

Weak-type (1,1) bounds for a class of operators with discrete kernel

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Abstract. In this paper we investigate the weak continuity of a certain class of operators with kernel defined on $\mathbb{Z} \times \mathbb{Z}$. We prove some results on the weak boundedness of discrete analogues of Calderón-Zygmund operators. The considered operators arise from the study of discrete pseudo-differential operators and discrete analogues of singular integral operators.

Keywords: L^p spaces, discrete operator, pseudo-differential operator, Calderón-Zygmund decomposition.

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Cotas del tipo débil (1,1) para una clase de operadores con núcleo discreto

Resumen. En este trabajo se investigará el tipo débil (1,1) de una cierta clase de operadores con núcleo definido sobre $\mathbb{Z} \times \mathbb{Z}$. Se estudiará la continuidad débil de operadores que son análogos discretos de los ahora conocidos, operadores singulares integrales de Calderón-Zygmund. Los operadores considerados surgen desde el estudio de operadores pseudo diferenciales de tipo discreto y versiones discretas de integrales singulares.

Palabras clave: Espacios L^p , operador discreto, operador pseudo diferencial, descomposición de Calderón-Zygmund.

1. Introduction

As it is well known, Marcel Riesz [14] in 1928 proved the $L^p(\mathbb{R})$ -boundedness of the Hilbert transform

$$Hf(x) = p.v. \int_{\mathbb{R}} \frac{f(x-t)}{t} dt. \quad (1)$$

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As a consequence of this result, M. Riesz further remarked that the L^p -boundedness of the Hilbert transform implies the boundedness of its discrete analogue

$$Hf(n) = \sum_{p \in \mathbb{Z}, p \neq 0} \frac{f(n-p)}{p}. \quad (2)$$

This observation of M. Riesz on the L^p -continuity of the discrete operator (2) was extended in 1952 by Alberto Calderón and Antoni Zygmund [2] to a more general class of singular integral operators. In this paper, Calderón and Zygmund observed that the L^p -boundedness of the singular integral operator

$$Tf(x) = \int_{\mathbb{R}^n} \mathfrak{K}(x-y)f(y)dy, \quad (3)$$

where \mathfrak{K} belongs to the broad class of singular kernel, implies the $L^p(\mathbb{Z}^n)$ -boundedness of its discrete counterpart

$$Tf(n) = \sum_{m \in \mathbb{Z}^n, m \neq n} \mathfrak{K}(n,m)f(m). \quad (4)$$

Kernels associated to integral singular operators (3) are now known as Calderón-Zygmund kernels. It is important to mention that the progress in the last years regarding discrete analogues of operators in harmonic analysis is related with Calderón-Zygmund analogues [5], [7], [9], discrete maximal operators and related problems with number theory (see [1], [4], [8]), translation invariant fractional integral operators, translation invariant singular Radon transforms, quasi-translation invariant operators, spherical averages (see, L. Pierce [13], E. Stein., S. Wainger [17], [18], [19]) and discrete rough maximal functions (M. Mirek [11], R. Urban, J. Zienkiewicz [20]). As done by K. J. Hughes [8] (and references therein) there is a nice connection of discrete operators with crucial problems in number theory (for example, the Waring's problem).

In this work we consider the classic problem of the weak continuity of discrete operators defined by the equation (4) by following Calderón-Zygmund's approach, but we deduce the weak continuity from discrete kernels imposing discrete conditions, instead of using any condition on Calderón-Zygmund's singular kernels as in the continuous case (or as in the Riesz's observation). On the other hand, pseudo-differential operators on \mathbb{Z} (see S. Molahajloo [12], M. W. Wong [21]) are operators with (non-singular) kernels defined on $\mathbb{Z} \times \mathbb{Z}$. In [12], [15] the authors have studied the L^p -continuity, L^p -compactness and almost diagonalization of these objects. Particularly, C. Rodriguez in [15] proved that sublinear operators on \mathbb{Z}^n of weak type (p, q) with $p > q \geq 1$ are bounded on $L^p(\mathbb{Z}^n)$.

The aim of this paper is to deduce the *weak*(1,1) continuity of some discrete operators with kernels defined on $\mathbb{Z} \times \mathbb{Z}$ in the spirit of the results proved by M. Riesz, A. Calderón, and A. Zygmund [7], [14], [13] on discrete analogues of integral singular operators. The operators considered also include pseudo-differential operators on \mathbb{Z} treated recently in S. Molahajloo [12] (see also [3] and [15]). The main results in this paper are the following:

Theorem I. Let T be a discrete operator with kernel. If T is strong (p, r) for $1 < p, r < \infty$, then T is strong $(1, r)$, i.e, T is a bounded operator from $L^1(\mathbb{Z})$ into $L^r(\mathbb{Z})$. Moreover, if $\varepsilon > 0$, $1 < p < \infty$ and $T : L^p(\mathbb{Z}^n) \rightarrow L^{1+\varepsilon}(\mathbb{Z}^n)$ is a bounded discrete operator (defined as in (4)), then T is of weak type $(1, 1 + \varepsilon)$.

Theorem II. Let T be a discrete operator (see (4)). If T is bounded from $L^p(\mathbb{Z})$ into $L^p(\mathbb{Z})$ for some $1 < p < \infty$, such that

$$C = \sup_{k, h \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |m-h| > 2|k-h|} |\mathfrak{K}(m, k) - \mathfrak{K}(m, h)| < \infty, \quad (5)$$

then T is of weak type (1,1), i.e, for any $\lambda > 0$,

$$\mu(\{m \in \mathbb{Z} : |Tf(m)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{Z})}, \quad (6)$$

where μ is the counting measure on \mathbb{Z} . As a consequence of the previous theorem, we obtain the following results:

Theorem II.A. Let T be a discrete operator (see (4)). If T is bounded from $L^p(\mathbb{Z})$ into $L^p(\mathbb{Z})$ for some $1 < p < \infty$, such that

$$\sup_{k, h \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |m-h| > 2|k-h|} |\mathfrak{K}(m, k) - \mathfrak{K}(m, h)| \leq C < \infty, \quad (7)$$

then T is bounded from $L^r(\mathbb{Z})$ into $L^r(\mathbb{Z})$ for all $1 < r \leq p$.

Theorem II.B. Let T be the discrete operator defined by

$$Tf(n) = \sum_{m \in \mathbb{Z}, m \neq n} \mathfrak{K}(n-m)f(m), \quad (8)$$

satisfying the following conditions:

$$|\widehat{\mathfrak{K}}(\xi)| \leq A, \quad (9)$$

$$C = \sup_{k, h \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |m-h| > 2|k-h|} |\mathfrak{K}(k-m) - \mathfrak{K}(h-m)| < \infty; \quad (10)$$

then T is bounded on $L^p(\mathbb{Z})$, $1 < p < \infty$ and *weak*(1,1).

The hypothesis (10) in Theorem II.B. is closely related with a discrete version of (12) (known as Hörmander condition) which is imposed in the following result due to A. Calderón and A. Zygmund (see [2], and pg. 23 in L. Pierce [13]).

Theorem 1.1. Let \mathfrak{K} be a tempered distribution in \mathbb{R}^n which coincides with a locally integrable function on \mathbb{R}^n and such that

$$|\widehat{\mathfrak{K}}(\xi)| \leq A, \quad (11)$$

$$\sup_{y \in \mathbb{R}^n} \int_{|x| > 2|y|} |\mathfrak{K}(x-y) - \mathfrak{K}(x)| dx \leq B. \quad (12)$$

Then the operator T (as in (3)) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ and *weak*(1,1). Moreover, the discrete counterpart of T is bounded on $L^p(\mathbb{Z}^n)$.

This paper is organized as follow: In section 2 we establish some preliminaries on harmonic analysis necessary to study the weak-boundedness of discrete operators. In section 3 we prove our main result about *weak*(1,1)-boundedness of discrete operators and we establish some consequences of this fact. Finally, in section 4, we want to identify the (L^p, L^q) -boundedness as a sufficient condition of the *weak*(1, q)-boundedness, in counterpart with the above result by C. Rodriguez.

2. Preliminaries

In this section we introduce the necessary background in harmonic analysis used in the remainder of this paper. In particular, we discuss the Marcinkiewicz interpolation theorem and the weak continuity of operators on Banach spaces. We first define the Fourier transform of certain discrete functions. Let $a \in L^2(\mathbb{Z})$. The Fourier transform $\widehat{a}(\theta)$ of a is the function on the unit circle \mathbb{S}^1 defined by

$$\widehat{a}(\theta) = \sum_{n \in \mathbb{Z}} e^{-in\theta} a(n). \quad (13)$$

The Fourier inversion formula for Fourier series gives

$$a(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \widehat{a}(\theta) d\theta. \quad (14)$$

The Plancherel formula for Fourier series gives

$$\sum_{m \in \mathbb{Z}} |a(m)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{a}(\theta)|^2 d\theta. \quad (15)$$

We now formulate some definitions on weak and strong continuity. Let (X, μ) and (Y, ρ) be measure spaces, and T be an operator from $L^p(X, \mu)$ into the space of measurable functions from Y to \mathbb{C} . We say that T is of weak type (p, q) , $1 \leq q < \infty$, if

$$\rho(\{y \in Y : |Tf(y)| > \lambda\}) \leq \left(\frac{C \|f\|_{L^p}}{\lambda} \right)^q. \quad (16)$$

We say that T is of strong type (p, q) if it is bounded from $L^p(X, \mu)$ to $L^q(Y, \rho)$. In this case, we denote the operator norm by $\|T\|_{(X, Y)}$. When $(X, \mu) = (Y, \rho)$ and T is the identity operator, the *weak* (p, p) inequality is the classical Chebyshev inequality. A remarkable result due to J. Marcinkiewicz and A. Zygmund on weak continuity is the following (see [6], [10], [22]):

Theorem 2.1. (*Marcinkiewicz interpolation*). *Let (X, μ) and (Y, ρ) be measure spaces, $1 \leq p_0 < p_1 \leq \infty$ and T be a sublinear operator from $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ to the measurable functions on Y that is weak (p_0, p_0) and weak (p_1, p_1) . Then T is strong (p, p) for $p_0 < p < p_1$.*

Remark 2.2. The interpolation of Marcinkiewicz Theorem was announced by J. Marcinkiewicz in [10]. However, he died in World War II and a complete proof was finally given by A. Zygmund (see [22]).

3. Weak(1,1)-continuity

As has been pointed out in [8], during the past score of years there has been renewed interest in the area of discrete analogues in harmonic analysis. This began with an

observation of M. Riesz in his work on the Hilbert transform in 1928 that was carried over in the work of A. Calderón and A. Zygmund on singular integrals in 1952. In this section we prove our result on weak(1, 1)-continuity in the spirit of these results. The following lemma will be applied to analyze local properties on functions in $L^1(\mathbb{Z})$. Moreover, Lemma 3.1 is a discrete version of a classical by A. Calderón and A. Zygmund (see [6]).

Lemma 3.1. *Let $f \in L^1(\mathbb{Z})$ be a positive function and $\lambda > 0$. Then, there exists $v, w_k \in L^1(\mathbb{Z})$ such that*

1. $f = v + \sum_{k \in \mathbb{Z}} w_k$; $\text{supp}(w_k) \subset Q_k$, $\|v\|_{L^1(\mathbb{Z})} + \sum_k \|w_k\|_{L^1(\mathbb{Z})} \leq 3\|f\|_{L^1(\mathbb{Z})}$.
2. $|v(x)| \leq 2\lambda$.
3. $\sum_k \mu(Q_k) \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{Z})}$.
4. $\sum_{m \in Q_k} w_k(m) = 0$.
5. For every k we can assume that Q_k is a dyadic cube (see p. 33 in [6]).

Proof. This proof is (essentially) a repetition of the proof of Theorem 2.11 in [6]. \square

We now formulate our main result and some consequences. We will use above definitions and discrete decomposition of functions in $L^1(\mathbb{Z})$.

Theorem 3.2. *Let T be a discrete operator (see (4)). If T is bounded from $L^p(\mathbb{Z})$ into $L^p(\mathbb{Z})$ for some $1 < p < \infty$, such that*

$$C = \sup_{k, h \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |m-h| > 2|k-h|} |\mathfrak{K}(m, k) - \mathfrak{K}(m, h)| < \infty, \quad (17)$$

then T is of weak type (1,1).

Proof. Fix $f \in L^1(\mathbb{Z})$ and $\lambda > 0$. We now decompose f as the sum of two functions v and $w = \sum w_k$ as in Lemma 3.1. Since $Tf = Tv + Tw$, we get

$$\begin{aligned} & \mu(\{x \in \mathbb{Z} : |Tf(x)| > \lambda\}) \\ & \leq \mu(\{x \in \mathbb{Z} : |Tv(x)| > \frac{\lambda}{2}\}) + \mu(\{x \in \mathbb{Z} : |Tw(x)| > \frac{\lambda}{2}\}) \\ & = I + II, \end{aligned}$$

where

$$I = \mu(\{x \in \mathbb{Z} : |Tv(x)| > \frac{\lambda}{2}\}).$$

For the estimate of I , we have

$$\begin{aligned} I &\leq \mu(\{x \in \mathbb{Z} : |Tv(x)|^p > \frac{\lambda^p}{2^p}\}) \\ &\leq \frac{2^p}{\lambda^p} \sum_{x \in \mathbb{Z}} |Tv(x)|^p \\ &\leq \frac{2^p}{\lambda^p} \|T\|_{B(L^p)} \sum_{x \in \mathbb{Z}} |v(x)|^p \\ &\leq \frac{2^p \lambda^{p-1}}{\lambda^p} \|T\|_{B(L^p)} \sum_{x \in \mathbb{Z}} |v(x)|. \end{aligned}$$

Hence,

$$I \leq \frac{2^p}{\lambda} \|T\|_{B(L^p)} \|v\|_{L^1(\mathbb{Z})}.$$

Let us consider (for every k) the cube Q_k^* containing Q_k with the same center as Q_k and whose side are 2 times longer. Let $\Omega^* = \cup_k Q_k^*$. Then,

$$\begin{aligned} II &\leq \mu(\Omega^*) + \mu(\{x \in \mathbb{Z} - \Omega^* : |Tw(x)| > \frac{\lambda}{2}\}) \\ &\leq \sum_k \mu(Q_k^*) + \mu(\{x \in \mathbb{Z} - \Omega^* : |Tw(x)| > \frac{\lambda}{2}\}) \\ &\leq 4 \sum_k \mu(Q_k) + \mu(\{x \in \mathbb{Z} - \Omega^* : |Tw(x)| > \frac{\lambda}{2}\}) \\ &\leq \frac{4}{\lambda} \|f\|_{L^1(\mathbb{Z})} + \frac{2}{\lambda} \sum_{m \in \mathbb{Z} - \Omega^*} |Tw(m)|. \end{aligned}$$

Now, $|Tw(m)| = |\sum_k Tw_k(m)|$. Hence, to complete the proof of the weak(1,1) inequality it will suffice to show that

$$\sum_{m \in \mathbb{Z} - \Omega^*} |\sum_k Tw_k(m)| \leq C \|f\|_{L^1(\mathbb{Z})}.$$

Let us choose a sequence $\{c_k\}_k$ satisfying $c_k \in Q_k$ for every $k \in \mathbb{Z}$. Then, we can write

$$\begin{aligned} \sum_k Tw_k(m) &= \sum_k \sum_{p \in Q_k} \mathfrak{K}(m, p) w_k(p) \\ &= \sum_k \sum_{p \in Q_k} \mathfrak{K}(m, p) w_k(p) - \mathfrak{K}(m, c_k) \sum_{p \in Q_k} w_k(p). \\ &= \sum_k \sum_{p \in Q_k} (\mathfrak{K}(m, p) - \mathfrak{K}(m, c_k)) w_k(p). \end{aligned}$$

Hence, we get

$$\begin{aligned} \sum_{m \in \mathbb{Z} - \Omega^*} \left| \sum_k T w_k(m) \right| &= \sum_{m \in \mathbb{Z} - \Omega^*} \left| \sum_k \sum_{p \in Q_k} (\mathfrak{K}(m, p) - \mathfrak{K}(m, c_k)) w_k(p) \right| \\ &\leq \sum_k \sum_p |w_k(p)| \sum_{m \in \mathbb{Z} - \Omega^*} |(\mathfrak{K}(m, p) - \mathfrak{K}(m, c_k))| \\ &\leq C \sum_k \sum_p |w_k(p)| \\ &\leq 3C \|f\|_{L^1(\mathbb{Z})}. \end{aligned} \quad \square$$

Theorem 3.3. Let T be a discrete operator (see (4)). If T is bounded from $L^p(\mathbb{Z})$ into $L^p(\mathbb{Z})$ for some $1 < p < \infty$, such that

$$C = \sup_{p, h \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |m-h| > 2|p-h|} |\mathfrak{K}(m, k) - \mathfrak{K}(m, h)| < \infty, \quad (18)$$

then, T is bounded from $L^r(\mathbb{Z})$ into $L^r(\mathbb{Z})$ for all $1 < r \leq p$.

Proof. We note that T is weak(1,1) and strong (p, p) . Therefore, T is weak(r, r) with $r = 1, p$. So, Marcinkiewicz interpolation (Theorem 2.1) implies the theorem. \square

Theorem 3.4. Let T be the discrete operator defined by

$$Tf(n) = \sum_{m \in \mathbb{Z}, m \neq n} \mathfrak{K}(n-m) f(m), \quad (19)$$

satisfying the following conditions:

$$|\widehat{\mathfrak{K}}(\xi)| \leq A, \quad (20)$$

$$\sup_{k, h \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, |m-h| > 2|k-h|} |\mathfrak{K}(k-m) - \mathfrak{K}(h-m)| \leq C < \infty; \quad (21)$$

then T is bounded on $L^p(\mathbb{Z})$, $1 < p < \infty$ and weak(1,1).

Proof. Weak(1,1) continuity follows from Theorem I. L^p -boundedness, $1 < p < 2$, follows by interpolation, and for $p > 2$ it follows by duality since the adjoint operator T^* has kernel $\mathfrak{K}^*(x) = \overline{\mathfrak{K}(-x)}$, which also satisfies (20) and (21). \square

In Theorem 3.4 we assumed that the operator was bounded on $L^2(\mathbb{Z})$ (via the hypothesis that $\widehat{\mathfrak{K}} \in L^\infty(\mathbb{Z})$).

Remark 3.5. Discrete pseudo-differential operators (i.e pseudo-differential operators defined on \mathbb{Z}) are discrete operators $T_\sigma = T + I$, where I is the identity operator and T has kernel defined by

$$\mathfrak{K}(n, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)\theta} \sigma(n, \theta) d\theta. \quad (22)$$

The discrete class $S_{1,0,u}^m(\mathbb{Z} \times \mathbb{S}^1)$ consists of those functions $\sigma(n, \theta)$ which satisfy discrete symbols inequalities

$$\forall r > 0, 0 \leq r \leq u, \exists C_r > 0, |\partial_\theta^r \sigma(n, \theta)| \leq C_r \langle n \rangle^{m-r}, \tag{23}$$

where $\langle n \rangle = (1 + n^2)^{\frac{1}{2}}$. Let $\sigma \in S_{1,0,2}^0(\mathbb{Z} \times \mathbb{S}^1)$. Integration by parts allows us to obtain the estimate

$$|\mathfrak{K}(n, k)| \leq C_q \langle n - k \rangle^{-2q}.$$

By Theorem II, $T = T_\sigma$ is of weak type (1,1).

4. Weak continuity and some properties

In this section we prove that (L^p, L^q) -continuity is a sufficient condition of weak continuity. We begin with some preparatory results.

Proposition 4.1. *Let T be a discrete operator. If T is strong (p, r) , then*

$$\left(\sum_{n \neq p} |\mathfrak{K}(n, p)|^r \right)^{\frac{1}{r}} \leq \|T\|_{(L^p, L^q)}. \tag{24}$$

Proof. Fix $k \in \mathbb{Z}$ and define $f_k \in L^p(\mathbb{Z})$ by $f_k(n) = 1$ if $k = n$ and $f_k(n) = 0$ if $k \neq n$. Then $Tf_k(n) = \mathfrak{K}(n, k)$ and

$$\|Tf_k\|_{L^r(\mathbb{Z})} = \left(\sum_{n \neq k} |\mathfrak{K}(n, k)|^r \right)^{\frac{1}{r}} \leq \|T\|_{(L^p, L^q)}. \quad \square$$

Theorem 4.2. *Let T be a discrete operator with kernel. If T is strong (p, r) for $1 < p, r < \infty$, then T is strong $(1, r)$, i.e., T is a bounded operator from $L^1(\mathbb{Z})$ into $L^r(\mathbb{Z})$.*

Proof. By Proposition 4.1,

$$\left(\sum_{n \neq k} |\mathfrak{K}(n, k)|^r \right)^{\frac{1}{r}} \leq \|T\|_{(L^p, L^q)}.$$

Now, If $f \in L^1(\mathbb{Z})$, by the Minkowski's integral inequality (discrete version) we obtain

$$\begin{aligned} \left(\sum_{m \in \mathbb{Z}} |Tf(m)|^r \right)^{\frac{1}{r}} &\leq \left(\sum_m \left| \sum_{p \neq m} |\mathfrak{K}(m, p) f(p)| \right|^r \right)^{\frac{1}{r}} \\ &\leq \sum_p \left(\sum_{m \neq p} |\mathfrak{K}(m, p) f(p)|^r \right)^{\frac{1}{r}} \\ &\leq \|f\|_{L^1(\mathbb{Z})} \|T\|_{(L^p, L^r)}. \end{aligned} \quad \square$$

Theorem 4.3. *Let $\varepsilon > 0$ and $1 < p < \infty$. If $T : L^p(\mathbb{Z}^n) \rightarrow L^{1+\varepsilon}(\mathbb{Z}^n)$ is a discrete operator as in (4), then T is of the weak type $(1, 1 + \varepsilon)$.*

Proof. Let μ be the counting measure on \mathbb{Z} . By the Theorem 4.2 we have,

$$\begin{aligned} \mu\{x \in \mathbb{Z} : |Tf(x)| > \lambda\} &\leq \mu\{x \in \mathbb{Z} : |Tf(x)|^{1+\varepsilon} > \lambda^{1+\varepsilon}\} \\ &\leq \frac{1}{\lambda^{1+\varepsilon}} \sum_{x \in \mathbb{Z}} |Tf(x)|^{1+\varepsilon} \\ &\leq \left(\frac{\|T\|_{(L^p, L^{1+\varepsilon})} \|f\|_{L^1}}{\lambda} \right)^{1+\varepsilon}. \end{aligned}$$

The above inequality implies the weak $(1, 1 + \varepsilon)$ -boundedness. \square

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