \in V$, we have

$$\int_\Omega \mathbf{w}^m \Delta \mathbf{\Psi} + \int_\Omega (\mathbf{v}_1^\epsilon \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{w}^m + \int_\Omega (\mathbf{w}^m \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{v}_2^\epsilon = 0.$$

Therefore, taking to the limit when $m \to \infty$, we obtain

$$\int_\Omega \mathbf{w}^0 \Delta \mathbf{\Psi} + \int_\Omega (\mathbf{v}_1^\epsilon \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{w}^0 + \int_\Omega (\mathbf{w}^0 \cdot \nabla)\mathbf{\Psi} \cdot \mathbf{v}_2^\epsilon = 0.$$

Analogously, for $\tau \in W^{1,3/2}(\Omega)$, $\int_\Omega \tau = 0$, we obtain:

$$\int_\Omega \mathbf{w}^m \Delta \tau = \int_\Gamma \tau(\mathbf{w}^m \cdot \mathbf{n}),$$

this implies that

$$\int_\Omega \mathbf{w}^0 \Delta \tau = \int_\Gamma \tau((\mathbf{g}_1 - \mathbf{g}_2) \cdot \mathbf{n}).$$

Hence $\mathbf{w}^0$ is a very weak solution of (P) with $\mathbf{f}_1 - \mathbf{f}_2 = 0$. The uniqueness is due to linearity and it implies that the whole sequence $\{\mathbf{w}^m\}$ converges to $\mathbf{w}^0$. The estimate (10), for the first case, follows from (9) and the lower semicontinuity of the norm $|\cdot|_3$. Finally, for the case $\mathbf{f}_1 - \mathbf{f}_2 \neq 0$, we consider the solution $\tilde{\mathbf{w}} \in \mathbf{H}^1(\Omega)$ of the problem

$$\begin{cases} -\Delta \tilde{\mathbf{w}} + \nabla \tilde{q} &= \mathbf{f}_1 - \mathbf{f}_2, \quad \text{in } \Omega, \\ \operatorname{div} \tilde{\mathbf{w}} &= \mathbf{0}, \quad \text{in } \Omega, \\ \tilde{\mathbf{w}} &= \mathbf{0}, \quad \text{on } \Gamma. \end{cases}$$

Hence, the unique very weak solution is obtained as $\mathbf{w} = \mathbf{w}^0 + \tilde{\mathbf{w}}$. As we have the inequality $|\tilde{\mathbf{w}}|_2 \le C|\mathbf{f}_1 - \mathbf{f}_2|_{H^{-1}}$, the estimate (10) is true.

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