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Hodge operator and asymmetric fluid in unbounded domains

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Abstract. A system of equations modeling the stationary flow of an incompressible asymmetric fluid is studied for bounded domains of an arbitrary form. Based on the methods of Clifford analysis, we write the system of asymmetric fluid in the hypercomplex formulation and represent its solution in Clifford operator terms. We have significantly used Clifford algebra, and in particular the Hodge operator of the Clifford algebra to demonstrate the existence and uniqueness of the strong solution for arbitrary unbounded domains.

1. Introduction

In this work we consider a boundary value problem for a system of equations modeling the stationary flow of a incompressible asymmetric fluid, in which Navier-Stokes equations are combined with equations of angular velocity of rotation of the fluid particles. Based on the methods of Clifford analysis and following the work of P. Cereijeiras and U. Kähler [2], where they develop a Clifford operator calculus over unbounded domains, we write the system of asymmetric fluid in the hypercomplex formulation, then represent its solutions in term of Clifford operators and prove the convergence of the iterative method used for our problem. The main technical difference between the Navier-Stokes equations

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studied in [2] and the system of the asymmetric fluid studied in this work is the term $\operatorname{curl} w^*$, where w^* is the angular velocity of rotation of the fluid particles. To write this term in the hypergeometric formulation we use the Hodge star operator.

2. Hodge operator in the Clifford Algebra approach

Let V be a vector space over the real field $\mathbb{R}$ of finite dimension, i.e., $\dim V = n, n \in \mathbb{N}$. By V^* we denote the dual space of V.

We recall that the space of k-tensors (denoted $T_k(V^*)$) are the set of all k-linear mappings τ_k such that

$$\tau_k : V^* \times \cdots \times V^* \to \mathbb{R}$$

and a multitensor τ of order $m \in \mathbb{N}$ is an element of $T(V)$, where

$$T(V) \equiv \sum_{k=0}^{\infty} \oplus T_k(V^*),$$

of the form $\tau = \sum_{k=0}^{m} \oplus \tau_k$, with $\tau_k \in T_k(V^*)$, such that all the components $\tau_k \in T_k(V^*)$ of τ are null for $k > m$. $T(V)$ is called the space of multitensors.

The Clifford algebra $\mathcal{C}l(V,g)$ of a metric vector space (V,g) is defined as the quotient algebra

$$\mathcal{C}l(V,g) = \frac{T(V)}{J_g},$$

where $J_g \subset T(V)$ is the bilateral ideal of $T(V)$ generated by the elements of the form $u \otimes v + v \otimes u - 2g(u,v)$, with $u, v \in V \subset T(V)$. The elements of $\mathcal{C}l(V,g)$ are called *Clifford numbers*.

Let $\rho_g : T(V) \to \mathcal{C}l(V,g)$ be the natural projection of $T(V)$ onto the quotient algebra $\mathcal{C}l(V,g)$. Multiplication in $\mathcal{C}l(V,g)$ is called Clifford product and defined as

$$AB = \rho_g(A \otimes B),$$

for all $A, B \in \mathcal{C}l(V,g)$. In particular, for $u, v \in V \subset \mathcal{C}l(V,g)$, we have

$$u \otimes v = \frac{1}{2}(u \otimes v - v \otimes u) + g(u,v) + \frac{1}{2}(u \otimes v + v \otimes u) - g(u,v),$$

and then

$$\rho_g(u \otimes v) \equiv uv = \frac{1}{2}(u \otimes v - v \otimes u) + g(u,v) = u \wedge v + g(u,v).$$

From here we derive the standard relation characterizing the Clifford algebra $\mathcal{C}l(V, g)$,

$$uv + vu = 2g(u, v).$$

In that follows we take $V = \mathbb{R}^n$, and we denote by $\mathbb{R}^{p,q}$ $(n = p + q)$ the real vector space $\mathbb{R}^n$ endowed with a non-degenerated metric $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that, if $\{e_i\}$, $(i = 1, 2, \ldots, n)$ is an orthonormal basis of $\mathbb{R}^{p,q}$, we have

$$g(e_i, e_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1, \ldots, p, \\ -1, & i = j = p + 1, \ldots, p + q = n, \\ 0, & i \neq j. \end{cases}$$

The Clifford algebra $\mathcal{C}l(\mathbb{R}^{p,q}, g) = \mathbb{R}_{p,q} = Cl_{p,q}$, is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{e_i\}$, $(i = 1, 2, \ldots, n)$ such that $e_i^2 = g(e_i, e_j)$, $e_ie_j = -e_je_i$ $(i \neq j)$, and $e_A = e_1e_2 \cdots e_n \neq \pm 1$.

Therefore the universal Clifford algebra $Cl_{p,q}$ has the dimension 2^n. Henceforth, each element $a \in Cl_{p,q}$ shall be written in the form

$$a = \sum_A a_A e_A,$$

where the coefficients a_A are real numbers.

Now, we briefly describe Hodge star operator, which will be used throughout these article. The Hodge star operator (or Hodge dual) is the linear mapping $\star : \bigwedge^r V \to \bigwedge^{n-r} V$ such that

$$A \wedge \star B = (A \cdot B)\tau_g,$$

for every $A, B \in \bigwedge^r V^*$ and where τ_g is the volume element in $\bigwedge^n V^*$. The inverse $\star^{-1} : \bigwedge^{n-r} V^* \to \bigwedge^r V^*$ of the Hodge star operator is given by

$$\star^{-1} = (-1)^{r(n-r)} \operatorname{sgn}(g)\star,$$

where $\operatorname{sgn} g = \det g / |\det g|$ denotes the sign of the determinant of the matrix $(g_{ij} = g(e_i, e_j))$.

An important property of the Hodge star operator, which we will use in the course of the article, is

$$\star A_r = \widetilde{A}_r \lrcorner \tau_g = \widetilde{A}_r \tau_g, \tag{1}$$

for any $A_r \in \bigwedge^r V^*$.

Here $\bigwedge^r V^* \equiv Cl^r$ denotes the space of k-forms, but the same results are obtained for k-vectors; for details see [6].

Let $\Omega \subset \mathbb{R}^n$ and $\Gamma = \partial\Omega$. Then functions u defined in Ω with values in $Cl_{0,n}$ ($p = 0$ and $q = n$) are considered. These functions may be written as

$$u(x) = \sum_A e_A u_A(x), \qquad x \in \Omega.$$

Properties such as continuity, differentiability, integrability, and so on, which are imposed on u have to be possessed by all components $u_A(x)$. In this way, the usual Banach space of these functions are denoted by $\mathcal{C}^\alpha(\Omega, Cl_{0,n}), \mathcal{L}_q(\Omega, Cl_{0,n})$ and $\mathcal{W}_q^k(\Omega, Cl_{0,n})$ or in abbreviated form $\mathcal{C}^\alpha(\Omega), \mathcal{L}_q(\Omega)$ and $\mathcal{W}_q^k(\Omega)$.

Let us now introduce the Dirac operator as

$$D = \sum_{K=1}^{n} e_k \frac{\partial}{\partial x_k}.$$

It is easy to prove that $D^2 = -\Delta$, where Δ is the Laplacian.

We remind that the subspace of $Cl_{0,n}$ generated by the basic element e_A with equal length k is denoted by $Cl_{0,n}^k$. Its elements are called k-vectors. It follows that $Cl_{0,n}^1$ is isomorphic to $\mathbb{R}^n$ ($Cl_{0,n}^1 \approx \mathbb{R}^n$). In this sense, we can identify each vector $u(x) \in \mathbb{R}^n$ with

$$u(x) = u_1(x)e_1 + \cdots + u_n(x)e_n \in Cl_{0,n}^1 \hookrightarrow Cl_{0,n}.$$

Then we can calculate $Du(x)$ when $u(x) \in Cl_{0,3}^1 \hookrightarrow Cl_{0,3}$, i.e., if $u(x) = u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3$, so that

$$\begin{aligned}
Du(x) &= \sum_{k=1}^{3} e_k \frac{\partial u(x)}{\partial x_k} \\
&= e_1 \frac{\partial}{\partial x_1}(u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3) \\
&\quad + e_2 \frac{\partial}{\partial x_2}(u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3) \\
&\quad + e_3 \frac{\partial}{\partial x_3}(u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3).
\end{aligned}$$

We can compute that

$$\begin{aligned}
Du(x) = &-\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} + e_1 \wedge e_2 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \\
&+ e_1 \wedge e_3 \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right) + e_2 \wedge e_3 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \\
= &-\operatorname{div}(u^*(x)) + \star \operatorname{curl}(u^*(x)),
\end{aligned} \tag{2}$$

where $u^*(x) \in \mathbb{R}^3$. Thus, we have

$$\operatorname{curl}(u^*(x)) = \star^{-1} \operatorname{div}(u^*(x)) + \star^{-1} Du(x). \tag{3}$$

Now, we can demonstrate that $\star^{-1} Du(x) = D \star^{-1} u(x)$. From equation (3) we derive (we remind that $\star^{-1} = (-1)^{r(n-r)}\mathrm{sgn}(g)\star$),

$$\star^{-1} Du(x) = (-1)^{r(n-r)}\mathrm{sgn}(g)(-\operatorname{div}(u^*(x)))e_1 \wedge e_2 \wedge e_3 + \operatorname{curl}(u^*(x)). \tag{4}$$

On the other hand,

$$D \star^{-1} u(x) = (-1)^{r(n-r)}\mathrm{sgn}(g) D \star u(x) \tag{5}$$

and

$$\star u(x) = (e_3 \wedge e_2)u_1(x) + (e_1 \wedge e_3)u_2(x) + (e_2 \wedge e_1)u_1(x); \tag{6}$$

then,

$$\begin{aligned} D \star u(x) &= \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k}((e_3 \wedge e_2)u_1(x) + (e_1 \wedge e_3)u_2(x) + (e_2 \wedge e_1)u_3(x)) \\ &= (e_1 \wedge e_3 \wedge e_2)\frac{\partial u_1}{\partial x_1} - (e_2 \wedge e_3 \wedge e_1)\frac{\partial u_2}{\partial x_2} + (e_3 \wedge e_2 \wedge e_1)\frac{\partial u_3}{\partial x_3} \\ &\quad + e_3\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) + e_2\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) + e_1\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right), \end{aligned}$$

or

$$D \star u(x) = -\operatorname{div}(u^*(x))\tau_g - \operatorname{curl}(u^*(x)), \tag{7}$$

where $\tau_g = e_1 \wedge e_2 \wedge e_3$. Thus we obtain

$$\begin{aligned} D \star^{-1} u(x) &= (-1)^{r(n-r)}\mathrm{sgn}(g)(-\operatorname{div}(u^*(x)))\tau_g \\ &\quad - ((-1)^{r(n-r)}\mathrm{sgn}(g))\operatorname{curl}(u^*(x)) \\ &= \operatorname{div}(u^*(x))\tau_g + \operatorname{curl}(u^*(x)), \end{aligned} \tag{8}$$

where

$$\begin{aligned} ((-1)^{r(n-r)}\mathrm{sgn}(g))\operatorname{curl}(u^*(x)) &= -\operatorname{curl}(u^*(x)), \\ ((-1)^{r(n-r)}\mathrm{sgn}(g))\operatorname{div}(u^*(x)) &= -\operatorname{div}(u^*(x)). \end{aligned}$$

Finally, from equations (4) and (8) we obtain that

$$\star^{-1} Du(x) = D \star^{-1} u(x); \tag{9}$$

then, the equation (3) can be written as

$$\begin{aligned} \operatorname{curl}(u^*(x)) &= \star^{-1}\operatorname{div}(u^*(x)) + D \star^{-1} u(x) \\ &= -(Du(x)) \wedge \tau_g + D \star^{-1} u(x). \end{aligned} \tag{10}$$

Other useful relations are listed below:

$$
\begin{aligned}
uDu &= -u \operatorname{div} u^* - \frac{1}{2} Du^2 - u^* \cdot \nabla u + u^* \operatorname{curl} u^* \tau_g,\\
(Du)u &= -u \operatorname{div} u^* + \frac{1}{2} Du^2 + u^* \cdot \nabla u + u^* \operatorname{curl} u^* \tau_g,\\
uDu - (Du)u &= -2u^* \cdot \nabla u^* - Du^2,\\
uDu + (Du)u &= -2 \operatorname{div} u^* + 2\tau_g u^* \operatorname{curl} u^*,\\
Du^2 &= (Du)u - uDu - 2u^* \cdot \nabla u,\\
D(u\omega) &= (Du)\omega - uD\omega - 2u^* \cdot \nabla \omega,\\
2u^* \cdot \nabla u &= (Du)u - uDu - Du^2,\\
2u^* \cdot \nabla \omega &= (Du)\omega - uD\omega - D(u\omega).
\end{aligned} \tag{11}
$$

3. Asymmetric Fluid and Hodge Operator

In this section we consider the stationary incompressible asymmetric fluid in bounded domains with density constant. A detailed study of this system can be viewed in [4] (see also [5]) and in exterior domains in [1]. Thus, let us denote by u^*, w^* and p the velocity field, the angular velocity of rotation of the fluid particles and the pressure distribution, respectively. The governing equations are the following:

$$
\begin{aligned}
-\Delta u^* + \frac{\rho}{\eta l_1}(u^* \cdot \nabla)u^* + \frac{1}{\rho l_1}\nabla p &= \frac{2\mu_r}{l_1} \operatorname{curl} w^* + f_1^*,\\
\operatorname{div} u^* &= 0,\\
-\Delta w^* + \frac{1}{l_2}(u^* \cdot \nabla)w^* - \frac{l_3}{l_2}\nabla \operatorname{div}\, w^* + \frac{4\mu_r}{l_2} w^* &= \frac{2\mu_r}{l_2} \operatorname{curl} u^* + g_1^*.
\end{aligned} \tag{12}
$$

For simplicity, they will be completed with the following boundary conditions:

$$
u^*(x) = 0, \quad w^*(x) = 0 \quad \text{on } \partial\Omega = \Gamma. \tag{13}
$$

In (12), f^* and g^* are known density functions of external sources for the linear and the angular momentum of particles, respectively. The positive constants l_1, l_2 and l_3 are given by

$$
l_1 = \frac{\mu + \mu_r}{\rho}; \quad l_2 = \frac{c_a + c_d}{\rho}; \quad l_3 = \frac{c_0 + c_d - c_a}{\rho},
$$

where μ, μ_r, c_0, c_a and c_d characterize the physical properties of the fluid. Thus, μ is the usual Newtonian viscosity; μ_r, c_0, c_a and c_d are additional viscosities related to the lack of symmetry of the stress tensor and, consequently, to the fact that the field of internal rotation w does not vanish. These constants must satisfy the inequality $c_0 + c_d > c_a$.

Now, we can write the system (12,13) in the Clifford formalism with

$$u(x), w(x) \in Cl^1_{0,3} \hookrightarrow Cl_{0,3}$$

as

$$\begin{aligned} D^2 u + \frac{\rho}{2\eta l_1}((Du)u - uDu - Du^2) + \frac{1}{\rho l_1} Dp - \frac{2\mu_r}{l_1}(D \star^{-1} w - (Dw) \wedge \tau_g) - f_1 = 0, \\ Du \wedge \tau_g = 0, \\ D^2 w + \frac{1}{2l_2}((Du)w - uDw - D(uw)) - \frac{l_3}{l_2} D \star (Dw \wedge \tau_g) + \frac{4\mu_r}{l_2} w = \frac{2\mu_r}{l_2} D \star^{-1} u + g_1, \\ u = 0, \quad w = 0 \quad \text{on } \partial\Omega = \Gamma. \end{aligned} \tag{14}$$

For further use, we introduce the operators $M(u)$ and $N(u, w)$ defined by

$$\begin{aligned} M(u) &= \tfrac{1}{2}((Du)u - uDu - Du^2) - \frac{\eta l_1}{\rho} f_1; \\ N(u, w) &= \tfrac{1}{2}((Du)w - uDw - D(uw)) - l_2 g_1. \end{aligned} \tag{15}$$

Although this section is dedicated to the model in bounded domains, we recall that in this paper we consider unbounded domains.

4. Hodge Operator and Projections

Now, we recall without proof the theorems and operators introduced in [2]. Let a fixed point z lying in the complement of the closure of Ω, which contains a non-empty open set. Then we can consider the operator

$$\widetilde{T} f(y) = \int_\Omega K_z(x, y) f(x) d\Omega_x, \tag{16}$$

with $K_z(x, y) = G(x - y) - G(x - z)$, where $G(x)$ is the so-called generalized Cauchy kernel, the Green function of the Dirac operator. This operator is a continuous mapping of $\mathcal{W}^k_q(\Omega)$ in $\mathcal{W}^{k+1}_q(\Omega)$, $1 < q < \infty$, $k = 0, 1, \ldots$ and is bounded operator of $\mathcal{W}^{-1}_q(\Omega)$ in $\mathcal{L}_q(\Omega)$, $1 < q < \infty$.

Theorem 4.1 (Borel-Pompeiu's formula)**.** *If $f \in \mathcal{W}^1_q(\Omega)$, $1 < q < \infty$, then we have*

$$\widetilde{F}_\Gamma f = f - \widetilde{T} D f,$$

with

$$\widetilde{F}_\Gamma f = \int_\Gamma K_z(x, y)\alpha(x) f(x) d\Gamma_x,$$

where $\alpha(x)$ is the outward pointing normal unit vector to Γ at the point x.

Proposition 4.2. *If $k \in \mathbb{N}$, then the operator*

$$\widetilde{F}_\Gamma : \mathcal{W}_q^{k-1/q}(\Gamma) \to \mathcal{W}_q^k(\Omega) \cap \ker D$$

is a continuous operator.

Theorem 4.3 (Plemelj-Sokhotzki's formula). *If $f \in \mathcal{W}_q^1(\Gamma)$, $1 < q < \infty$, $l > 0$, then we have*

$$\mathrm{tr}\widetilde{F}_\Gamma f = \frac{1}{2} f + \frac{1}{2}\widetilde{S}_\Gamma f,$$

whereby

$$\widetilde{S}_\Gamma f = 2\int_\Gamma K_z(x,y)\alpha(x) f(x) d\Gamma_x$$

is the singular integral operator of Cauchy type over the boundary.

Theorem 4.4. *The space $\mathcal{L}_q(\Omega)$, $1 < q < \infty$, allows the direct decomposition*

$$\mathcal{L}_q(\Omega) = \ker D(\Omega) \cap \mathcal{L}_q(\Omega) \oplus D(\overset{0}{\mathcal{W}}{}_q^1(\Omega)).$$

The above theorem allows to obtain the projections

$$\mathbf{P} : \mathcal{L}_q(\Omega) \to \ker D(\Omega) \cap \mathcal{L}_q(\Omega)$$

and

$$\mathbf{Q} : \mathcal{L}_q(\Omega) \to D(\overset{0}{\mathcal{W}}{}_q^1(\Omega));$$

for $q = 2$ these projections are orthoprojections. Moreover, in [2] *it is shown that*

$$\mathbf{Q} f = D\Delta_0^{-1} D f,$$

where Δ_0^{-1}, is the solution operator of the Dirichlet problem of the Poisson equation with homogeneous boundary data

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma,$$

for $f \in \mathcal{W}_q^{-1}(\Omega)$, $1 < q < \infty$.

Theorem 4.5. *Suppose $f_1, g_1 \in \mathcal{W}_q^{-1}(\Omega)$, $p \in \mathcal{L}_q(\Omega, \mathbb{R})$, $1 < q < \infty$; then any solution of the system* (14) *has the representation*

$$
\begin{aligned}
u+\frac{\rho}{\eta l_1}\widetilde{T}Q\widetilde{T}M(u)+\frac{1}{\rho l_1}\widetilde{T}Qp &= \frac{2\mu_r}{l_1}(\widetilde{T}Q\star^{-1}w-\widetilde{T}Q\widetilde{T}Dw\wedge\tau_g)\\
\tau_g\wedge(\frac{\rho}{\eta l_1}Q\widetilde{T}M(u)+\frac{1}{\rho l_1}Qp) &= -\frac{2\mu_r}{l_1}\tau_g\wedge(Q\star^{-1}w-Q\widetilde{T}Dw\wedge\tau_g)\\
w+\frac{1}{l_2}\widetilde{T}Q\widetilde{T}N(u,w) &= \frac{l_3}{l_2}\widetilde{T}Q(\star(Dw\wedge\tau_g))-\frac{4\mu_r}{l_2}\widetilde{T}Q\widetilde{T}w+\frac{2\mu_r}{l_2}\widetilde{T}Qu\tau_g
\end{aligned}
\tag{17}
$$

Proof. Recall that $Qf = D\Delta_0^{-1}Df$ and $D\widetilde{T}f(y) = f(y)$, and the Borel-Pompeiu's formula

$$\widetilde{T}Du = u - \widetilde{F}_\Gamma u = u, \qquad u \in \overset{0}{\mathcal{W}}{}^1_q(\Omega),$$

we can write

$$\widetilde{T}Q\widetilde{T}DDu = \widetilde{T}(D\Delta_0^{-1}D)\widetilde{T}DDu = \widetilde{T}D\Delta_0^{-1}DDu = \widetilde{T}Du = u. \tag{18}$$

On the other hand,

$$
\begin{aligned}
\widetilde{T}Q\widetilde{T}(D\star^{-1}w - Dw\wedge\tau_g) &= \widetilde{T}Q\widetilde{T}D\star^{-1}w - \widetilde{T}Q\widetilde{T}Dw\wedge\tau_g\\
&= \widetilde{T}Q\star^{-1}w - \widetilde{T}Q\widetilde{T}Dw\wedge\tau_g;
\end{aligned}
\tag{19}
$$

then by applying the $\widetilde{T}Q\widetilde{T}$ operator to system (14) and using the formulas (18) and (19), we obtain the expected result. ☑

Lemma 4.6. 1. *Let $n/2 \le q < \infty$. Then the operator $M : \overset{\circ}{\mathcal{W}}{}^1_q(\Omega) \to \mathcal{W}_q^{-1}(\Omega)$ is a continuous operator and we have*

$$\|[\star(uD\wedge\tau_g)]\,u\|_{\mathcal{W}_q^{-1}(\Omega)} \le C_2\,\|u\|^2_{\mathcal{W}_q^1(\Omega)}\,.$$

2. *Let $n/2 \le q < \infty$. Then the operator $N_u : \overset{\circ}{\mathcal{W}}{}^1_q(\Omega)\times\overset{\circ}{\mathcal{W}}{}^1_q(\Omega) \to \mathcal{W}_q^{-1}(\Omega)$ is a continuous operator and we have*

$$\|[\star(uD\wedge\tau_g)]\,w\|_{\mathcal{W}_q^{-1}(\Omega)} \le C_2\,\|u\|_{\mathcal{W}_q^1(\Omega)}\,\|w\|_{\mathcal{W}_q^1(\Omega)}\,.$$

Proof. See [2]. ☑

On the other hand, the system (17) can be solved by the following iterative method similar to [2]:

$$
\begin{aligned}
u_i+\frac{\rho}{\eta l_1}\widetilde{T}Q\widetilde{T}M(u_{i-1})+\frac{1}{\rho l_1}\widetilde{T}Qp_i-\frac{2\mu_r}{l_1}(\widetilde{T}Q\star^{-1}w_i-\widetilde{T}Q\widetilde{T}Dw_i\wedge\tau_g) &= 0\\
\tau_g\wedge(\frac{\rho}{\eta l_1}Q\widetilde{T}M(u_{i-1})+\frac{1}{\rho l_1}Qp_i)-\frac{2\mu_r}{l_1}\tau_g\wedge(Q\star^{-1}w_i-Q\widetilde{T}Dw_i\wedge\tau_g) &= 0\\
w_i+\frac{1}{l_2}\widetilde{T}Q\widetilde{T}N(u_i,w_i)-\frac{l_3}{l_2}\widetilde{T}Q(\star(Dw_i\wedge\tau_g))+\frac{4\mu_r}{l_2}\widetilde{T}Q\widetilde{T}w_i-\frac{2\mu_r}{l_2}\widetilde{T}Q\star^{-1}u_i &= 0
\end{aligned}
\tag{20}
$$

Note that the first two equations above represent an interaction similar to the case of Navier-Stokes equations and their treatment is identical to [2]. Indeed,

$$\|u_i - u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq \frac{\rho}{\eta l_1}\left\|\widetilde{T}Q\widetilde{T}(M(u_{i-1}) - M(u_{i-2}))\right\|_{\mathcal{W}_q^1(\Omega)} + \frac{1}{\rho l_1}\left\|\widetilde{T}Q(p_i - p_{i-1})\right\|_{\mathcal{W}_q^1(\Omega)}$$
$$+ \frac{2\mu_r}{l_1}\left\|\widetilde{T}Q\star^{-1}(w_i - w_{i-1})\tau_g\right\|_{\mathcal{W}_q^1(\Omega)} + \frac{2\mu_r}{l_1}\left\|\widetilde{T}Q\widetilde{T}D(w_{i-1} - w_{i-2})\wedge\tau_g\right\|_{\mathcal{W}_q^1(\Omega)}$$

Now, using the second equation of (20), we have

$$\|u_i - u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq 2C_1\|M(u_{i-1}) - M(u_{i-2})\|_{\mathcal{W}_q^{-1}(\Omega)},$$

where

$$C_1 = \frac{\rho}{\eta l_1}\left\|\widetilde{T}\right\|_{\left[\mathcal{L}_q(\Omega)\cap imQ, \mathcal{W}_q^1(\Omega)\right]}\|Q\|_{[\mathcal{L}_q(\Omega), \mathcal{L}_q(\Omega)\cap imQ]}\left\|\widetilde{T}\right\|_{\left[\mathcal{W}_q^{-1}(\Omega), \mathcal{L}_q(\Omega)\right]}.$$

Now, due to previous Lemma, we have

$$\|[Sc(uD)]\,u\|_{\mathcal{W}_q^{-1}(\Omega)} \leq C_2\|u\|^2_{\mathcal{W}_q^1(\Omega)},$$
$$\|[Sc(uD)]\,w\|_{\mathcal{W}_q^{-1}(\Omega)} \leq C_2\|u\|_{\mathcal{W}_q^1(\Omega)}\|w\|_{\mathcal{W}_q^1(\Omega)};$$

then, similarly as in [2], this results in

$$\|M(u_{i-1}) - M(u_{i-2})\|_{\mathcal{W}_q^{-1}} \leq C_2\|u_{i-1} - u_{i-2}\|_{\mathcal{W}_q^1(\Omega)}\left(\|u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} + \|u_{i-2}\|_{\mathcal{W}_q^1(\Omega)}\right).$$

With $L_i = 2C_1C_2(\|u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} + \|u_{i-2}\|_{\mathcal{W}_q^1(\Omega)})$ we obtain

$$\|u_i - u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq L_i\|u_{i-1} - u_{i-2}\|_{\mathcal{W}_q^1(\Omega)}.$$

Furthermore, by using the second equation of (20) and consider that the element of volume, τ_g, has norm bounded, we have

$$\begin{aligned}
\|u_i\|_{\mathcal{W}_q^1(\Omega)} &\leq \frac{\rho}{\eta l_1}\left\|\widetilde{T}Q\widetilde{T}M(u_{i-1})\right\|_{\mathcal{W}_q^1(\Omega)} \\
&\quad + \left\|\frac{1}{\rho l_1}\widetilde{T}Qp_i + \frac{2\mu_r}{l_1}(\widetilde{T}Qw_i - \widetilde{T}Q\widetilde{T}(\star(Dw_i\wedge\tau_g)))\tau_g\right\|_{\mathcal{W}_q^1(\Omega)} \\
&\leq \frac{\rho}{\eta l_1}\left\|\widetilde{T}Q\widetilde{T}M(u_{i-1})\right\|_{\mathcal{W}_q^1(\Omega)} + \frac{\rho}{\eta l_1}\left\|\widetilde{T}\right\|_{\mathcal{W}_q^1}\left\|Q\widetilde{T}M(u_{i-1})\right\|_{\mathcal{W}_q^1(\Omega)} \\
&\leq \frac{2\rho}{\eta l_1}\left\|\widetilde{T}Q\widetilde{T}M(u_{i-1})\right\|_{\mathcal{W}_q^1(\Omega)} \\
&\leq 2C_1C_2\|u_{i-1}\|^2_{\mathcal{W}_q^1(\Omega)} + C_1\frac{\rho}{\eta}\|f_1\|_{\mathcal{W}_q^{-1}(\Omega)};
\end{aligned}$$

then, by using arguments similar to [2] (see page 97), to ensure that $\|u_i\|_{\mathcal{W}_q^1(\Omega)} \leq \|u_{i-1}\|_{\mathcal{W}_q^1(\Omega)}$, we must have that $(\rho/\eta l_1)\|f_1\|_{\mathcal{W}_q^{-1}(\Omega)} \leq (16C_1^2C_2^2)^{-1}$; then,

$$\left| \|u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} - \frac{1}{4C_1C_2} \right| \leq W,$$

with $W = \left[(4C_1C_2)^{-2} - \rho\|f_1\|_{\mathcal{W}_q^{-1}(\Omega)}/(\eta l_1 C_2) \right]^{1/2}$. Finally, it can be shown that

$$\|u_i - u_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq (1 - 4C_1C_2W)\|u_{i-1} - u_{i-2}\|_{\mathcal{W}_q^1(\Omega)},$$

with the condition $L_i \leq (1 - 4C_1C_2W) \equiv L < 1$. Then, by Banach's fixed point theorem the iterative method (20) converges. Now we demonstrate convergence for the third equation of (20) in which w_i is calculated by the third equation of (20) in analogy to [2] placing

$$w_i^j = -\frac{1}{l_2}\widetilde{T}Q\widetilde{T}N(u_i, w_i^{j-1}) + \frac{l_3}{l_2}\widetilde{T}Q(\star(Dw_i^j \wedge \tau_g)) + \frac{4\mu_r}{l_2}\widetilde{T}Q\widetilde{T}w_i^j - \frac{2\mu_r}{l_2}\widetilde{T}Qu_i\tau_g;$$

thus,

$$\begin{aligned} \left\| w_i^j - w_i^{j-1} \right\|_{\mathcal{W}_q^1(\Omega)} \leq & \frac{1}{l_2}\left\| \widetilde{T}Q\widetilde{T}(N(u_i, w_i^{j-1}) - N(u_i, w_i^{j-2})) \right\|_{\mathcal{W}_q^1} \\ & + \frac{l_3}{l_2}\left\| \widetilde{T}Q(\star(D(w_i^j - w_i^{j-1}) \wedge \tau_g)) \right\|_{\mathcal{W}_q^1} \\ & + \frac{4\mu_r}{l_2}\left\| \widetilde{T}Q\widetilde{T}(w_i^j - w_i^{j-1}) \right\|_{\mathcal{W}_q^1}, \end{aligned} \tag{21}$$

where

a)

$$\begin{aligned} \frac{1}{l_2}\left\| \widetilde{T}Q\widetilde{T}N(u_i, w_i^{j-1}) - N(u_i, w_i^{j-2}) \right\|_{\mathcal{W}_q^1} & \leq \overline{P_1}\left\| N(u_i, w_i^{j-1}) - N(u_i, w_i^{j-2}) \right\|_{\mathcal{W}_q^{-1}} \\ & = \overline{P_1}\left\| u_i\nabla w_i^{j-1} - u_i\nabla w_i^{j-2} \right\|_{\mathcal{W}_q^{-1}} \\ & = \overline{P_1}\left\| u_i(\nabla w_i^{j-1} - \nabla w_i^{j-2}) \right\|_{\mathcal{W}_q^{-1}} \\ & \leq P_1\|u_i\|_{\mathcal{W}_q^1}\left\| w_i^{j-1} - w_i^{j-2} \right\|_{\mathcal{W}_q^1}, \end{aligned}$$

where $P_1 = \overline{P_1}C$ and

$$\overline{P_1} = \frac{1}{l_2}\left\|\widetilde{T}\right\|_{[\mathcal{L}_q(\Omega)\cap imQ, \mathcal{W}_q^1(\Omega)]}\|Q\|_{[\mathcal{L}_q(\Omega), \mathcal{L}_q(\Omega)\cap imQ]}\left\|\widetilde{T}\right\|_{[\mathcal{W}_q^{-1}(\Omega), \mathcal{L}_q(\Omega)]}.$$

b)

$$\frac{l_3}{l_2}\left\| \widetilde{T}Q \star (D(w_i^j - w_i^{j-1})) \wedge \tau_g \right\|_{\mathcal{W}_q^1} \leq P_2\left\| w_i^j - w_i^{j-1} \right\|_{\mathcal{W}_q^1},$$

where

$$P_2 = \frac{l_3}{l_2} \left\| \widetilde{T} \right\|_{\left[\mathcal{L}_q \cap imQ, \mathcal{W}_q^1\right]} \|Q\|_{[\mathcal{L}_q, \mathcal{L}_q \cap imQ]} \|\nabla\|_{[\mathcal{W}_q^1, \mathcal{L}_q]} .$$

c)

$$\frac{4\mu_r}{l_2} \left\| \widetilde{T} Q \widetilde{T}(w_i^j - w_i^{j-1}) \right\|_{\mathcal{W}_q^1} \leq P_3 \left\| w_i^j - w_i^{j-1} \right\|_{\mathcal{W}_q^1} ,$$

where

$$P_3 = \frac{4\mu_r}{l_2} \left\| \widetilde{T} \right\|_{\left[\mathcal{L}_q \cap imQ, \mathcal{W}_q^1\right]} \|Q\|_{[\mathcal{L}_q, \mathcal{L}_q \cap imQ]} \left\| \widetilde{T} \right\|_{\left[\mathcal{W}_q^1, \mathcal{W}_q^2\right]} .$$

Then, from (21) and the above estimates, we can write

$$\left\| w_i^j - w_i^{j-1} \right\|_{\mathcal{W}_q^1(\Omega)} \leq P_1 \|u_i\|_{\mathcal{W}_q^1} \left\| w_i^{j-1} - w_i^{j-2} \right\|_{\mathcal{W}_q^1} + 2P \left\| w_i^j - w_i^{j-1} \right\|_{\mathcal{W}_q^1} ,$$

where $P = \max\{P_2, P_3\}$. Thus, if $2P < 1$ we can write

$$\left\| w_i^j - w_i^{j-1} \right\|_{\mathcal{W}_q^1(\Omega)} \leq \frac{P_1}{1-2P} \|u_i\|_{\mathcal{W}_q^1} \left\| w_i^{j-1} - w_i^{j-2} \right\|_{\mathcal{W}_q^1} .$$

Then, if $\frac{P_1}{1-2P} \|u_i\|_{\mathcal{W}_q^1} < 1$, we have that $\left\{ w_i^j \right\}$ converges in $\overset{0}{\mathcal{W}}{}_q^1(\Omega)$.

Consequently, we have proved that the system (14) has a unique solution.

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