

Fixed point property for nonexpansive mappings and nonexpansive semigroups on the unit disk

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Abstract. For closed convex subsets D of a Banach spaces, in 2009, Tomonari Suzuki [11] proved that the fixed point property (FPP) for nonexpansive mappings and the FPP for nonexpansive semigroups are equivalent. In this paper some relations between the aforementioned properties for mappings and semigroups defined on D , a closed convex subset of the hyperbolic metric space (\mathbb{D}, ρ) , are studied. This work arises as a generalization to the space (\mathbb{D}, ρ) of the study made by Suzuki.

Keywords: ρ -nonexpansive mappings, fixed point property, semigroups.

MSC2010: 47H09, 47H10, 30C99.

Propiedad del punto fijo para funciones y semigrupos no expansivos en el disco unidad

Resumen. Para subconjuntos D cerrados y convexos de espacios de Banach, Tomonari Suzuki [11] demostró en 2009 que la propiedad del punto fijo (PPF) para funciones no expansivas y la PPF para semigrupos de funciones no expansivas son equivalentes. En este trabajo se estudian algunas relaciones entre dichas propiedades, cuando D es un subconjunto del espacio métrico (\mathbb{D}, ρ) . Este trabajo surge como una generalización al espacio (\mathbb{D}, ρ) de los resultados de Suzuki.

Palabras clave: Funciones ρ -no expansivas, propiedad del punto fijo, semigrupos.

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Received: 11 December 2014, Accepted: 19 March 2015.

To cite this article: L. Benítez-Babilonia, Fixed point property for nonexpansive mappings and nonexpansive semigroups on the unit disk, *Rev. Integr. Temas Mat.* 33 (2015), no. 1, 41-50.

1. Introduction

We denote by \mathbb{D} the unit disk in the complex plane \mathbb{C} . Further, let us recall the Poincaré hyperbolic metric on \mathbb{D} (see, for example, [10]):

$$\rho(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - \bar{z}w} \right| \quad \text{if } z, w \in \mathbb{D}.$$

Throughout this paper D will represent a closed convex subset of the metric space (\mathbb{D}, ρ) with $\text{Int}D \neq \emptyset$, where $\text{Int}D$ denotes the interior of D . A mapping $f : D \rightarrow \mathbb{D}$ is called a ρ -**nonexpansive mapping** on D if

$$\rho(f(x), f(y)) \leq \rho(x, y) \quad \text{for all } x, y \in D. \quad (1)$$

This means that f is nonexpansive with respect to the Poincaré hyperbolic metric ρ . We denote by $N_\rho(D, \mathbb{D})$ the set of all ρ -nonexpansive mappings $f : D \rightarrow \mathbb{D}$, and by $\mathcal{F}(f)$ the set of all fixed points of f . The set $\mathcal{F}(f)$ has been studied in many settings ([1], [2], [6], [4], [7], [11] and [8]), when f is a nonexpansive mapping. For example, in 1965, Browder [1] proved that $\mathcal{F}(f)$ is nonempty provided D is a bounded subset of a Hilbert space E . For more details about the study of fixed point of nonexpansive mappings we suggest the reader to review [7], [9], where the authors have given a summary of this topic.

A family of mappings $S = \{\phi_t : D \rightarrow D : t \geq 0\}$ is called a ρ -**nonexpansive semigroup** in $N_\rho(D)$, the set of all ρ -nonexpansive self-mappings of D , if the following conditions are satisfied:

- S1. For all $s, t \geq 0$, $\phi_{s+t} = \phi_s \circ \phi_t$.
- S2. For each $x \in D$, the mapping $t \rightarrow \phi_t(x)$ from $[0, +\infty)$ into D is strongly continuous.

The family S is also called a strongly continuous semigroup of ρ -nonexpansive mappings on D . Furthermore, S is called a ρ -**nonexpansive semigroup with identity** if S1, S2 and the following additional condition holds:

- S3. For every $z \in D$, $\phi_0(z) = z$.

We denote by $\mathcal{F}(S)$ the set of all common fixed points of S , i.e.,

$$\mathcal{F}(S) = \bigcap_{t \geq 0} \mathcal{F}(\phi_t). \quad (2)$$

The set D is said to **have the (uniqueness) fixed point property for ρ -nonexpansive mappings** if every ρ -nonexpansive mapping on D has a (unique respectively) fixed point in D . Also, D is said to **have the (uniqueness) fixed point property for ρ -nonexpansive semigroups (with identity, respectively)** if every ρ -nonexpansive semigroup (with identity, respectively) on D has a (unique respectively) common fixed point in D .

Fixed point theorems for families of nonexpansive mappings were studied in [6], [5], [3], [12], [8] et. al. Indeed, in [6], the weak fixed point property (i.e., any weakly compact convex subset of a Banach space E has the fixed point property) of some subsets of the Banach space of all real continuous functions defined on K , where K is a compact metrizable space, is studied. In [8], the fixed point property for semigroups of nonexpansive mappings on weakly compact convex subset of a Banach space has been studied.

It is well known that every Banach space $(E, \|\cdot\|)$ is a metric space with metric defined by $d(x, y) = \|x - y\|$ for $x, y \in E$. Further, a d -nonexpansive mapping on E means nonexpansive mapping on E with respect to the metric d . Therefore, it is interesting to investigate whether results developed in another contexts, for example in Banach spaces, can be extended to metric spaces or to the hyperbolic geometry. Such a question involves omitting the condition of linear structure, or sometimes, as in our case, the metric $d(x, y) = \|x - y\|$ is inappropriate for extending the results which are given for Banach spaces.

When the metric is unsuitable, one way to eliminate the deficiency is to define another metric, which implies a change in the geometry. As a matter of fact, for the study of fixed point of holomorphic mappings on \mathbb{D} , it is precise to define the so-called Poincaré hyperbolic metric on \mathbb{D} . Such a metric provides the needed properties in the construction of a non-Euclidean geometry (hyperbolic) on \mathbb{D} and appears as the more suitable one from the point of view of the Riemann metric theory and measurement of lengths of tangent vectors. Moreover, the hyperbolic metric on \mathbb{D} has the property that each holomorphic mapping on \mathbb{D} becomes nonexpansive with respect to such a metric.

For example, an extension of the Wolff-Denjoy fixed point theory to the hyperbolic metric can be found in Shoikhet [10]. This theory gives the behavior of the iterates of $f \in N_\rho(\mathbb{D})$ or the convergence of the sequence $\{f^n(z)\}_{n \in \mathbb{N}}$ for every $z \in \mathbb{D}$.

On the other hand, recently, Suzuki [11] proved the following.

Theorem 1.1. *Let D be a closed convex subset of a Banach space $(E, \|\cdot\|)$. Then, the following conditions are equivalent:*

- A. The set D has the fixed point property for d -nonexpansive mappings.*
- B. The set D has the fixed point property for d -nonexpansive semigroups.*
- C. The set D has the fixed point property for d -nonexpansive semigroups with identity.*

The aim of this paper is to study the connection between the three conditions in the above mentioned theorem, when D is considered to be a subset of the hyperbolic metric space (\mathbb{D}, ρ) . Indeed, we prove that **B** implies **A** (see Theorem 3.5) by proving, in our setting, analogous results which are given in [11] and [12]. The equivalence is established when the uniqueness of the fixed point is assumed (see Corollary 3.8). Also, we study the set of common fixed points of a one-parameter semigroup of nonexpansive mappings in D (see Theorem 3.1 and Corollary 3.2).

When we omit the uniqueness property, the difficulty in extending some results of [11] and [12] (**A** implies **B**) relies on the convex structure of \mathbb{D} given by the hyperbolic metric, more precisely, the failure of the inequality

$$\rho(\alpha z + (1 - \alpha)w, \alpha u + (1 - \alpha)v) \leq \alpha \rho(z, u) + (1 - \alpha)\rho(w, v), \quad z, w, u, v \in \mathbb{D}, \quad \alpha \in (0, 1).$$

2. Preliminaries

In this section, we recall some results required throughout this work.

For a real number t , we denote by $[|t|]$ the maximum integer not exceeding t .

Lemma 2.1 (Suzuki [12]). *Let $\lambda \in (0, 1)$ and $\theta \in [0, 1]$. Define a sequence $\{A_n\}$ of subsets of $[0, 1]$ by $A_1 = \{\theta\}$, $A_2 = \{|1 - \theta|, |\lambda - \theta|\}$ and*

$$A_{n+1} = \bigcup_{t \in A_n} \{|1 - t|, |\lambda - t|\} \quad \text{for } n \geq 2,$$

and put $A(\theta) = \bigcup_{n \geq 1} A_n$. Then,

$$A(\theta) \setminus \{1\} = \bigcup_{m \in \mathbb{Z}} \{m\lambda + \kappa\theta - [m\lambda + \kappa\theta] : \kappa = \pm 1\}.$$

Moreover, if $h \in A(\theta)$, then $A(h) = A(\theta)$.

By Lemma 2.1, the following lemmas can be proved.

Lemma 2.2 (Suzuki [12]). *Let λ and θ be as in Lemma 2.1. If $\lambda \in \mathbb{I}$ is an irrational number, then the closure of $A(\theta)$ is $[0, 1]$.*

Lemma 2.3 (Suzuki [12]). *Let λ and θ be as in Lemma 2.1. If $\lambda \in \mathbb{Q}$ is a rational number, then $A(\theta)$ is a finite set.*

Next, we prove a lemma which is a modification of a result (Lemma 6) given in [12].

Lemma 2.4. *Let $S = \{\phi_t : t \geq 0\}$ be a ρ -nonexpansive semigroup in $N_\rho(D)$ and $\theta \in [0, 1]$ as before. Assume that there exist $z_0 \in D$, $\lambda \in (0, 1)$ and $\tau \in A(\theta)$ such that*

$$\rho_\lambda(z_0) = \phi_1(z_0) = z_0 \quad \text{and} \quad \rho(\phi_\tau(z_0), z_0) = \max\{\rho(\phi_s(z_0), z_0) : s \in A(\theta)\}.$$

Then,

$$\rho(\phi_\tau(z_0), z_0) = \rho(\phi_s(z_0), z_0) \quad \text{for all } s \in A(\theta).$$

Proof. From Lemma 2.1, we have $A(\theta) = A(\tau) = \bigcup_{n \in \mathbb{N}} B_n$, where $B_1 = \{\tau\}$, $B_2 = \{|1 - \tau|, |\lambda - \tau|\}$ and $B_{n+1} = \bigcup_{t \in B_n} \{|1 - t|, |\lambda - t|\}$ for $n \geq 2$. Now, we claim that

$$\rho(\phi_\tau(z_0), z_0) = \rho(\phi_s(z_0), z_0) \quad \text{for all } s \in B_n, \tag{3}$$

holds for each $n \in \mathbb{N}$ and prove it by induction.

Equation (3) holds for $n = 1$ because $B_1 = \{\tau\}$. We assume that (3) holds for some $n \in \mathbb{N}$. For $t \in B_n$, we have $|1 - t|, |\lambda - t| \in B_{n+1} \subset A(\theta)$. Since $t \in B_n$, we obtain

$$\rho(\phi_\tau(z_0), z_0) = \rho(\phi_t(z_0), z_0) = \rho(\phi_t(z_0), \phi_1(z_0)) \leq \rho(\phi_{|1-t|}(z_0), z_0) \leq \rho(\phi_\tau(z_0), z_0).$$

In a similar way, we prove

$$\rho(\phi_\tau(z_0), z_0) = \rho(\phi_t(z_0), z_0) = \rho(\phi_t(z_0), \phi_\lambda(z_0)) \leq \rho(\phi_{|\lambda-t|}(z_0), z_0) \leq \rho(\phi_\tau(z_0), z_0).$$

Hence

$$\rho(\phi_{|1-t|}(z_0), z_0) = \rho(\phi_{|\lambda-t|}(z_0), z_0) = \rho(\phi_\tau(z_0), z_0).$$

Since $t \in B_n$ is arbitrary, Equation (3) holds for $n + 1$. Therefore, we obtain the desired result. \square

Now, we recall the definition of the nonlinear resolvent for a continuous function $F : D \rightarrow D$ (see, for example, [10]). This resolvent method is useful to study one-parameter semigroups in $N_\rho(\mathbb{D})$.

Definition 2.5. Let $F : D \rightarrow \mathbb{C}$ be a continuous mapping. The mapping F is said to **satisfy the range condition (RC)** on D if for every $r > 0$ the nonlinear resolvent $J_r := (I - rF)^{-1}$ is well defined on D and belongs to $N_\rho(D)$.

In other words, F satisfies the range condition if for each $r > 0$ and $z \in D$ the equation $w - rF(w) = z$ has a unique solution $w = J_r(z)$ in D such that

$$\rho(J_r(z_1), J_r(z_2)) \leq \rho(z_1, z_2) \quad \text{provided} \quad z_1, z_2 \in D.$$

Proposition 2.6 (Shoikhet [10]). *If $f \in N_\rho(D)$, then $F = f - I$ satisfies the range condition on D .*

Proof. Let $x \in D$ and $r > 0$ be given, and define

$$H_r(z) = \frac{1}{1+r}x + \frac{r}{1+r}f(z), \quad z \in D.$$

Note that $H_r : D \rightarrow D$ is a contraction map (see [10]). So, it has a unique fixed point due to Banach's contraction principle. Then, the equation $z - r(f(z) - z) = x$, has a unique solution $z_r(x)$ in D . Therefore, $J_r = (I - rF)^{-1} : D \rightarrow D$ is well defined by $J_r(x) = z_r(x)$. Furthermore,

$$J_r(z) = \lim_{n \rightarrow +\infty} H_r^n(z), \quad \text{with} \quad H_r^n(z) = H_r(H_r^{n-1}(z)).$$

But if $n \in \mathbb{N}$, then by properties of ρ we have

$$\begin{aligned} \rho(H_r^n(z), H_r^n(w)) &\leq \max\{\rho(f(H_r^{n-1}(z)), f(H_r^{n-1}(w))), \rho(z, w)\} \\ &\leq \max\{\rho(H_r^{n-1}(z), H_r^{n-1}(w)), \rho(z, w)\} \leq \dots \leq \rho(z, w). \end{aligned}$$

Hence,

$$\rho(J_r(z), J_r(w)) = \lim_{n \rightarrow +\infty} \rho(H_r^n(z), H_r^n(w)) \leq \rho(z, w).$$

Thus, J_r is ρ -nonexpansive. \square

The proof of the following result can also be found in [10].

Theorem 2.7 (Shoikhet [10]). *Let $F : \mathbb{D} \rightarrow \mathbb{C}$ be a continuous function on \mathbb{D} . Then, F satisfies the range condition on \mathbb{D} if and only if F is the infinitesimal generator of*

one-parameter semigroup in $N_\rho(\mathbb{D})$, i.e., there exists $S = \{\phi_t : t \geq 0\}$ a one-parameter semigroup of ρ -nonexpansive self-mappings of D such that

$$F(z) = \lim_{t \rightarrow 0^+} \frac{\phi_t(z) - z}{t}, \quad \text{for every } z \in \mathbb{D}. \tag{4}$$

Furthermore, for every $z \in \mathbb{D}$ and $T > 0$ the sequence $\{J_{t/n}^n(z)\}$ converges to $\phi_t(z)$ uniformly on $[0, T]$ as $n \rightarrow +\infty$ for each $z \in \mathbb{D}$, i.e.,

$$\phi_t(z) = \lim_{n \rightarrow +\infty} J_{t/n}^n(z). \tag{5}$$

The following theorem is not a direct consequence of Theorem 2.7, but follows from its proof after some modifications. Since the proof of Theorem 2.7 is rather long, we do not present it here.

Theorem 2.8. *Let $F : D \rightarrow \mathbb{C}$ be a continuous function on D that satisfies the range condition on D . Then, F is the infinitesimal generator of a one-parameter semigroup $S = \{\phi_t : t \geq 0\}$ in $N_\rho(D)$, and the following exponential formula holds:*

$$\phi_t(z) = \lim_{n \rightarrow +\infty} J_{t/n}^n(z) \quad z \in D, \quad t > 0. \tag{6}$$

We give a sketch of the proof of Theorem 2.8. To do that, first, we present a list of properties that the nonlinear resolvent satisfies. We recommend the reader to see Shoikhet [10], where the proofs of such properties are given in the case when $D = \mathbb{D}$.

Proof. (Theorem 2.8) Since $F : D \rightarrow \mathbb{C}$ is a continuous function on D that satisfies the range condition on D , the nonlinear resolvent $J_r : D \rightarrow D$ is well defined and is nonexpansive on D . Then J_r satisfies the following properties:

1. For each compact set $K \subset D$ and $\epsilon > 0$, there is $\mu = \mu_{K,\epsilon} > 0$ such that for all $r \in (0, \mu)$ and each $n = 0, 1, 2, 3, \dots$ the following inequalities hold:

$$\left| F(z) - \frac{J_{r/n}^{(n)}(z) - z}{r} \right| < \epsilon, \quad \text{and} \quad \left| J_r(z) - J_{r/n}^{(n)}(z) \right| < 2r\epsilon,$$

whenever $z \in K$.

2. For $0 \leq v \leq u$ the following resolvent identity holds:

$$J_u(z) = J_v \left(\frac{v}{u}z + \left(1 - \frac{v}{u}\right) J_u(z) \right), \quad z \in D.$$

3. For each $t \geq 0$ the sequence $\{J_{t/n}^{(n)}\}_{n \in \mathbb{N}}$ is a locally uniformly Cauchy sequence on D endowed with the hyperbolic metric ρ .

4. The map $r \rightarrow J_{r/n}^{(n)}(z)$ is continuous for r sufficiently small.

From 3, we have that the limit in (6) exists and the mapping $\phi_t : D \rightarrow D$ defined by this formula belongs to $N_\rho(D)$. The condition S2 follows from 4. The semigroup property of $S = \{\phi_t : t \geq 0\}$ can be proved in a standard way, using (6) and passing from rational $s \geq 0, t \geq 0$ to real numbers by the continuity of S . Finally, that F is the infinitesimal generator of S is obtained from 1. ☑

3. The fixed point property for nonexpansive semigroups

Our main results will be proved in this section. One of them (See Theorem 3.1) is an analogous result to the main Theorem in [13] and studies the set of common fixed points of a one-parameter semigroup of nonexpansive mappings in D . On the other hand, Theorems 3.3 and 3.5 are modifications of Theorems in [11] for the metric space (\mathbb{D}, ρ) . The first theorem is proved following similar arguments as given in the proofs in [11]. In the second one we use the set of common fixed points of a one-parameter semigroup and the uniqueness fixed point property for ρ -nonexpansive mappings.

Theorem 3.1. *Let $S = \{\phi_t : t \geq 0\}$ be a strongly continuous semigroup of ρ -nonexpansive mappings on D . If $\lambda \in (0, 1) \setminus \mathbb{Q}$, then*

$$\mathcal{F}(S) = \mathcal{F}(\phi_\lambda) \cap \mathcal{F}(\phi_1). \tag{7}$$

Proof. It is clear that $\mathcal{F}(S) \subset \mathcal{F}(\phi_\lambda) \cap \mathcal{F}(\phi_1)$. For the reciprocal, we assume that $z_0 \in \mathcal{F}(\phi_\lambda) \cap \mathcal{F}(\phi_1)$.

Since $\{\phi_t(z_0) : t \in [0, 1]\}$ is a compact subset of D , there exists $\tau \in [0, 1]$ such that

$$\rho(\phi_\tau(z_0), z_0) = \max\{\rho(\phi_s(z_0), z_0) : s \in [0, 1]\}.$$

It is obvious that $\tau \in A(\tau) \subset [0, 1]$, where $A(\tau)$ is given by Lemma 2.1, with some $\lambda \in (0, 1)$. Then, $\rho(\phi_\tau(z_0), z_0) = \max\{\rho(\phi_s(z_0), z_0) : s \in A(\tau)\}$.

Hence, by Lemma 2.4, we obtain $\rho(\phi_\tau(z_0), z_0) = \rho(\phi_s(z_0), z_0)$ for all $s \in A(\tau)$.

From Lemma 2.2, we know that the closure of $A(\tau)$ is $[0, 1]$. Then, the continuity of $t \rightarrow \phi_t(z_0)$ implies that

$$\rho(\phi_\tau(z_0), z_0) = \rho(\phi_s(z_0), z_0) \quad \text{for all } s \in [0, 1]. \tag{8}$$

Since $\lambda, 1 \in [0, 1]$, from Equation (8) we have

$$2\rho(\phi_\tau(z_0), z_0) = \rho(\phi_\tau(z_0), z_0) + \rho(\phi_\tau(z_0), z_0) = \rho(\phi_\lambda(z_0), z_0) + \rho(\phi_1(z_0), z_0) = 0.$$

This implies that $\phi_\tau(z_0) = z_0$. Therefore, $\phi_s(z_0) = z_0$ for all $s \in [0, 1]$. If $t > 1$, then there exist $n \in \mathbb{N}$ and $s \in [0, 1)$ such that $t = n + s$ and

$$\phi_t(z_0) = \phi_n(\phi_s(z_0)) = \phi_n(z_0) = \phi_{n-1}(\phi_1(z_0)) = \phi_{n-1}(z_0) = \dots = z_0.$$

That is, $z_0 \in \mathcal{F}(\phi_t)$ for all $t \geq 0$. This completes the proof. \(\square\)

Corollary 3.2. *Let $S = \{\phi_t : t \geq 0\}$ be a strongly continuous semigroup of ρ -nonexpansive mappings on D . Let α and β be positive real numbers satisfying $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$. Then,*

$$\mathcal{F}(S) = \mathcal{F}(\phi_\alpha) \cap \mathcal{F}(\phi_\beta). \tag{9}$$

Proof. Without loss of generality, we may assume that $\alpha < \beta$. It is easy to see that $S_1 = \{\psi_t : t \geq 0\}$, where $\psi_t = \phi_{\beta t}$, is a strongly continuous semigroup of ρ -nonexpansive mappings on D . Further, $\mathcal{F}(S) = \mathcal{F}(S_1)$. Finally, applying Theorem 3.1 with the semigroup S_1 and $\lambda = \alpha/\beta$, the proof is completed. \(\square\)

The following theorem plays an important role in the development of further results. Also, from this theorem it is evident that extending results to the case of hyperbolic metric is not straightforward.

Theorem 3.3. *Let us assume that $f \in N_\rho(D)$ and define $F = f - I$. If $S = \{\phi_t : t \geq 0\}$ is the ρ -nonexpansive semigroup given by (6), then $\mathcal{F}(f) = \mathcal{F}(S)$ holds.*

Proof. Since $F = f - I$ satisfies the range condition, $J_r : D \rightarrow D$ is well defined for every $r > 0$. So, for every $x \in D$ and $r > 0$, the equation

$$z = \frac{1}{1+r}x + \frac{r}{1+r}f(z) \quad (10)$$

has a unique solution given by $x = J_r(z)$. If $x_0 \in \mathcal{F}(f)$; then,

$$x_0 = \frac{1}{1+r}x_0 + \frac{r}{1+r}f(x_0). \quad (11)$$

Hence, $J_r(x_0) = x_0$ for all $r > 0$. Thus, $\phi_t(x_0) = x_0$ for all $t \geq 0$, which implies that $x_0 \in \mathcal{F}(S)$. We have shown that $\mathcal{F}(f) \subset \mathcal{F}(S)$.

Conversely, let us assume that $x_0 \in \mathcal{F}(S)$ and fix $\epsilon > 0$. From Corollary 2.8, we see that for every $t \in [0, T]$, there exists $n_0 \in \mathbb{N}$ such that

$$\rho(x_0, J_{t/n}^n(x_0)) = \rho(\phi_t(x_0), J_{t/n}^n(x_0)) < \epsilon \quad \text{if } n \geq n_0. \quad (12)$$

In particular, by setting $T = 2$ and $t = k/n$ with $n, k \in \mathbb{N}$, where $n \leq k \leq 2n$, we have

$$\rho(x_0, J_{1/n}^k(x_0)) < \epsilon, \quad (13)$$

for all $n_0 \leq n \leq k \leq 2n$.

Now, for $n \geq n_0$ and $r = 1/n$, from (10) and (13), we get

$$\begin{aligned} |f(x_0) - J_{1/n}^{2n}(x_0)| &= \left| f(x_0) - \frac{1}{1+r}J_{1/n}^{2n-1}(x_0) - \frac{r}{1+r}f(J_{1/n}^{2n}(x_0)) \right| \\ &\leq \frac{1}{1+r} \left| f(x_0) - J_{1/n}^{2n-1}(x_0) \right| + \frac{r}{1+r} \left| f(x_0) - f(J_{1/n}^{2n}(x_0)) \right| \\ &\leq \frac{1}{1+r} \left| f(x_0) - J_{1/n}^{2n-1}(x_0) \right| + \frac{r}{(1+r)\mathbf{m}_K} \rho(f(x_0), f(J_{1/n}^{2n}(x_0))) \\ &\leq \frac{1}{1+r} \left| f(x_0) - J_{1/n}^{2n-1}(x_0) \right| + \frac{r}{(1+r)\mathbf{m}_K} \rho(x_0, J_{1/n}^{2n}(x_0)) \\ &\leq \frac{1}{1+r} \left| f(x_0) - J_{1/n}^{2n-1}(x_0) \right| + \left(1 - \frac{1}{(1+r)} \right) \frac{\epsilon}{\mathbf{m}_K}. \end{aligned}$$

Here, we have used the equivalence between the hyperbolic metric and the Euclidean metric on compact subsets K of \mathbb{D} , that is, there exist constants $\mathbf{m}_K, \mathbf{M}_K > 0$, such that $\mathbf{m}_K|z - w| \leq \rho(z, w) \leq \mathbf{M}_K|z - w|$ for all $z, w \in K$. Hence,

$$\begin{aligned} |f(x_0) - J_{2/2n}^{2n}(x_0)| &= |f(x_0) - J_{1/n}^{2n}(x_0)| \\ &\leq \frac{1}{(1+1/n)^n} \left| f(x_0) - J_{1/n}^n(x_0) \right| + \left(1 - \frac{1}{(1+1/n)^n} \right) \frac{\epsilon}{\mathbf{m}_K}. \end{aligned}$$

Then, by taking limit as $n \rightarrow +\infty$, from (6), we obtain

$$|f(x_0) - \phi_2(x_0)| \leq \frac{1}{e} |f(x_0) - \phi_1(x_0)| + \left(1 - \frac{1}{e}\right) \frac{\epsilon}{\mathbf{m}_K}.$$

Since $x_0 \in \mathcal{F}(S)$ and ϵ is arbitrary, we conclude $e|f(x_0) - x_0| \leq |f(x_0) - x_0|$, which implies that $f(x_0) = x_0$. Therefore, $\mathcal{F}(S) \subset \mathcal{F}(f)$. \checkmark

Remark 3.4. It is obvious that the following two propositions are equivalent:

- The set D has the fixed point property for ρ -nonexpansive semigroups.
- The set D has the fixed point property for ρ -nonexpansive semigroups with identity.

Now, we prove some relations between the fixed point property for ρ -nonexpansive mappings and the fixed point property for ρ -nonexpansive semigroups. The next result is proved following similar arguments from the proofs in [11].

Theorem 3.5. *Let D be a closed convex subset of the metric space (\mathbb{D}, ρ) . If D has the fixed point property for ρ -nonexpansive semigroups, then D has the fixed point property for ρ -nonexpansive mappings.*

Proof. Let us assume that D has the fixed point property for nonexpansive semigroups. If f is a nonexpansive mapping on D , then $F = f - I$ satisfies the range condition due to Proposition 2.6. Let us define $S = \{\phi_t : t \geq 0\}$ by using (6). Then, $\mathcal{F}(f) = \mathcal{F}(S)$ holds from Theorem 3.3. But $\mathcal{F}(S)$ is nonempty from the assumption. Hence, $\mathcal{F}(f)$ is nonempty. \checkmark

Remark 3.6. It is easy to see that the above theorem holds if we assume that D has the fixed point property for ρ -nonexpansive semigroups with identity.

Theorem 3.7. *Let D be a closed convex subset of the metric space (\mathbb{D}, ρ) . If D has the uniqueness fixed point property for ρ -nonexpansive mappings, then D has the fixed point property for ρ -nonexpansive semigroups. Moreover, D has the uniqueness fixed point property for ρ -nonexpansive semigroups.*

Proof. Assume that D has the uniqueness fixed point property for nonexpansive mapping. Let $S = \{\phi_t : t \geq 0\}$ be a nonexpansive semigroup on D . Let us prove that $\mathcal{F}(S)$ is nonempty. But, $\mathcal{F}(\phi_\lambda)$ and $\mathcal{F}(\phi_1)$ are nonempty for all $\lambda \geq 0$. In particular, if $\lambda_0 \in (0, 1) \setminus \mathbb{Q}$, then from Theorem 3.1 we have $\mathcal{F}(S) = \mathcal{F}(\phi_{\lambda_0}) \cap \mathcal{F}(\phi_1)$.

Now, if $z_0 \in \mathcal{F}(\phi_{\lambda_0})$, then $w_0 = \phi_1(z_0) \in \mathcal{F}(\phi_{\lambda_0})$, because of the fact that

$$\phi_{\lambda_0}(w_0) = \phi_{\lambda_0}(\phi_1(z_0)) = \phi_{\lambda_0+1}(z_0) = \phi_1(\phi_{\lambda_0}(z_0)) = \phi_1(z_0) = w_0.$$

The uniqueness of the fixed point of ϕ_{λ_0} implies that $z_0 = w_0 = \phi_1(z_0)$. So, $z_0 \in \mathcal{F}(\phi_1)$. Therefore, $z_0 \in \mathcal{F}(S)$, which means that D has the fixed point property for ρ -nonexpansive semigroups.

The uniqueness fixed point property for ρ -nonexpansive semigroups of D follows since $\mathcal{F}(S)$ has a unique element. \checkmark

Corollary 3.8. *Let D be a closed convex subset of the metric space (\mathbb{D}, ρ) . Then, the following two propositions are equivalent:*

1. *The set D has the uniqueness fixed point property for ρ -nonexpansive mappings.*
2. *The set D has the uniqueness fixed point property for ρ -nonexpansive semigroups.*

Acknowledgment

The author is thankful to the referees for their valuable suggestions. Also, the author acknowledges support from CONACyT grant 106923.

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