Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 22, Nos. 1 y 2, 2004, pág. 1–6

Minimal hypersurfaces in $\mathbb{R}^n$ as regular values of a function*

Óscar Mario Perdomo**

Abstract. In this paper we prove that if $M = f^{-1}(0)$ is a minimal hypersurface of $\mathbb{R}^n$, where $f : V \subset \mathbb{R}^n \to \mathbb{R}$ is a smooth function defined on a open set V, then f must satisfy the equation $|\nabla f|^2 \Delta f = \frac{1}{2}\langle \nabla|\nabla f|^2, \nabla f\rangle$ for every $x \in M$. We will also prove that if M is the zero level set of a homogeneous 2 polynomial, then M must be a Clifford minimal hypersurface.

1. Introduction and preliminaries

In this paper we will consider hypersurfaces $M \subset \mathbb{R}^n$ that are level sets of functions, i.e. we will assume that $M = \{x \in V : f(x) = 0\}$ where $f : V \to \mathbb{R}$ is a smooth function defined in an open set V of $\mathbb{R}^n$ and $|\nabla f(x)| \neq 0$ for all $x \in M$. For these hypersurfaces, we have that the Gauss map can be written as $\nu(x) = |\nabla f(x)|^{-1}\nabla f(x)$ for all $x \in M$. Clearly, the tangent space of M at a point x is the space of vectors $v \in \mathbb{R}^n$ such that $\langle v, \nabla f(x)\rangle = 0$. Notice that the mean curvature of M at x is given by

$$-\sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i\rangle, \tag{1}$$

where $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal bases of the vector space T_xM.

It is worth mentioning some elementary facts about real value functions on $\mathbb{R}^n$ that will be used later on.

Lemma 1.1. *If $f : V \to \mathbb{R}$ is a smooth function defined in an open set V of $\mathbb{R}^n$, then*

Keywords: minimal hypersurfaces in $\mathbb{R}^n$, Clifford minimal hypersurface.
MSC2000: Primary: 49Q05. Seconday: 53A10.
* The author wishes to thank Colciencias and Universidad del Valle for their financial support.
** Departamento de Matemáticas, Universidad del Valle, Cali, Colombia,
e-mail: `osperdom@univalle.edu.co`.

(a)
$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^{n} \langle \operatorname{Hess}(f)_x(w_j), w_j \rangle,$$

where $\{w_1, \ldots, w_n\}$ *is any orthonormal basis of* $\mathbb{R}^n$ *and* $\operatorname{Hess}(f)$ *is the* $n \times n$ *Hessian matrix of* f.

(b) $\langle \operatorname{Hess}(f) \nabla f, \nabla f \rangle = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$.

(c) $\dfrac{d \nabla f(\alpha(t))}{dt} = \operatorname{Hess}(f)_{\alpha(t)} \alpha'(t)$ *for any smooth curve* $\alpha : (a, b) \to V$.

Proof. **(a)** holds true because $\Delta f(x)$ is the trace of the matrix $\operatorname{Hess}(f)_x$, and the trace of a matrix is invariant under change of basis.
(b) Is a direct computation and **(c)** follows from the chain rule. ☑

2. Main result

In this section we will state and prove one of the main results of this paper.

Theorem 2.1. *Let* $M = \{x \in V : f(x) = 0\}$ *where* $f : V \to \mathbb{R}$ *is a smooth function defined in an open set* V *of* $\mathbb{R}^n$ *with* $|\nabla f(x)| \neq 0$ *for all* $x \in M$. M *is minimal if and only if* $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$ *for every* $x \in M$.

Proof. We are going to compute the mean curvature H of M in terms of the function f and its partial derivatives. Let us start computing $\langle d\nu_x(v), v \rangle$ for any $v \in T_x M$. Let us take a smooth curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = x$ and $\alpha'(0) = v$. We have that

$$\begin{aligned}
\langle d\nu_x(v), v \rangle &= \left\langle \frac{d\nu(\alpha(t))}{dt}\Big|_{t=0}, v \right\rangle \\
&= \left\langle \frac{d|\nabla f(\alpha(t))|^{-1} \nabla f(\alpha(t))}{dt}\Big|_{t=0}, v \right\rangle \\
&= \frac{d|\nabla f(\alpha(t))|^{-1}}{dt}\Big|_{t=0} \langle \nabla f(x), v \rangle + |\nabla f(x)|^{-1} \left\langle \frac{d\nabla f(\alpha(t))}{dt}\Big|_{t=0}, v \right\rangle \\
&= 0 + |\nabla f(x)|^{-1} \langle \operatorname{Hess}(f)_x v, v \rangle \;=\; |\nabla f(x)|^{-1} \langle \operatorname{Hess}(f)_x v, v \rangle .
\end{aligned}$$

Now, if $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal bases of $T_x M$, then by the equation (1) in section 1, we get that

$$\begin{aligned} H &= -\sum_{i=1}^{n-1}\langle d\nu_x(v_i), v_i\rangle = -|\nabla f(x)|^{-1}\langle \operatorname{Hess}(f)_x v_i, v_i\rangle \\ &= |\nabla f(x)|^{-1}(-\Delta f(x) + \langle \operatorname{Hess}(f)_x |\nabla f(x)|^{-1}\nabla f(x), |\nabla f(x)|^{-1}\nabla f(x)\rangle) \\ &= |\nabla f(x)|^{-1}(-\Delta f(x) + |\nabla f(x)|^{-2}\frac{1}{2}\langle \nabla|\nabla f|^2, \nabla f\rangle). \end{aligned}$$

Therefore we get that M is minimal, if and only if, for every $x \in M$, we have that $|\nabla f(x)|^2\Delta f(x) = \frac{1}{2}\langle\nabla|\nabla f|^2, \nabla f\rangle$. ☑

Example 2.2 (Clifford minimal cones). *Let k and l be two positive integers such that $k + l = n - 2$, and let $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be the function given by*

$$f(x) = f(x_1, \dots, x_n) = k(x_1^2 + \cdots + x_{l+1}^2) - l(x_{1+2}^2 + \cdots + x_n^2).$$

Let us check that $M_{lk} = f^{-1}(0)$ is a minimal hypersurface. A direct computation shows that

$$\begin{aligned} \nabla f(x) =& 2(kx_1, \dots, kx_{l+1}, -lx_{l+2}, \dots, -lx_n), \\ |\nabla f(x)|^2 =& 4k^2(x_1^2 + \cdots + x_{l+1}^2) + 4l^2(x_{l+2}^2 + \cdots + x_n^2), \\ \nabla|\nabla f(x)|^2 =& 8(k^2x_1, \dots, k^2x_{l+1}, l^2x_{l+2}, \dots, ^2 lx_n), \\ \frac{1}{2}\langle\nabla|\nabla f|^2, \nabla f\rangle =& 8k^3(x_1^2 + \cdots + x_{l+1}^2) - 8l^3(x_{l+2}^2 + \cdots + lx_n^2), \\ \Delta f(x) =& 2k(l+1) - 2l(k+1) = 2k(k-l). \end{aligned}$$

Therefore, we have that if $x \in M$, i.e. if

$$k(x_1^2 + \cdots + x_{l+1}^2) = l(x_{1+2}^2 + \cdots + x_n^2),$$

then,

$$\begin{aligned} |\nabla f(x)|^2\Delta f =& 2(k-l)(4k^2 - 4lk)(x_1^2 + \cdots + x_{l+1}^2) \\ =& 8k(k^2 - l^2)(x_1^2 + \cdots + x_{l+1}^2) \\ =& 8k^3x_1^2 + \cdots + x_{l+1}^2) - 8l^3(x_{1+2}^2 + \cdots + x_n^2) \\ =& \frac{1}{2}\langle\nabla|\nabla f|^2, \nabla f\rangle. \end{aligned}$$

We will say that M is a Clifford minimal cone if $M = M_{kl}$ up to a rigid motion in $\mathbb{R}^n$.

Let us assume now that (N, g) is a riemannian n dimensional manifold, V is an open subset of N and $f : V \to \mathbb{R}$ is a smooth function such that $M = f^{-1}(0)$ is a hypersurface of N, i.e. 0 is a regular value of f. In this case we will denote by $\operatorname{Hess}(f)_x : T_xM \times T_xM \to \mathbb{R}$ the bilinear form defined by $\operatorname{Hess}(f)_x(v, w) = \langle D_v\nabla f, w\rangle$, where D is the Levi-Civita connection on N.

The exact same proof of the previous theorem gives us the following result.

Theorem 2.3. *Let N be a riemannian manifold and let $M = \{x \in V : f(x) = 0\}$, where $f : V \to \mathbb{R}$ is a smooth function defined in an open set V of N with $|\nabla f(x)| \neq 0$ for all $x \in M$. M is minimal if and only if $|\nabla f|^2 \Delta f = \frac{1}{2}\langle \nabla |\nabla f|^2, \nabla f\rangle$ for every $x \in M$.*

3. Minimal hypersurfaces of $\mathbb{R}^n$ given by quadratic form

In this section we characterize the minimal Clifford cones as the only hypersurfaces that are level sets of quadratic forms. More precisely we prove,

Theorem 3.1. *Let $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be the function defined by $f(x) = \langle Bx, x\rangle$, where B is a $n \times n$ symmetric matrix and take $M = \{x \in \mathbb{R}^n : \langle Bx, x\rangle = 0\} \setminus \{\mathbf{0}\} = f^{-1}(0)$. The value 0 is a regular value of f and M is a minimal hypersurface if and only if M is a Clifford minimal cone.*

Before we prove this theorem we will need the following lemma.

Lemma 3.2. *Let B be an invertible symmetric matrix with both, positive and negative eigenvalues. If C is a matrix that commutes with B such that $\langle Cx, x\rangle = 0$ whenever $\langle Bx, x\rangle = 0$, then $C = \lambda B$ for some real number λ.*

Proof. Since B and C commutes, after an orthogonal change of coordinates, we can assume that

$$C = \begin{pmatrix} c_1 & & & & & \\ & \ddots & & & & \\ & & c_r & & & \\ & & & c_{r+1} & & \\ & & & & \ddots & \\ & & & & & c_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & & & & & \\ & \ddots & & & & \\ & & b_r & & & \\ & & & b_{r+1} & & \\ & & & & \ddots & \\ & & & & & b_n \end{pmatrix},$$

where $0 < r < n$, $b_1, \ldots, b_r$ are positive real numbers and $b_{r+1}, \ldots, b_n$ are negative real numbers. Let us denote $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ the canonical bases for $\mathbb{R}^n$, and for $1 \leq i \leq r$ and $r < j \leq n$ we will denote $x_{ij} = \sqrt{-b_j}e_i + \sqrt{b_i}e_j$. Note that $\langle Bx_{ij}, x_{ij}\rangle = 0$, therefore $\langle Cx_{ij}, x_{ij}\rangle = 0$, i.e.,

$$-b_j c_i + b_i c_j = 0;$$

from the equation above we get that $c_i = \frac{c_n}{b_n} b_i$ for $1 \leq i \leq r$, and for $r < j \leq n$ we have, using the expression for c_1,

$$c_j = \frac{c_1}{b_1} b_j = \frac{c_n}{b_n} b_1 \frac{1}{b_1} b_j = \frac{c_n}{b_n} b_j.$$

Therefore $C = \frac{c_n}{b_n} B$, and this completes the proof. ☑

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. A Direct computation shows that $\nabla f(x) = 2Bx$. Since we are assuming that 0 is a regular value, then $\nabla f(x) \neq \mathbf{0}$ for all $x \in M$; in particular, $Bx_0 \neq \mathbf{0}$ for every $x_0 \neq \mathbf{0}$, because if $Bx_0 = \mathbf{0}$ for some $x_0 \neq \mathbf{0}$ we would have that $x_0 \in M$ and $\nabla f(x_0) = \mathbf{0}$. Therefore B is an invertible matrix. We also have that, since $M \neq \emptyset$, then B must have positive eigenvalues and negative eigenvalues. We have them

$$|\nabla f(x)|^2 = \langle 2Bx, 2Bx\rangle = 4\langle Bx, Bx\rangle = 4\langle B^2x, x\rangle.$$

Since B^2 is symmetric, then $\nabla|\nabla f|^2(x) = 8B^2x$. A direct computation shows that $\Delta f = 2\,\mathrm{trace}(B)$. Using Theorem 2.1, we have that M is minimal if and only if $|\nabla f|^2\Delta f = \frac{1}{2}\langle\nabla|\nabla f|^2, \nabla f\rangle$ for every $x \in M$, i.e. if, for every $x \neq \mathbf{0}$ such that $\langle Bx, x\rangle = 0$ we have that

$$4\langle B^2x, x\rangle(2\,\mathrm{trace}(B)) = \frac{1}{2}\langle 8B^2x, 2Bx\rangle = 8\langle B^3x, x\rangle.$$

In others words, if we define $C = \mathrm{trace}(B)B^2 - B^3$, we have that M is minimal if and only if $\langle Cx, x\rangle = 0$ for every x such that $\langle Bx, x\rangle = 0$. Using Lemma 3.2 we concluded that there exists a real number a such that $\mathrm{trace}(B)B^2 - B^3 = aB$. Since B is an invertible matrix, we get that B satisfies the polynomial equation

$$B^2 - \mathrm{trace}(B)B + aI = 0. \tag{2}$$

Therefore B can only have two eigenvalues. Since B has negative and positive eigenvalues, we can assume that the eigenvalues of B are $\lambda_1 > 0$ with multiplicity $r \geq 1$ and $\lambda_2 < 0$ with multiplicity $n - r \geq 1$. Note that $\mathrm{trace}(B) = r\lambda_1 + (n-r)\lambda_2$. The equation (2) is equivalent to the following system of equations for λ_1, λ_2 and a:

$$\begin{aligned} \lambda_1^2 - (r\lambda_1 + (n-r)\lambda_2)\lambda_1 + a =& (1-r)\lambda_1^2 - (n-r)\lambda_1\lambda_2 + a = 0, \\ \lambda_2^2 - (r\lambda_1 + (n-r)\lambda_2)\lambda_2 + a =& -(n-r-1)\lambda_2^2 - r\lambda_1\lambda_2 + a = 0; \end{aligned}$$

combining these two equations we get

$$(1-r)\lambda_1^2 - (n-2r)\lambda_1\lambda_2 + (n-r-1)\lambda_2^2. \tag{3}$$

From this equation we get that $r = 1$ or $r = n-1$ implies that $\lambda_1 = \lambda_2$ which is impossible because $\lambda_1\lambda_2 < 0$. Therefore $1 < r < n-1$. From equation (3) we get that $t = \frac{\lambda_2}{\lambda_1}$ satisfies the equation

$$(1-r) - (n-2r)t + (n-r-1)t^2.$$

Therefore, $t = 1$ or $t = \dfrac{r-1}{n-r-1}$; since we have that t must be negative, then t cannot be 1. Therefore, up to a constant we may take $\lambda_1 = n-r-1$ and $\lambda_2 = r-1$. I.e., up to a rigid motion $f(x) = \langle Bx, x\rangle$ must be a multiple of the function given in the example 2.2. This implies that M must be a Clifford minimal cone. ☑

Remark on the construction of minimal hypersurfaces using homogeneous polynomials of degree k: Let $f:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ be a homogeneous polynomial of degree k such that $f^{-1}(0)=M$ is not empty and such that for every $x\in M$, $\nabla f(x)\neq\mathbf{0}$. By theorem 2.1 we have that M is minimal if and only if

$$g(x)=|\nabla f|^2\Delta f-\frac{1}{2}\langle\nabla|\nabla f|^2,\nabla f\rangle=0 \tag{4}$$

for every x such that $f(x)=0$. Notice that the left hand side of the equation (4) is a homogeneous polynomial of degree $3k-4$; also notice that if $g(x)=h(x)f(x)$ for some homogeneous polynomial h of degree $2k-4$, then M will be minimal.

It is easy to prove the veracity of the following:

Conjecture. *Let $f:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ be a homogeneous polynomial of degree k such that $f^{-1}(0)=M$ is not empty and such that for every $x\in M$, $\nabla f(x)\neq\mathbf{0}$. If $g(x)$ is a polynomial of degree m with $m\geq k$ such that $g(x)=0$ for every $x\in M$, then there exists a homogeneous polynomial h of degree $m-k$ such that $g(x)=h(x)f(x)$.*

That conjeture implies the following result:

> *"Let $M=f^{-1}(0)\neq\emptyset$, where $f:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ is a homogeneous polynomial. M is minimal if and only if*
>
> $$|\nabla f|^2\Delta f-\frac{1}{2}\langle\nabla|\nabla f|^2,\nabla f\rangle=hf$$
>
> *for some homogeneous polynomial h".*

So far the only known examples of these minimal hypersurfaces are the isoparametric minimal hypersurfaces; the degree of f in these examples are $k=1,2,3,4,6$. Note that Theorem 2.1 proves the conjecture when $k=2$.

References

[1] Manfredo P. Do Carmo. *Riemannian Geometry*, Birkhäuser, 1992.

Óscar Mario Perdomo
Departamento de Matemáticas
Universidad del Valle
Cali, Colombia.
e-mail: osperdom@univalle.edu.co.