Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 32, No. 2, 2014, pág. 139–152

Jacobson's conjecture and skew PBW extensions

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Abstract. The aim of this paper is to compute the Jacobson's radical of skew PBW extensions over domains. As a consequence of this result we obtain a direct relation between these extensions and the Jacobson's conjecture, which implies that skew PBW extensions over domains satisfy this conjecture.

Keywords: Noncommutative rings, Jacobson's radical, skew PBW extensions.
MSC2010: 16N20, 16N40, 16W70, 16N60, 16S32, 16S36.

Conjetura de Jacobson y extensiones PBW torcidas

Resumen. El propósito de este artículo es calcular el radical de Jacobson de las extensiones PBW torcidas sobre dominios. Como consecuencia de este resultado obtenemos una relación directa entre estas extensiones y la conjetura de Jacobson, lo cual nos permite mostrar que las extensiones PBW torcidas sobre dominios satisfacen esta conjetura.

Palabras clave: Anillos no conmutativos, radical de Jacobson, extensiones PBW torcidas.

1. Introduction

The Jacobson's radical introduced by N. Jacobson is the analog of the Frattini subgroup in group theory. For a ring B, its Jacobson radical $J(B)$ is defined as the intersection of maximal left ideals in B. It is a remarkable fact that $J(B)$ coincides with the intersection of maximal left ideals (see [18] for more details). If $J(B) = \{0\}$, then B is called *Jacobson semisimple*.

For several commutative and noncommutative rings the Jacobson's radical has been computed (see [14, 15, 18]). The main purpose of this paper is to find this radical for a

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Received: 17 April 2014, Accepted: 24 May 2014.
To cite this article: A. Reyes, Jacobson's conjecture and skew PBW extensions, *Rev. Integr. Temas Mat.* 32 (2014), no. 2, 139-152.

kind of noncommutative rings known as skew PBW (PBW denotes Poincaré-Birkhoff-Witt) extensions which were introduced in [3]. Once we have calculated their Jacobson's radical, we show immediately the relation between these extensions and the Jacobson's conjecture.

Conjecture (Jacobson's conjecture). *The intersection of the powers of the Jacobson radical in a Noetherian ring A equals zero, i.e., $\bigcap_{n=1}^{\infty} J(A)^n = 0$.*

Our interest in this conjecture is motivated by its advances in noncommutative algebra, since it is known that this conjecture is true in a commutative Noetherian ring (this is a consequence of the Krull Intersection Theorem, see Kaplansky [14], Theorem 79). The noncommutative question was formulated for one-sided Noetherian rings B, by Jacobson in [8], p. 200. He had earlier introduced transfinite powers of $J(B)$ (the intersection of the finite powers being $J(B)^{\omega}$) and had shown that some transfinite power of $J(B)$ must be zero (see [7], Theorem 11). However, counterexamples to the one-sided question were presented by Herstein [5] and Jategaonkar [9], Example 1. Jategaonkar also constructed counterexamples by showing that arbitrarily high transfinite powers of $J(B)$ are needed (cf. [10], Theorem 4.6).

The question in the two-sided Noetherian case has been answered positively for FBN rings (see [4], Theorem 9.13) by Cauchon ([1], Theorem 5, [2], Theorem I 2, p. 36), and Jategaonkar ([11], Theorem 3.7); see also Schelter ([21]). For Noetherian rings of Krull dimension one by Lenagan ([16], Theorem 4.4), and for Noetherian rings satisfying the second layer condition (cf. [4], Theorem 14.8) by Jategaonkar ([12], Theorem H; [13], Theorem 1.8).

With all above results in mind, we consider that this paper contributes to the study of this conjecture for a considerable number of noncommutative rings which include rings and algebras of interest for modern mathematical physics such as PBW extensions, well-known classes of Ore algebras, operator algebras, diffusion algebras, quantum algebras, quadratic algebras in 3-variables, skew quantum polynomials, among many others (see Section 4).

The paper is organized as follows. In Section 2 we present the definition and some key results of skew PBW extensions. In Section 3 we compute the Jacobson's radical of these extensions (Theorem 3.2). As a consequence we also establish its prime radical (Proposition 3.3). From Theorem 3.2 we show that skew PBW extensions over domains satisfy the Jacobson's conjecture (Remark 3.5). Section 4 illustrates this result with some remarkable examples of skew PBW extensions.

2. Definitions and key results

In this section we present some results about skew PBW extensions.

Definition 2.1 ([3], Definition 1)**.** Let R and A be rings. We say that A is a *skew PBW extension of R* (also called a σ-PBW extension of R) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finite elements $x_1, \dots, x_n \in A \setminus R$ such A is a left R-free module with basis

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case we also say that A *is a left polynomial ring over* R with respect to the set of variables $\{x_1, \dots, x_n\}$ and $\mathrm{Mon}(A)$ is the set of standard monomials of A. In addition, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n \rangle$.

For the next proposition, we recall that if B is a ring and σ is a ring endomorphism $\sigma : B \to B$, a σ-derivation $\delta : B \to B$ satisfies by definition $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$. If y is an indeterminate, and $yb = \sigma(b)y + \delta(b)$, for any $b \in B$, we denote this noncommutative ring as $B[y; \sigma, \delta]$, and it is called a *skew polynomial ring*.

Proposition 2.2 ([3], Proposition 3). *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$.*

Definition 2.3 ([3], Definition 4). Let A be a skew PBW extension.

(a) A is *quasi-commutative* if conditions (iii) and (iv) in Definition 2.1 are replaced by:

(iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i. \tag{3}$$

(iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{4}$$

(b) A is *bijective* if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Definition 2.4 ([3], Definition 6). Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.2.

(i) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^{\alpha} := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) For $X = x^{\alpha} \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) Let $0 \neq f \in A$. We will denote by $t(f)$ the finite set of terms that conform f, i.e., if $f = c_1 X_1 + \cdots + c_t X_t$ with $X_i \in \mathrm{Mon}(A)$ and $c_i \in R \setminus \{0\}$, then $t(f) := \{c_1 X_1, \ldots, c_t X_t\}$.

(iv) Let f be as in (iii). Then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

Skew PBW extensions can be characterized as follows.

Theorem 2.5 ([3], Theorem 7). *Let A be a polynomial ring over R with respect to $\{x_1, \ldots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(a) *for each $x^{\alpha} \in \mathrm{Mon}(A)$ and all $0 \neq r \in R$, there exist unique elements $r_{\alpha} := \sigma^{\alpha}(r) \in R - \{0\}$, $p_{\alpha,r} \in A$ such that*

$$x^{\alpha} r = r_{\alpha} x^{\alpha} + p_{\alpha,r}, \tag{5}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, then r_{α} is also invertible.

(b) *For each $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*

$$x^{\alpha} x^{\beta} = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{6}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

A *filtered ring* is a ring B with a family $FB = \{F_n B \mid n \in \mathbb{Z}\}$ of additive subgroups of B where we have the ascending chain $\cdots \subset F_{n-1}B \subset F_n B \subset \cdots$ such that $1 \in F_0 B$ and $F_n B F_m B \subseteq F_{n+m} B$ for all $n, m \in \mathbb{Z}$. From a filtered ring B it is possible to construct its associated graded ring $G(B)$ taking $G(B)_n := F_n B / F_{n-1} B$.

The first theorem of this section characterizes the graded associated ring of a skew PBW extension. $G(B)$ is a ring, which is known in the literature as the *associated graded ring* of B.

Theorem 2.6 ([17], Theorem 2.2). *If A is a skew PBW extension of a ring R, then A is a filtered ring with filtration FA given by*

$$F_m A := \begin{cases} R & \text{if} \quad m = 0, \\ \{f \in A \mid \deg(f) \leq m\} & \text{if} \quad m \geq 1. \end{cases} \tag{7}$$

With this filtration the graded associated ring $G(A)$ is a quasi-commutative skew PBW extension of R. If the skew PBW extension A is bijective, then $G(A)$ is a quasi-commutative bijective extension of R.

Next theorem establishes the relation between skew PBW extensions and iterated skew polynomial rings in the sense of Proposition 2.2.

Theorem 2.7 ([17], Theorem 2.3). *Let A be a quasi-commutative skew PBW extension of a ring R. Then* (i) *A is isomorphic to an iterated skew polynomial ring, and* (ii) *if A is bijective, each endomorphism of the skew polynomial ring in* (i) *is an isomorphism.*

3. Jacobson's radical

As we mentioned above, the *Jacobson's radical* of a ring B, denoted by $J(B)$, is the intersection of maximal left ideals of B. If $B \neq 0$, maximal left ideals always exist by Zorn's Lemma, and so $J(B) \neq B$. If $B = 0$, then there are no maximal left ideals; in this case, we define $J(B) = 0$. A ring B is called *Jacobson semisimple* (or J-semisimple) if $J(B) = 0$. If $\mathrm{rad}(B) = \{0\}$, where $\mathrm{rad}(-)$ denotes the prime radical (the intersection of prime left ideals of B), then B is called *semiprime*.

In the noncommutative setting an *integral domain*, briefly called a *domain*, is defined as a ring in which the product of any two nonzero elements is nonzero ([18]). Proposition 3.1 establishes necessary and sufficient conditions to guarantee that skew PBW extensions are domains.

Proposition 3.1 ([17], Proposition 4.1). *Let A be a skew PBW extension of a ring R. If R is a domain, then A is also a domain.*

Proof. In $\mathrm{Mon}(A)$ we define the order

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but } \exists\, i \text{ with } \quad \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases} \tag{8}$$

This order is total on $\mathrm{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Each element $f \in A$ can be written in a unique way as $f = c_1x^{\alpha_1} + \cdots + c_tx^{\alpha_t}$, with $c_i \in R \setminus \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that x^{α_1} is the *leading monomial* of f, which is denoted $\mathrm{lm}(f) := x^{\alpha_1}$; c_1 is the *leading coefficient* of f, written $\mathrm{lc}(f) := c_1$, and that $c_1x^{\alpha_1}$ is the *leading term* of f, denoted by $\mathrm{lt}(f) := c_1x^{\alpha_1}$. Note that

$$x^\alpha \succ x^\beta \Rightarrow \mathrm{lm}(x^\gamma x^\alpha x^\lambda) \succ \mathrm{lm}(x^\gamma x^\beta x^\lambda), \text{ for every } x^\gamma, x^\lambda \in \mathrm{Mon}(A).$$

Let $f = cx^\alpha + p$, $g = dx^\beta + q$ be nonzero elements of A, with $cx^\alpha = \mathrm{lt}(f)$, $dx^\beta = \mathrm{lt}(g)$, i.e., $c, d \neq 0$, $x^\alpha \succ \mathrm{lm}(p)$ and $x^\beta \succ \mathrm{lm}(q)$. Theorem 2.5 implies

$$\begin{aligned} fg &= (cx^\alpha + p)(dx^\beta + q) \\ &= cx^\alpha dx^\beta + cx^\alpha q + pdx^\beta + pq \\ &= c(d_\alpha x^\alpha + p_{\alpha,d})x^\beta + cx^\alpha q + pdx^\beta + pq, \end{aligned}$$

with $0 \neq d_\alpha = \sigma^\alpha(d) \in R$, $p_{\alpha,d} \in A$, $p_{\alpha,d} = 0$, or $\deg(p_{\alpha,d}) < |\alpha|$ thanks to Proposition 2.2. Therefore,

$$\begin{aligned} fg &= cd_\alpha x^\alpha x^\beta + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \\ &= cd_\alpha(c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}) + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \\ &= cd_\alpha c_{\alpha,\beta}x^{\alpha+\beta} + cd_\alpha p_{\alpha,\beta} + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq, \end{aligned}$$

where $0 \neq c_{\alpha,\beta} \in R$, $p_{\alpha,\beta} \in A$, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$. Note that $cd_\alpha c_{\alpha,\beta} \neq 0$ and $h := cd_\alpha p_{\alpha,\beta} + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \in A$ is an element such that $h = 0$ or $x^{\alpha+\beta} \succ \mathrm{lm}(h)$. This shows that $fg \neq 0$, which concludes the proof. ☑

Theorem 3.2 ([19], Theorem 1.3.2). *Let R be a domain. If A is a bijective skew PBW extension of R, then A is a semisimple Jacobson ring, that is, $J(A) = \{0\}$. Moreover* $\mathrm{rad}(A) = 0$.

Proof. Consider the associated graded ring $G(A)$ of filtration (7). Let f be a nonzero element of A with the notation given in Definition 2.4. We define $\alpha(f) := f + F_{\deg(f)-1}A$ if and only if $f \neq 0$.

Let $0 \neq f \in J(A)$. Then there exists a nonzero $g \in A$ for which $(1 - f)g = 1$ (note that $\deg(1 - f) = \deg(f)$). Suppose that $\deg(f) \geq 1$. It is clear that $\alpha(1 - f), \alpha(g) \neq 0$. Then

$$\begin{aligned} \alpha(1 - f)\alpha(g) &= (1 - f + F_{\deg(1-f)-1}A)(g + F_{\deg(g)-1}A) \\ &= (1 - f)g + F_{\deg(1-f)+\deg(g)-1}A \\ &= 1 + F_{\deg(1-f)+\deg(g)-1}A. \end{aligned}$$

Since $\deg(1 - f) \geq 1$ then $\deg(1 - f) + \deg(g) - 1 \geq \deg(g) \geq 0$, which implies $1 + F_{\deg(1-f)+\deg(g)-1}A = 0 + F_{\deg(1-f)+\deg(g)-1}A$. Hence $\alpha(1 - f)\alpha(g) = 0 + F_{\deg(1-f)+\deg(g)-1}A$, which is a contradiction since $G(A)$ is a domain (Theorem 2.6 and Proposition 3.1) and the fact that $\alpha(1 - f), \alpha(g)$ are nonzero elements of $G(A)$. Therefore $\deg(f) \leq 0$ which means that $J(A) \subseteq R$.

Next we show that $J(A) \subseteq \{0\}$. Let $g \in A$ with $\deg(g) \geq 1$ and let $f \in J(A)$. Then $fg \in J(A) \subseteq R$, that is, $fg \in F_0A$. Since $\deg(f) + \deg(g) - 1 \geq 0$ it follows that $\alpha(f)\alpha(g) = (f + F_{\deg(f)-1}A)(g + F_{\deg(g)-1}A) = fg + F_{\deg(f)+\deg(g)-1}A = 0 + F_{\deg(f)+\deg(g)-1}A$, but $\alpha(g) \neq 0$ so $\alpha(f) = 0$, that is, $f = 0$. This concludes the proof. ☑

In the noncommutative setting, a prime ideal in a ring B is any proper ideal P of B such that, whenever I and J are ideals of B with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A *prime ring* is a ring in which 0 is a prime ideal. The next proposition establishes sufficient conditions to obtain skew PBW extensions which are prime rings.

Proposition 3.3 ([19], Proposition 1.3.3). *If A is a bijective skew PBW extension of a prime ring R, then A is also a prime ring and* $\mathrm{rad}(A) = 0$.

Proof. By Theorem 2.6 one has that $G(A)$ is a quasi-commutative skew PBW extension of R, and by assumption it is also bijective. Now, Theorem 2.7 implies that $G(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$ where θ_i is bijective for $1 \leq i \leq n$. Hence by [18], Theorem 1.2.9, $G(A)$ is a prime ring. Finally, from [18], Theorem 1.6.9, it follows the assertion. ☑

Remark 3.4 ([19], Remark 1.3.4)**.** (i) Theorem 3.2 generalizes the result that establishes that if k is a division ring, then a polynomial ring $k[\{x_i\}]$ in commuting variables $\{x_i\}$ is J-semisimple. This also applies for the Ore extension of bijective type $k[x;\sigma]$ and the ring of type derivation $k[x;\delta]$, which are also J-semisimple (cf. [15], Corollary 4.17).

(ii) We recall that if T is a set of commuting variables, then the polynomial ring $B = R[T]$ is prime if and only if R is prime. The same statement holds for the ring of Laurent polynomials $R[T, T^{-1}]$ ([15], Proposition 10.18). In this way, bijective skew PBW extensions of prime rings (for instance fields, polynomial rings and Laurent polynomial rings over prime rings, or iterated skew PBW extensions over prime rings) have prime radical zero (see Section 4).

Remark 3.5. It follows from Theorem 3.2 that Jacobson's conjecture is true for all examples of skew PBW extensions over a domain presented in [17], Section 3. The next section illustrates this result with some remarkable examples of skew PBW extensions.

4. Some examples of skew PBW extensions over domains

In this section we present some examples of skew PBW extensions over domains. Hence its Jacobson's radical is trivial and examples verify the Jacobson's conjecture. For a more complete list of examples and a detailed description and reference of each ring (see [17], Section 3 and [19], Chapter 2.)

4.1. PBW extensions

Any PBW extension is a bijective skew PBW extension since in this case $\sigma_i = \mathrm{id}_R$, for every $1 \leq i \leq n$, and $c_{i,j} = 1$, for every $1 \leq i, j \leq n$. Thus, for PBW extensions we have $A = i(R)\langle x_1, \dots, x_n\rangle$. The following are examples of PBW extensions.

(a) *Polynomial rings over domains*: $A = R[t_1, \dots, t_n]$ is a skew PBW extension of R.

(b) Any *skew polynomial ring of derivation type* $A = R[x;\sigma,\delta]$, i.e., with $\sigma = \mathrm{id}_R$. In general, any *Ore extension of derivation type* $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$, that is, $\sigma_i = \mathrm{id}_R$, for any $1 \leq i \leq n$.

(c) Let k be a commutative ring and $\mathfrak{g}$ a finite-dimensional Lie algebra over k with basis $\{x_1, \dots, x_n\}$; the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $\mathcal{U}(\mathfrak{g})$, is a PBW extension of k, since $x_ir - rx_i = 0$, $x_ix_j - x_jx_i = [x_i, x_j] \in \mathfrak{g} = k + kx_1 + \cdots + kx_n$,

$r \in k$, for $1 \leq i, j \leq n$. In fact, the *universal enveloping algebra of a Kac-Moody Lie algebra* is a PBW extension of a polynomial ring.

(d) The *twisted or smash product differential operator ring* $k \#_\sigma \mathcal{U}(\mathfrak{g})$ studied by McConnell [18] and others, where $\mathfrak{g}$ is a finite-dimensional Lie algebra acting on k by derivations, and σ is Lie 2-cocycle with values in k.

4.2. Ore extensions of bijective type

Any *skew polynomial ring* $R[x;\sigma,\delta]$ *of bijective type* is a bijective skew PBW extension. In this case we have $R[x;\sigma,\delta] \cong \sigma(R)\langle x\rangle$. If additionally $\delta = 0$, then $R[x;\sigma]$ is quasi-commutative. In a general way, let $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ be an *iterated skew polynomial ring of bijective type*, i.e., the following conditions hold:

- for $1 \leq i \leq n$, σ_i is bijective;
- for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$;
- for $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c, d \in R$ and c has a left inverse;
- for $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$;

then, $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a bijective skew PBW extension. Under these we have $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] \cong \sigma(R)\langle x_1,\ldots,x_n\rangle$. Some remarkable examples of this kind of noncommutative rings are the following:

(a) *Quantum plane* $\mathcal{O}_q(\Bbbk^2)$. Let $q \in \Bbbk^*$. The *quantized coordinate ring of* $\Bbbk^2$ is a $\Bbbk$-algebra, denoted by $\mathcal{O}_q(\Bbbk^2)$, presented by two generators x, y and the relation $xy = qyx$. We have $\mathcal{O}_q(\Bbbk^2) \cong \sigma(\Bbbk)\langle x, y\rangle$.

(b) *The algebra of q-differential operators* $D_{q,h}[x,y]$. Let $q, h \in \Bbbk, q \neq 0$; consider the ring $\Bbbk[y][x;\sigma,\delta]$, where $\sigma(y) := qy$, $\delta(y) := h$. Then $xy = \sigma(y)x + \delta(y) = qyx + h$, so $xy - qyx = h$, and hence $D_{q,h}[x,y] \cong \sigma(\Bbbk)\langle x, y\rangle$.

(c) The *mixed algebra* D_h. It is defined by $D_h := \Bbbk[t][x;\mathrm{id}_{\Bbbk[t]},\frac{d}{dt}][x_h;\sigma_h]$, where $h \in \Bbbk$ and $\sigma_h(x) := x$. Then $D_h \cong \sigma(\Bbbk)\langle t, x, x_h\rangle$.

4.3. Operator algebras

In this subsection we recall some important and well-known operator algebras. We will see that these algebras are skew PBW extensions of Ore extensions and hence some operator algebras are iterated skew PBW extensions.

(a) *Algebra of linear partial differential operators.* The nth Weyl algebra $A_n(\Bbbk)$ over $\Bbbk$ coincides with the $\Bbbk$-algebra of linear partial differential operators with polynomial coefficients $\Bbbk[t_1,\ldots,t_n]$. As we have seen, the generators of $A_n(\Bbbk)$ satisfy the following relations: $t_it_j = t_jt_i$, $\partial_i\partial_j = \partial_j\partial_i$, for $1 \leq i < j \leq n$, and

$\partial_j t_i = t_i \partial_j + \delta_{ij}$, for $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker symbol. Therefore $\sigma(\mathbb{k})\langle t_1, \dots, t_n; \partial_1, \dots, \partial_n\rangle$.

(b) *Algebra of linear partial q-differential operators.* For a fixed $q \in \mathbb{k} \setminus \{0\}$, this is the $\mathbb{k}$-algebra $\mathbb{k}[t_1, \dots, t_n][D_1^{(q)}, \dots, D_m^{(q)}]$, $n \geq m$, subject to the relations:

$$\begin{aligned} t_j t_i &= t_i t_j, && 1 \leq i < j \leq n, \\ D_i^{(q)} t_i &= q t_i D_i^{(q)} + 1, && 1 \leq i \leq m, \\ D_j^{(q)} t_i &= t_i D_j^{(q)}, && i \neq j, \\ D_j^{(q)} D_i^{(q)} &= D_i^{(q)} D_j^{(q)}, && 1 \leq i < j \leq m. \end{aligned}$$

If $n = m$, this operator algebra coincides with the additive analogue $A_n(q_1, \dots, q_n)$ of the Weyl algebra $A_n(q)$ (Section 4.5, Example (a)). This algebra can be expressed as the skew PBW extension $\sigma(\mathbb{k})\langle t_1, \dots, t_n; D_1^{(q)}, \dots, D_m^{(q)}\rangle$.

(c) *Operator differential rings.* Let R be an algebra over a commutative ring k and let $\Delta = \{\delta_1, \dots, \delta_n\}$ be a set of commuting derivations of R. Let $T = R[\theta_1, \dots, \theta_n; \delta_1, \dots, \delta_n]$ be the operator differential ring. The elements of T can be written in a unique way as left R-linear combinations with the ordered monomials in $\theta_1, \dots, \theta_n$. The product on T is defined extending the product from R subject to the relation $\theta_i r - r\theta_i = \delta_i(r)$, $r \in R$, $i = 1, \dots, n$, and $\theta_i\theta_j - \theta_j\theta_i = 0$, $i, j = 1, \dots, n$, and $T = \sigma(R)\langle\theta_1, \dots, \theta_n\rangle$.

4.4. Diffusion algebras

Diffusion algebras arose in physics as a possible way to understand a large class of 1-dimensional stochastic process [6]. A *diffusion algebra* $\mathcal{A}$ with parameters $a_{ij} \in \mathbb{C} \setminus \{0\}, 1 \leq i, j \leq n$ is a $\mathbb{C}$-algebra generated by indeterminates $x_1, \dots, x_n$ subject to relations $a_{ij}x_i x_j - b_{ij}x_j x_i = r_j x_i - r_i x_j$, whenever $i < j$, $b_{ij}, r_i \in \mathbb{C}$ for all $i < j$. Therefore $\mathcal{A}$ admits a PBW-basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$, that is, $\mathcal{A}$ is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for $\mathcal{A}$. From Definition 2.1, (iii) and (iv), it is clear that the family of skew PBW extensions are more general than diffusion algebras.

In the applications to physics the parameters a_{ij} are strictly positive reals and the parameters b_{ij} are positive reals as they are unnormalised measures of probability [6]. We will denote $q_{ij} := \frac{b_{ij}}{a_{ij}}$. The parameter q_{ij} is a root of unity if and only if it equals to 1. It is therefore reasonable to assume that these parameters are not roots of unity different from 1 ([6], p. 22). If all coefficients q_{ij} are nonzero, then the corresponding diffusion algebra have a PBW basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$, and hence these algebras are skew PBW extensions. More precisely, $\mathcal{A} \cong \sigma(\mathbb{C})\langle x_1, \dots, x_n\rangle$. Note that diffusion algebras cannot be expressed as Ore extensions which follows from the definition.

4.5. Quantum algebras

(a) *Additive analogue of the Weyl algebra.* The $\Bbbk$-algebra $A_n(q_1, \dots, q_n)$ is by definition generated by the indeterminates $x_1, \dots, x_n$, $y_1, \dots, y_n$ subject to the relations:

$$x_j x_i = x_i x_j, \qquad 1 \leq i, j \leq n, \tag{9}$$
$$y_j y_i = y_i y_j, \qquad 1 \leq i, j \leq n, \tag{10}$$
$$y_i x_j = x_j y_i, \qquad i \neq j, \tag{11}$$
$$y_i x_i = q_i x_i y_i + 1, \qquad 1 \leq i \leq n, \tag{12}$$

where $q_i \in \Bbbk \setminus \{0\}$. We can see that $A_n(q_1, \dots, q_n) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$.

(b) *Multiplicative analogue of the Weyl algebra.* The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by the indeterminates $x_1, \dots, x_n$ subject to the relations: $x_j x_i = \lambda_{ji} x_i x_j$, $1 \leq i < j \leq n$, $\lambda_{ji} \in \Bbbk \setminus \{0\}$. Thus $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$.

(c) *Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$.* This algebra is the q-analogue of the universal enveloping algebra $\mathfrak{so}(3, \Bbbk)$. By definition it is the $\Bbbk$-algebra generated by the variables I_1, I_2, I_3 subject to relations $I_2 I_1 - q I_1 I_2 = -q^{1/2} I_3, I_3 I_1 - q^{-1} I_1 I_3 = q^{-1/2} I_2, I_3 I_2 - q I_2 I_3 = -q^{1/2} I_1$, with $q \in \Bbbk \setminus \{0\}$. Then $\mathcal{U}'(\mathfrak{so}(3, \Bbbk)) \cong \sigma(\Bbbk)\langle I_1, I_2, I_3\rangle$.

(d) *q-Heisenberg algebra.* The $\Bbbk$-algebra $\mathbf{H}_n(q)$ is generated by the set of indeterminates $\{x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n\}$ subject to the relations:

$$x_j x_i = x_i x_j, \qquad z_j z_i = z_i z_j, \qquad y_j y_i = y_i y_j, \quad 1 \leq i, j \leq n, \tag{13}$$
$$z_j y_i = y_i z_j, \qquad z_j x_i = x_i z_j, \qquad y_j x_i = x_i y_j, \quad i \neq j, \tag{14}$$
$$z_i y_i = q y_i z_i, \qquad z_i x_i = q^{-1} x_i z_i + y_i, \qquad y_i x_i = q x_i y_i, \; 1 \leq i \leq n, \tag{15}$$

with $q \in \Bbbk \setminus \{0\}$. Then $\mathbf{H}_n(q) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n\rangle$.

(e) *Algebra of quantum matrices $\mathcal{O}_q(M_n(\Bbbk))$.* This algebra was introduced by Faddeev, Reshetikhin and Takhtadjian. It is by definition generated by $\Bbbk$ and the variables x_{ij}, $1 \leq i, j \leq n$, subject to

$$x_{im} x_{ik} = q^{-1} x_{ik} x_{im}, \qquad 1 \leq k < m \leq n,$$
$$x_{jk} x_{ik} = q^{-1} x_{ik} x_{jk}, \qquad 1 \leq i < j \leq n,$$
$$x_{im} x_{jk} = x_{jk} x_{im}, \qquad 1 \leq i < j, k < m \leq n,$$
$$x_{jm} x_{im} = q^{-1} x_{im} x_{jm}, \qquad 1 \leq i < j \leq n,$$
$$x_{jm} x_{jk} = q^{-1} x_{jk} x_{jm}, \qquad 1 \leq k < m \leq n,$$
$$x_{ik} x_{jm} - x_{jm} x_{ik} = (q - q^{-1}) x_{im} x_{jk}, \qquad 1 \leq i < j, k < m \leq n.$$

From these relations we can see that $\mathcal{O}_q(M_n(\Bbbk)) \cong \sigma(\Bbbk[x_{im}, x_{jk}])\langle x_{ik}, x_{jm}\rangle$, for $1 \leq i < j, k < m \leq n$. If $n = 2$, and $x_{11} := y, x_{12} := u, x_{21} := v$ and $x_{22} := x$, we obtain $\mathcal{O}_q(M_2(\Bbbk))$, the coordinate algebra of the quantum matrix space $M_2(\Bbbk)$ (this algebra is also known as Manin algebra of 2×2 quantum matrices).

(f) *Quantum enveloping algebra of* $\mathfrak{sl}(2,\Bbbk)$. $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))$ is defined as the $\Bbbk$-algebra generated by the variables x, y, z, z^{-1} with relations $zz^{-1} = z^{-1}z = 1, xz = q^{-2}zx, yz = q^2zy, xz^{-1} = q^2z^{-1}x, yz^{-1} = q^{-2}z^{-1}y$, and $xy - yx = \frac{z-z^{-1}}{q-q^{-1}}$, with $q \neq 1, -1$. From these relations we can see that $\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk)) = \sigma(\Bbbk[z^{\pm 1}])\langle x, y\rangle$.

(g) *Hayashi's algebra* $W_q(J)$. $W_q(J)$ is the algebra generated by the variables $x_i, y_i, z_i, 1 \leq i \leq n$, $i \in J$, where $\mid J \mid= n$, and relations (13)-(15), replacing $z_ix_i = q^{-1}x_iz_i + y_i$ by $(z_ix_i - qx_iz_i)y_i = 1 = y_i(z_ix_i - qx_iz_i)$, $\ i = 1, \dots, n$, $\ q \in \Bbbk \setminus \{0\}$. Since $x_iy_j^{-1} = y_j^{-1}x_i,\ z_iy_j^{-1} = y_j^{-1}z_i,\ y_jy_j^{-1} = y_j^{-1}y_j = 1,\ z_ix_i = qx_iz_i + y_i^{-1}$, for $1 \leq i, j \leq n$, then $W_q(J) \cong \sigma(\Bbbk[y_1^{\pm 1}, \dots, y_n^{\pm 1}])\langle x_1, \dots, x_n; z_1, \dots, z_n\rangle$.

(h) *The complex algebra* $V_q(\mathfrak{sl}_3(\mathbb{C}))$. Let q be a complex number such that $q^8 \neq 1$. Consider the complex algebra generated by $e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2$ with the following relations:

$$
\begin{array}{lll}
e_{13}e_{12} = q^{-2}e_{12}e_{13}, & f_{13}f_{12} = q^{-2}f_{12}f_{13}, & \\
e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, & f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13}, & \\
e_{23}e_{13} = q^{-2}e_{13}e_{23}, & f_{23}f_{13} = q^{-2}f_{13}f_{23}, & \\
e_{12}f_{12} = f_{12}e_{12} + \dfrac{k_1^2 - l_1^2}{q^2 - q^{-2}}, & e_{12}k_1 = q^{-2}k_1e_{12}, & k_1f_{12} = q^{-2}f_{12}k_1, \\
e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, & e_{12}k_2 = qk_2e_{12}, & k_2f_{12} = qf_{12}k_2, \\
e_{12}f_{23} = f_{23}e_{12}, & e_{13}k_1 = q^{-1}k_1e_{13}, & k_1f_{13} = q^{-1}f_{13}k_1, \\
e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, & e_{13}k_2 = q^{-1}k_2e_{13}, & k_2f_{13} = q^{-1}f_{13}k_2, \\
e_{13}f_{13} = f_{13}e_{13} - \dfrac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, & e_{23}k_1 = qk_1e_{23}, & k_1f_{23} = qf_{23}k_1, \\
e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, & e_{23}k_2 = q^{-2}k_2e_{23}, & k_2f_{23} = q^{-2}f_{23}k_2, \\
e_{23}f_{12} = f_{12}e_{23}, & e_{12}l_1 = q^{2}l_1e_{12}, & l_1f_2 = q^{2}f_{12}l_1, \\
e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, & e_{12}l_2 = q^{-1}l_2e_{12}, & l_2f_{12} = q^{-1}f_{12}l_2, \\
e_{23}f_{23} = f_{23}e_{23} + \dfrac{k_2^2 - l_2^2}{q^2 - q^{-2}}, & e_{13}l_1 = ql_1e_{13}, & l_1f_{13} = qf_{13}l_1, \\
e_{13}l_2 = ql_2e_{13}, & l_2f_{13} = qf_{13}l_2, & e_{23}l_1 = q^{-1}l_1e_{23}, \\
l_1f_{23} = q^{-1}f_{23}l_1, & e_{23}l_2 = q^{2}l_2e_{23}, & l_2f_{23} = q^{2}f_{23}l_2, \\
l_1k_1 = k_1l_1, & l_2k_1 = k_1l_2, & k_2k_1 = k_1k_2, \\
l_1k_2 = k_2l_1, & l_2k_2 = k_2l_2, & l_2l_1 = l_1l_2.
\end{array}
$$

This algebra is a bijective skew PBW extension of the polynomial ring $\mathbb{C}[l_1, l_2, k_1, k_2]$. That is, $V_q(\mathfrak{sl}_3(\mathbb{C})) \cong \sigma(\mathbb{C}[l_1, l_2, k_1, k_2])\langle e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}\rangle$.

(i) *The algebra of differential operators* $D_{\mathbf{q}}(S_{\mathbf{q}})$ *on a quantum space* $S_{\mathbf{q}}$. Let k be a commutative ring and let $\mathbf{q} = [q_{ij}]$ be a matrix with entries in k^*, such that $q_{ii} = 1 = q_{ij}q_{ji}$ for all $1 \leq i, j \leq n$. The k-algebra $S_{\mathbf{q}}$ is generated by $x_i, 1 \leq i \leq n$, subject to the relations $x_ix_j = q_{ij}x_jx_i$. The algebra $S_{\mathbf{q}}$ is regarded as the *algebra* *of functions on a quantum space*. The algebra $D_{\mathbf{q}}(S_{\mathbf{q}})$ of $\mathbf{q}$-differential operators on $S_{\mathbf{q}}$ is defined by $\partial_i x_j - q_{ij}x_j\partial_i = \delta_{ij}$, for all i, j, and $\partial_i\partial_j = q_{ij}\partial_j\partial_i$, for all i, j. Therefore, $D_{\mathbf{q}}(S_{\mathbf{q}}) \cong \sigma(\sigma(k)\langle x_1, \ldots, x_n\rangle)\langle \partial_1, \ldots, \partial_n\rangle$.

(j) *Quantum Weyl algebra* $A_n(q, p_{i,j})$. The ring $A_n(q, p_{i,j})$ can be viewed as a quantization of the usual Weyl algebra $A_n(\Bbbk)$. By definition, $A_n(q, p_{i,j})$ is the ring generated over the field $\Bbbk$ by the variables x_i, ∂_j with $i, j = 1, \ldots, n$ and subject to relations

$$\begin{aligned}
x_i x_j &= p_{ij} q x_j x_i, && \text{for all } i < j,\\
\partial_i\partial_j &= p_{ij} q^{-1}\partial_j\partial_i, && \text{for all } i < j,\\
\partial_i x_j &= p_{ij}^{-1} q x_j\partial_i, && \text{for all } i \neq j,\\
\partial_i x_i &= 1 + q^2 x_i\partial_i + (q^2 - 1)\sum_{i<j} x_j\partial_j, && \text{for all } i.
\end{aligned}$$

We can check that $A_n(q, p_{i,j}) \cong \sigma(\sigma(\sigma(\sigma(\cdots\sigma(\sigma(\Bbbk)\langle x_n\rangle)\langle\partial_n\rangle)\cdots)\langle x_2\rangle)\langle y_2\rangle)\langle x_1\rangle)\langle y_1\rangle$. Note that if $q, p_{ij} = 1$ we obtain the usual Weyl algebra $A_n(\Bbbk)$.

4.6. 3-dimensional skew polynomial algebras

Definition 4.1. *A 3-dimensional skew polynomial algebra* $\mathcal{A}$ is a $\Bbbk$-algebra generated by the indeterminates x, y, z restricted to $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$ and $xy - \gamma yx = \nu$ such that

1. $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk^*$;
2. Standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\Bbbk$-basis of the algebra.

It is clear that 3-dimensional skew polynomial ring are skew PBW extensions of the field $\Bbbk$.

Next proposition establishes a classification of 3-dimensional skew polynomial algebras.

Proposition 4.2 ([20], Theorem C.4.3.1, p. 101)**.** *Let $\mathcal{A}$ be a 3-dimensional skew polynomial algebra. Then $\mathcal{A}$ is one of the following algebras:*

(a) *if* $|\{\alpha, \beta, \gamma\}| = 3$, *then* $\mathcal{A}$ *is defined by*

$$yz - \alpha zy = 0, \qquad zx - \beta xz = 0, \qquad xy - \gamma yx = 0. \tag{16}$$

(b) *if* $|\{\alpha, \beta, \gamma\}| = 2$ *y* $\beta \neq \alpha = \gamma = 1$, $\mathcal{A}$ *is one of the following algebras:*

- (i) $yz - zy = z, \quad zx - \beta xz = y, \quad xy - yx = x;$
- (ii) $yz - zy = z, \quad zx - \beta xz = b, \quad xy - yx = x;$
- (iii) $yz - zy = 0, \quad zx - \beta xz = y, \quad xy - yx = 0;$

(iv) $yz - zy = 0, \quad zx - \beta xz = b, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x;$

(vi) $yz - zy = z, \quad zx - \beta xz = 0, \quad xy - yx = 0.$

Here a and b are any elements of $\Bbbk$. All nonzero values of b give isomorphic algebras.

(c) *If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then $\mathcal{A}$ is one of the following algebras:*

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$

(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

In this case b is an arbitrary element of $\Bbbk$. Again, any nonzero values of b given isomorphic algebras.

(d) *If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is the algebra*

$$yz - \alpha zy = a_1 x + b_1, \qquad zx - \alpha xz = a_2 y + b_2, \qquad xy - \alpha yx = a_3 z + b_3.$$

If $a_i = 0, i = 1, 2, 3$, all nonzero values of b_i give isomorphic algebras.

(e) *If $\alpha = \beta = \gamma = 1$, $\mathcal{A}$ is isomorphic to one of the following algebras*

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$

(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$

(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$

(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

Parameters $a, b \in \Bbbk$ are arbitrary and all nonzero values of b generates isomorphic algebras.

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