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Separation axioms on enlargements of generalized topologies

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Abstract. The aim of this paper is to characterize the κ_μ.closure of any subset A of X and study under what conditions a subset A of X is g.κ_μ-closed. We also introduce the notions of κ-T_i $(i = 0, 1/2, 1, 2)$ and study some properties of them.
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Axiomas de separación en ampliaciones de topologías generalizadas

Resumen. El objetivo de este trabajo es caracterizar la κ_μ.clausura de cualquier subconjunto A de X y estudiar en qué condiciones un subconjunto A de X es g.κ_μ-cerrado. También introducimos las nociones de κ-T_i $(i = 0, 1/2, 1, 2)$ y el estudio de algunas propiedades de ellas.
Palabras claves: Topología generalizada, ampliaciones.

1. Introduction

In 2002, Császár [1] introduced the notions of generalized topology and generalized continuity. In 2008, Császár [3] defined an enlargement and construct the generalized topology induced by an enlargement; introduced the concept of (κ, λ)-continuity and $(\kappa_\mu, \lambda_\mu)$-continuity on enlargements. In 2008, Császár [4] defined and studied the notions of product of generalized topologies. In 2010, S. Maragathavalli et al. in [5] studied the g.κ_μ-closed sets in generalized topological spaces and gave some characterization and properties. Also V. Renukadevi in [6] gave a characterization of g.κ_μ-closed using enlargements. In this paper we characterize the κ_μ-closure of any subset A of X, compare the sets c_κ defined in [3] and c_{κ_μ}, study under what conditions a subset A of X is g.κ_μ-closed) and introduce the notions of κ-T_i $(i = 0,\ 1/2,\ 1,\ 2)$ and study some properties of them, finally we study some notions related with the product of generalized topologies.

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2. Preliminaries

Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a generalized topology on X if and only if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $\bigcup_{i\in I} G_i \in \mu$. We call the pair (X, μ) a generalized topological space on X. The elements of μ are called μ-open sets [1] and the complements are called μ-closed sets. The generalized-closure of a subset A of X, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A; and the generalized-interior of A, denoted by $i_\mu(A)$, is the union of μ-open sets included in A. Let μ be a generalized topology on X. A mapping $\kappa : \mu \to P(X)$ is called an enlargement [3] on X if $M \subseteq \kappa M$ ($= \kappa(M)$) whenever $M \in \mu$. Let μ be a generalized topology on X and $\kappa : \mu \to P(X)$ an enlargement of μ. Let us say that a subset $A \subseteq X$ is κ_μ-open [3] if and only if $x \in A$ implies the existence of a μ-open set M such that $x \in M$ and $\kappa M \subseteq A$. The collection of all κ_μ-open sets is a generalized topology on X [3]. A subset $A \subseteq X$ is said to be κ_μ-closed if and only if $X\backslash A$ is κ_μ-open [3]. The set c_κ (briefly $c_\kappa A$) is defined in [3] as the following:

$$c_\kappa(A) = \{x \in X : \kappa(M) \cap A \neq \emptyset \text{ for every } \mu\text{-open set } M \text{ containing } x\}.$$

Definition 2.1 ([3]). Let (X, μ) and (Y, ν) be generalized topological spaces. A function $f : (X, \mu) \to (Y, \nu)$ is said to be (κ, λ)-continuous if $x \in X$ and $N \in \nu$, $f(x) \in N$ imply the existence of $M \in \mu$ such that $x \in M$ and $f(\kappa M) \subset \lambda N$.

Theorem 2.2 ([3]). *Let (X, μ) and (Y, ν) be generalized topological spaces and $f : (X, \mu) \to (Y, \nu)$ a (κ, λ)-continuous function. Then the following hold:*

1. *$f(c_\kappa(A)) \subset c_\lambda(f(A))$ holds for every subset A of (X, μ).*
2. *for every λ_ν-open set B of (Y, ν), $f^{-1}(B)$ is κ_μ-open in (X, μ).*

3. Enlargement-separation axioms

Definition 3.1. Let $\kappa : \mu \to P(X)$ be an enlargement and A a subset of X. Then the κ_μ-closure of A is denoted by $c_{\kappa_\mu}(A)$, and it is defined as the intersection of all κ_μ-closed sets containing A.

Remark 3.2. Since the collection of all κ_μ-open sets is a generalized topology on X, then for any $A \subset X$, $c_{\kappa_\mu}(A)$ is a κ_μ-closed set.

Proposition 3.3. *Let $\kappa : \mu \to P(X)$ be an enlargement and A a subset of X. Then $c_{\kappa_\mu}(A) = \{y \in X : V \cap A \neq \emptyset$ for every $V \in \kappa_\mu$ such that $y \in V\}$.*

Proof. Denote $E = \{y \in X : V \cap A \neq \emptyset$ for every $V \in \kappa_\mu$ such that $y \in V\}$. We shall prove that $c_{\kappa_\mu}(A) = E$. Let $x \notin E$. Then there exists a κ_μ-open set V containing x such that $V \cap A = \emptyset$. This implies that $X\backslash V$ is κ_μ-closed and $A \subset X\backslash V$. Hence $c_{\kappa_\mu}(A) \subset X\backslash V$. It follows that $x \notin c_{\kappa_\mu}(A)$. Thus we have that $c_{\kappa_\mu}(A) \subset E$. Conversely, let $x \notin c_{\kappa_\mu}(A)$. Then there exists a κ_μ-closed set F such that $A \subset F$ and $x \notin F$. Then we have that $x \in X\backslash F$, $X\backslash F \in \kappa_\mu$ and $(X\backslash F) \cap A = \emptyset$. This implies that $x \notin E$. Hence $E \subset c_{\kappa_\mu}(A)$. Therefore $c_{\kappa_\mu}(A) = E$. ☑

Example 3.4. *Let* $X = \{a, b, c, d\}$ *and* $\mu = P(X)\backslash\{$*all proper subsets of* X *which contains* $d\}$. *The enlargement* κ *adds the element* d *to each nonempty* μ*-open set. Then* $\kappa_\mu = \{\emptyset, X\}$. *Now put* $A = \{a\}$. *Obviously* $c_{\kappa_\mu}(A) = X$ *and* $c_\kappa(A) = \{a, d\}$. *This example shows that* $c_\kappa \subsetneq c_{\kappa_\mu}$.

Example 3.5. *Let* $X = \mathbb{R}$ *be the real line and* $\mu = \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R}\backslash\{x\}, x \neq 0\}$. *The enlargement* κ *is defined as* $\kappa(A) = c_\mu(A)$. *Then* $\kappa_\mu = \{\emptyset, X\}$.

Example 3.6. *Let* $X = \mathbb{R}$ *and* $\mu = \{\emptyset, \mathbb{R}\} \cup \{A_a = (a, +\infty)$ *for all* $a \in \mathbb{R}\}$. *The enlargement map* κ *is defined as follows:*

$$\kappa(A) = \begin{cases} A & \text{if } A = (0, +\infty), \\ \mathbb{R} & \text{if } A \neq (0, +\infty), \\ \emptyset & \text{if } A = \emptyset. \end{cases}$$

The generalized κ_μ *topology on* X *is* $\{\emptyset, \mathbb{R}, (0, +\infty)\}$.

Definition 3.7. An enlargement κ on μ is said to be open, if for every μ-neighborhood U of $x \in X$, there exists a κ_μ-open set B such that $x \in B$ and $\kappa(U) \supset B$.

Example 3.8. *Let* $X = \{a, b, c\}$ *and* $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. *Define* $\kappa : \mu \to P(X)$ *as follows:*

$$\kappa(A) = \begin{cases} A & \text{if } b \in A, \\ c_\mu(A) & \text{if } b \notin A. \end{cases}$$

The enlargement κ *on* μ *is open.*

Proposition 3.9. *If* $\kappa : \mu \to P(X)$ *is an open enlargement and* A *a subset of* X, *then* $c_\kappa(A) = c_{\kappa_\mu}(A)$ *and* $c_\kappa(c_\kappa(A)) = c_\kappa(A)$ *hold, and* $c_\kappa(A)$ *is* κ_μ*-closed in* (X, μ).

Proof. Suppose that $x \notin c_\kappa(A)$. Then there exists a μ-open set U containing x such that $\kappa(U) \cap A = \emptyset$. Since κ is an open enlargement, by Definition 3.7, there exists a κ_μ-open set V such that $x \in V \subset \kappa(U)$ and so $V \cap A = \emptyset$. By Proposition 3.3, $x \notin c_{\kappa_\mu}(A)$; it follows that $c_{\kappa_\mu}(A) \subset c_\kappa(A)$. By Corollary 1.7 of [3], we have $c_\kappa(A) \subset c_{\kappa_\mu}(A)$. In consequence, we obtain that $c_\kappa(c_\kappa(A)) = c_\kappa(A)$. By Proposition 1.3 of [3], we obtain that $c_\kappa(A)$ is a κ_μ-closed in (X, μ). ☑

Definition 3.10 ([6]). Let μ be a generalized topology on X and $\kappa : \mu \to P(X)$ an enlargement of μ. Then a subset A of a generalized topological space (X, μ) is said to be a generalized κ_μ-closed (abbreviated by g.κ_μ-closed) set in (X, μ), if $c_\kappa(A) \subset U$ whenever $A \subset U$ and $U \in \kappa_\mu$.

Proposition 3.11. *Every* κ_μ*-closed set is g.*κ_μ*-closed.*

Proof. Straightforward. ☑

Remark 3.12. A subset A is g.id_μ-closed if and only if A is g_μ-closed in the sense of Maragathavalli et. al. [5].

Theorem 3.13 ([6]). *Let* κ *be an enlargement of a generalized topological space* (X, μ). *If* A *is g.*κ_μ*-closed in* (X, μ), *then* $c_\kappa(\{x\}) \cap A \neq \emptyset$ *for every* $x \in c_\kappa(A)$.

Proof. Let A be a g.κ_μ-closed set of (X, μ). Suppose that there exists a point $x \in c_\kappa(A)$ such that $c_\kappa(\{x\}) \cap A = \emptyset$. By Proposition 1.3 of [3], $c_\kappa(\{x\})$ is μ-closed. Put $U = X \backslash c_\kappa(\{x\})$. Then, we have that $A \subset U$, $x \notin U$ and U is a μ-open set of (X, μ). Since A is a g.κ_μ-closed set, $c_\kappa(A) \subset U$. Thus, we have $x \notin c_\kappa(A)$. This is a contradiction. ☑

The converse of the above theorem is not necessarily true, as we can see.

Example 3.14. *Let N be the set of all natural numbers and μ the discrete topology on N. Let i_0 be a fixed odd number. Define $\kappa : \mu \to P(N)$ as follows:*

$$\kappa(\{n\}) = \begin{cases} \{2i : i \in N\} & \text{if } n \text{ is an even number,} \\ \{2i+1 : i \in N\} & \text{if } n = i_0, \\ \{n\} & \text{if } n \text{ is an odd number } \neq i_0, \end{cases}$$

and $\kappa(A) = N$ for the rest.

Clearly, κ is an enlargement on μ. Take $A = \{2, 4\}$. It is easy to see that $c_\kappa(A) = \{2i : i \in N\}$ and $c_\kappa(\{x\}) \cap A \neq \emptyset$ for every $x \in c_\kappa(A)$, but A is not a g.κ_μ-closed set.

Theorem 3.15. *Let μ be a generalized topology on X and $\kappa : \mu \to P(X)$ an enlargement on μ.*

1. *If a subset A is g.κ_μ-closed in (X, μ), then $c_\kappa(A) \backslash A$ does not contain any nonempty κ_μ-closed set.*
2. *If $\kappa : \mu \to P(X)$ is an open enlargement on (X, μ), then the converse of (1) is true.*

Proof. (1). Suppose that there exists a κ_μ-closed set F such that $F \subset c_\kappa(A) \backslash A$. Then, we have that $A \subset X \backslash F$ and $X \backslash F$ is κ_μ-open. It follows from assumption that $c_\kappa(A) \subset X \backslash F$ and so $F \subset (c_\kappa(A) \backslash A) \cap (X \backslash c_\kappa(A))$. Therefore, we have that $F = \emptyset$. (2). Let U be a κ_μ-open set such that $A \subset U$. Since κ is an open enlargement, it follows from Proposition 3.9 that $c_\kappa(A)$ is κ_μ-closed in (X, μ). Thus using Proposition 1.1 of [3], we have that $c_{\kappa_\mu}(A) \cap X \backslash U$, say F, is a κ_μ-closed set in (X, μ). Since $X \backslash U \subset X \backslash A$, $F \subset c_{\kappa_\mu}(A) \backslash A$. Using the assumption of the converse of (1) above, $F = \emptyset$ and hence $c_{\kappa_\mu}(A) \subset U$. ☑

Remark 3.16. The Theorem 4.1 of [6] is not true, because the condition that κ is an open enlargement can not be omitted, as we show in the following example.

Example 3.17. *In the Example 3.14, μ is not an open enlargement. If we take $A = \{2, 4\}$, it is easy to see that $c_\kappa(A) \backslash A$ does not contain any nonempty κ_μ-closed set and A is not a g.κ_μ-closed set.*

Lemma 3.18 ([6]). *Let A be a subset of a generalized topological space (X, μ) and $\kappa : \mu \to P(X)$ an enlargement on (X, μ). Then, for each $x \in X$, $\{x\}$ is κ_μ-closed or $(X \backslash \{x\})$ is a g.κ_μ-closed set of (X, μ).*

Proof. Suppose that $\{x\}$ is not κ_μ-closed. Then $X \backslash \{x\}$ is not κ_μ-open. Let U be any κ_μ-open set such that $X \backslash \{x\} \subset U$. Then, since $U = X$, $c_\kappa(X \backslash \{x\}) \subset U$. Therefore, $X \backslash \{x\}$ is g.κ_μ-closed. ☑

Definition 3.19. A generalized topological space (X, μ) is said to be a κ-$T_{1/2}$ space, if every g.κ_μ-closed set of (X, μ) is κ_μ-closed.

Theorem 3.20. *A generalized topological space (X, μ) is κ-$T_{1/2}$ if and only if, for each $x \in X$, $\{x\}$ is κ_μ-closed or κ_μ-open in (X, μ).*

Proof. Necessity: It is obtained by Lemma 3.18 and Definition 3.19. Sufficiency: Let F be g.κ_μ-closed in (X, μ). We shall prove that $c_{\kappa_\mu}(F) = F$. It is sufficient to show that $c_{\kappa_\mu}(F) \subset F$. Assume that there exists a point x such that $x \in c_{\kappa_\mu}(F) \backslash F$. Then, by assumption, $\{x\}$ is κ_μ-closed or κ_μ-open.

Case(i): $\{x\}$ is κ_μ-closed set. For this case, we have a κ_μ-closed set $\{x\}$ such that $\{x\} \subset c_{\kappa_\mu}(F) \backslash F$. This is a contradiction to Theorem 3.15 (1).

Case(ii): $\{x\}$ is κ_μ-open set. Using Corollary 1.7 of [3], we have $x \in c_{\kappa_\mu}(F)$. Since $\{x\}$ is κ_μ-open, it implies that $\{x\} \cap F \neq \emptyset$. This is a contradiction. Thus, we have that $c_\kappa(F) = F$, and so, by Proposition 1.4 of [3], F is κ_μ-closed. ☑

Definition 3.21. Let $\kappa : \mu \to P(X)$ be an enlargement. A generalized topological space (X, μ) is said to be:

1. κ-T_0 if for any two distinct points $x, y \in X$ there exists a μ-open set U such that either $x \in U$ and $y \notin \kappa(U)$ or $y \in U$ and $x \notin \kappa(U)$.
2. κ-T_1 if for any two distinct points $x, y \in X$ there exist two μ-open sets U and V containing x and y, respectively such that $y \notin \kappa(U)$ and $x \notin \kappa(V)$.
3. κ-T_2 if for any two distinct points $x, y \in X$ there exist two μ-open sets U and V containing x and y, respectively such that $\kappa(U) \cap \kappa(V) = \emptyset$.

Theorem 3.22. *Let A be a subset of a generalized topological space (X, μ) and $\kappa : \mu \to P(X)$ an open enlargement on (X, μ). Then (X, μ) is a κ-T_0 space if and only if for each pair $x, y \in X$ with $x \neq y$, $c_\kappa(\{x\}) = c_\kappa(\{y\})$ holds.*

Proof. Let x and y be any two distinct points of a κ-T_0 space. Then, by Definition 3.21, there exists a μ-open set U such that $x \in U$ and $y \notin \kappa(U)$. It follows that there exists a μ-open set S such that $x \in S$ and $S \subset \kappa(U)$. Hence, $y \in X \backslash \kappa(U) \subset X \backslash S$. Because $X \backslash S$ is a μ-closed set, we obtain that $c_\kappa(\{y\}) \subset X \backslash S$, and so $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. Conversely, suppose that $x \neq y$ for any $x, y \in X$. Then, we have that $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. Thus, we assume that there exists $z \in c_\kappa(\{x\})$ but $z \notin c_\kappa(\{y\})$. If $x \in c_\kappa(\{y\})$, then we obtain $c_\kappa(\{x\}) \subset c_\kappa(\{y\})$. This implies that $z \in c_\kappa(\{y\})$. This is a contradiction; in consequence, $x \in c_\kappa(\{y\})$. Therefore, there exists a μ-open set W such that $x \in W$ and $\kappa(W) \cap \{y\} = \emptyset$. Thus, we have that $x \in W$ and $y \notin \kappa(W)$. Hence, (X, μ) is a κ-T_0 space. ☑

Example 3.23. *In the Example 3.14, take $A = \{2, 4\}$; then $c_\kappa(A) - A = \{2i : i \in N - \{1, 2\}\}$ does not contain any nonempty κ_μ-open set, and A is not a g.κ_μ-closed set.*

Theorem 3.24. *A generalized topological space (X, μ) is κ-T_1 if and only if every singleton set of X is κ_μ-closed.*

Proof. The proof follows from the respective definitions. ☑

From Theorems 3.20, 3.24 and Definition 3.21, we obtain the following:

$$\kappa\text{-}T_2 \to \kappa\text{-}T_1 \to \kappa\text{-}T_{1/2} \to \kappa\text{-}T_0.$$

Definition 3.25. Let (X, μ) be a generalized topological space. Then the sequence $\{x_k\}$ is said to be κ-converge to a point $x_0 \in X$, denoted $x_k \underset{\rightarrow}{\kappa} x_0$, if for every μ-open set U containing x_0 there exists a positive integer n such that $x_k \in \kappa(U)$ for all $k \geq n$.

Theorem 3.26. *Let (X, μ) be a κ-T_2 space. If $\{x_k\}$ is a κ-converge sequence, then it κ-converges to at most one point.*

Proof. Let $\{x_k\}$ be a sequence in X κ-converging to x and y. Then by definition of κ-T_2 space, there exist $U, V \in \mu$ such that $x \in U, y \in V$ and $\kappa(U) \cap \kappa(V) = \emptyset$. Since $x_k \underset{\rightarrow}{\kappa} x$, there exists a positive integer n_1 such that $x_k \in \kappa(U)$ for all $k \geq n_1$. Also $x_k \underset{\rightarrow}{\kappa} y$, therefore there exists a positive integer n_2 such that $x_k \in \kappa(V)$, for all $k \geq n_2$. Let $n_0 = \max(n_1, n_2)$. Then $x_k \in \kappa(U)$ and $x_k \in \kappa(V)$, for all $k \geq n_0$ or $x_k \in \kappa(U) \cap \kappa(V)$, for all $k \geq n_0$. This contradiction proves that $\{x_k\}$ κ-converges to at most one point. ☑

Remark 3.27. Note that the above results generalize the well known separation axioms in general topology in an structure more weaker than a topology.

4. Additional Properties

Proposition 4.1. *Let $f : (X, \mu) \to (Y, \nu)$ be a (κ, λ)-continuous injection. If (Y, ν) is λ-T_1 (resp. λ-T_2), then (X, μ) is κ-T_1 (resp. κ-T_2).*

Proof. Suppose that (Y, ν) is λ-T_2. Let x and x' be distinct points of X. Then there exist two open sets V and W of Y such that $f(x) \in V, f(x') \in W$ and $\lambda(V) \cap \lambda(W) = \emptyset$. Since f is (κ, λ)-continuous, for V and W there exist two open sets U, S such that $x \in U, x' \in S, f(\kappa(U)) \subset \lambda(V)$ and $f(\kappa(S)) \subset \lambda(W)$. Therefore, we have $\kappa(U) \cap \kappa(S) = \emptyset$, and hence (X, μ) is κ-T_2. The proof of the case of λ-T_1 is similar. ☑

In [4] the notion of product of generalized topologies is defined. Let μ and ν be two generalized topologies, and β the collection of all sets $U \times V$, where $U \in \mu$ and $V \in \nu$. Clearly $\emptyset \in \beta$, so we can define a generalized topology $\mu \times \nu = \mu \times \nu(\beta)$ having β for base. We call $\mu \times \nu$ the product of the generalized topologies μ and ν.

Definition 4.2. An enlargement $\kappa : \mu \times \nu \to P(X \times Y)$ is said to be associated with κ_1 and κ_2, if $\kappa(U \times V) = \kappa_1(U) \times \kappa_2(V)$ holds for each $(\neq \emptyset)U \in \mu$, $(\neq \emptyset)V \in \nu$.

Definition 4.3. An enlargement $\kappa : \mu \times \nu \to P(X \times Y)$ is said to be regular with respect to κ_1 and κ_2, if for each point $(x, y) \in X \times Y$ and each $\mu \times \nu$-open set W containing (x, y), there exists $U \in \mu$ and $V \in \nu$ such that $x \in U$, $y \in V$ and $\kappa_1(U) \times \kappa_2(V) \subset \kappa(W)$.

Proposition 4.4. *Let $\kappa : \mu \times \mu \to P(X \times X)$ be an enlargement associated with κ_1 and κ_2. If $f : (X, \mu) \to (Y, \nu)$ is (κ_1, κ_2)-continuous and (Y, ν) is a κ_2-T_2 space, then the set $A = \{(x, y) \in X \times X : f(x) = f(y)\}$ is a κ-closed set of $(X \times X, \mu \times \mu)$.*

Proof. We show that $c_\kappa(A) \subset A$. Let $(x, y) \in X \times X \backslash A$. Then, there exist $U, V \in \nu$ such that $f(x) \in U, f(y) \in V$ and $\kappa_2(U) \cap \kappa_2(V) = \emptyset$. Moreover, for U and V there exist $W, S \in \mu$ such that $x \in W$, $y \in S$, $f(\kappa_1(W)) \subset \kappa_2(U)$ and $f(\kappa_1(S)) \subset \kappa_2(V)$. Therefore, we have $\kappa(W \times S) \cap A = \emptyset$. This shows that $(x, y) \notin c_\kappa(A)$. ☑

Corollary 4.5. *If $\kappa : \mu \times \mu \to P(X \times X)$ is an enlargement associated with κ_1 and κ_2 and it is regular with respect to κ_1 and κ_2. A generalized topological space (X, μ) is κ_1-T_2 if and only if the diagonal set $\Delta = \{(x, x) : x \in X\}$ is κ-closed in $(X \times X, \mu \times \mu)$.*

Proposition 4.6. *Let $\kappa : \mu \times \nu \to P(X \times Y)$ be an enlargement associated with κ_1 and κ_2. If $f : (X, \mu) \to (Y, \nu)$ is (κ_1, κ_2)-continuous and (Y, ν) is a κ_2-T_2 space, then the graph of f, $G(f) = \{(x, f(x)) \in X \times Y\}$ is a κ-closed set of $(X \times Y, \mu \times \nu)$.*

Proof. The proof is similar to that of Proposition 4.4. ☑

Definition 4.7. An enlargement κ on μ is said to be regular, if for any μ-open neighborhoods U, V of $x \in X$, there exists a μ-open neighborhood W of x such that $\kappa(U) \cap \kappa(V) \supset \kappa(W)$.

Theorem 4.8. *Suppose that κ_1 is a regular enlargement and $\kappa : \mu \times \nu \to P(X \times Y)$ is regular with respect to κ_1 and κ_2. Let $f : (X, \mu) \to (Y, \nu)$ be a function whose graph $G(f)$ is κ-closed in $(X \times Y, \mu \times \nu)$. If a subset B is κ_2-compact in (Y, ν), then $f^{-1}(B)$ is κ_1-closed in (X, μ).*

Proof. Suppose that $f^{-1}(B)$ is not κ_1-closed. Then, there exists a point x such that $x \in c_{\kappa_1}(f^{-1}(B))$ and $x \notin f^{-1}(B)$. Since $(x, b) \notin G(f)$ for each $b \in B$ and $G(f) \supset c_\kappa(G(f))$, there exists a $\mu \times \nu$-open set W such that $(x, b) \in W$ and $\kappa(W) \cap G(f) = \emptyset$. By the regularity of κ, for each $b \in B$ we can take two sets $U(b) \in \mu$ and $V(b) \in \nu$ such that $x \in U(b), b \in V(b)$ and $\kappa_1(U(b)) \times \kappa_2(V(b)) \subset \kappa(W)$. Then we have $f(\kappa_1(U(b))) \cap \kappa_2(V(b)) = \emptyset$. Since $\{V(b) : b \in B\}$ is a ν-open cover of B, there exists a finite number of points $b_1, ..., b_n \in B$ such that $B \subset \bigcup_{i=1}^n \kappa_2(V(b_i))$, by the κ_2-compactness of B. By the regularity of κ_1, there exists $U \in \mu$ such that $x \in U$, $\kappa_1(U) \subset \bigcap_{i=1}^n \kappa_1(U(b_i))$. Therefore, we have $\kappa_1(U) \cap f^{-1}(B) \subset \bigcup_{i=1}^n \kappa_1(U(b_i)) \cap f^{-1}(\kappa_2(V(b_i))) = \emptyset$. This shows that $x \notin c_{\kappa_1}(f^{-1}(B))$, thus we have a contradiction. ☑

Theorem 4.9. *Let $f : (X, \mu) \to (Y, \nu)$ be a function whose graph $G(f)$ is κ-closed in $(X \times Y, \mu \times \nu)$, and suppose that the following conditions hold:*

1. *$\kappa_1 : \mu \to P(X)$ is open,*
2. *$\kappa_2 : \nu \to P(Y)$ is regular, and*
3. *$\kappa : \mu \times \nu \to P(X \times Y)$ is an enlargement associated with κ_1 and κ_2, and κ is regular with respect to κ_1 and κ_2.*

If every cover of A by κ_1-open sets of (X, μ) has a finite subcover, then $f(A)$ is κ_2-closed in (Y, ν).

Proof. The proof is similar to that of Theorem 4.8 ☑

Proposition 4.10. *Let $\kappa : \mu \times \nu \to P(X \times Y)$ be an enlargement associated with κ_1 and κ_2. If $f : (X, \mu) \to (Y, \nu)$ is (κ_1, κ_2)-continuous and (Y, ν) is a κ_2-T_2, then the graph of f, $G(f) = \{(x, f(x)) \in X \times Y\}$ is a $\kappa_{\mu\times\nu}$-closed set of $(X \times Y, \mu \times \nu)$.*

Proof. The proof is similar to that of Proposition 4.4. ☑

Definition 4.11. A function $f : (X, \mu) \to (Y, \nu)$ is said to be (κ, λ)-closed, if for any κ_μ-closed set A of (X, μ), $f(A)$ is λ_ν-closed in (Y, ν).

Theorem 4.12. *Suppose that f is (κ, λ)-continuous and (id, λ)-closed. If for every g.κ_μ-closed set A of (X, μ), then the image $f(A)$ is g.λ_ν-closed.*

Proof. Let V be any λ_ν-open set of (Y, ν) such that $f(A) \subset V$. By the Theorem 2.2 (2), $f^{-1}(V)$ is κ_μ-open. Since A is g.κ_μ-closed and $A \subset f^{-1}(V)$, we have $c_\kappa(A) \subset f^{-1}(V)$, and hence $f(c_\kappa(A)) \subset V$. It follows from Proposition 1.3 of [3] and our assumption that $f(c_\kappa(A))$ is λ_ν-closed. Therefore we have $c_\lambda(f(A)) \subset c_\lambda(f(c_\kappa(A))) = f(c_\kappa(A)) \subset V$. This implies $f(A)$ is g.λ_ν-closed. ☑

Theorem 4.13. *If $f : (X, \mu) \to (Y, \nu)$ is (κ, λ)-continuous and (id, λ)-closed, if f is injective and (Y, ν) is λ-$T_{1/2}$, then (X, μ) is κ-$T_{1/2}$.*

Proof. Let A be a g.κ_μ-closed set of (X, μ). We show that A is κ_μ-closed. By Theorem 4.12 and our assumptions it is obtained that $f(A)$ is g.λ_ν-closed, and hence $f(A)$ is λ_μ-closed. Since f is (κ, λ)-continuous, $f^{-1}(f(A))$ is κ_μ-closed by using Theorem 2.2 (2). ☑

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