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Is the process of finding f' chaotic?

Héctor Méndez-Lango*

Abstract. Let $(H(\mathbb{C}), \rho)$ be the metric space of all entire functions f where the metric ρ induces the topology of uniform convergence on compact subsets of the complex plane. Let $D : H(\mathbb{C}) \to H(\mathbb{C})$ be the linear mapping that assigns to each f its derivative, $D(f) = f'$. We show in this note that the set of entire functions that are periodic under this map is dense in $(H(\mathbb{C}), \rho)$. It implies that $D : H(\mathbb{C}) \to H(\mathbb{C})$ is chaotic in the sense of Devaney.

1. Introduction

Finding the derivative f' of a given function f is an everyday task. The aim of this note is to show that there is a setting where the mapping $f \to f'$ is chaotic in the sense of Devaney. It is the set of entire functions with the topology of uniform convergence on compact subsets of the complex plane $\mathbb{C}$, $(H(\mathbb{C}), \rho)$. Some signs of this fact appeared in 1983 (see [2]). At that time C. Blair and L. A. Rubel proved the existence of a entire function f such that the set $\{f^{(n)} : n \geq 0\}$ of all derivatives of f is dense in the space $(H(\mathbb{C}), \rho)$. Since the existence of a dense orbit implies transitivity, in order to show that the mapping $f \to f'$ is chaotic (see [1]) it is enough to prove that the set of entire functions that are periodic under this map is dense in $(H(\mathbb{C}), \rho)$. Such is our main goal.

2. Some properties of the space $(H(\mathbb{C}), \rho)$

We consider functions from $\mathbb{C}$ into $\mathbb{C}$. An *entire function* is a function which is analytic in the whole complex plane. If f is an entire function, then f has a power series expression

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

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* Departamento de Matemáticas, Facultad de Ciencias, UNAM, Ciudad Universitaria, C.P. 04510, D. F. México, *e-mail*: `hml@fciencias.unam.mx`.

with infinite radius of convergence. Let $H(\mathbb{C})$ be the set of all entire functions. We denote with z_0 the constant function $f(z) = z_0$ (when this does not cause any confusion.) Notice that $H(\mathbb{C})$ is a vector space. Sometimes we refer to $f \in H(\mathbb{C})$ as a point in $H(\mathbb{C})$. Given f and g in $H(\mathbb{C})$ and a positive integer n, we define

$$\rho_n(f,g) = \sup\{|f(z) - g(z)| : |z| \leq n\}$$

and

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

It is known (see [3, chapter VII]) that ρ is a metric defined in $H(\mathbb{C})$, and $(H(\mathbb{C}), \rho)$ is a complete space. The following lemma contains some properties of ρ. The proof is not difficult, so we leave it to the reader.

Lemma 2.1. *Let f, g, φ and γ be entire functions, and c be a complex number. Then*

i) $\rho(f,g) = \rho(f-g,0)$.

ii) $\rho(f+g,0) \leq \rho(f,0) + \rho(g,0)$.

iii) $\rho(cf,cg) \leq \max\{|c|\,\rho(f,g),\ \rho(f,g)\}$.

iv) If for some $\varepsilon > 0$, we have that $\rho(f,\varphi) < \varepsilon$ and $\rho(g,\gamma) < \varepsilon$, then

$$\rho(f+g, \varphi+\gamma) < 2\varepsilon.$$

The proof of the next useful assertion is in [3, chapter VII, Lemma 1.7].

Lemma 2.2. *If $\varepsilon > 0$ is given, then there exist a $\delta > 0$ and a positive integer n such that for f and g in $H(\mathbb{C})$, $\rho_n(f,g) < \delta$ implies $\rho(f,g) < \varepsilon$.*

Let A denote the subset of $H(\mathbb{C})$ that consist of all polynomial functions.

Proposition 2.3. *A is dense in $H(\mathbb{C})$.*

Proof. Let $f \in H(\mathbb{C})$ and $\varepsilon > 0$. There exist $\delta > 0$ and $N \in \mathbb{N}$ such that for $g \in H(\mathbb{C})$

$$\sup\{|f(z) - g(z)| : |z| \leq N\} < \delta$$

implies $\rho(f,g) < \varepsilon$.

Consider the power series expression for f,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since

$$f_i(z) = \sum_{n=0}^{i} a_n z^n$$

converges uniformly to $f(z)$ in the set $\{z : |z| \leq N\}$, there exists $k \in \mathbb{N}$ such that

$$\sup\{|f(z) - f_k(z)| : |z| \leq N\} < \delta.$$

Therefore $\rho(f, f_k) < \varepsilon$.

Since $f_k \in A$, the proof is complete. ☑

3. Some properties of the mapping $f \to f'$

Given a mapping $F : H(\mathbb{C}) \to H(\mathbb{C})$ and $n \in \mathbb{N}$, F^n denotes the composition of F with itself n times. The *orbit of f under F* is the sequence

$$\left\{ f, F(f), F^2(f), \dots \right\},$$

and it is denoted by $o(f, F)$. It is said that f is a *periodic point* of F if there exists $n \in \mathbb{N}$ such that $F^n(f) = f$. The set of all periodic points of F is denoted by $\mathrm{Per}(F)$. We say that F is *transitive* if for each pair of nonempty open subsets U and W of $H(\mathbb{C})$ there exist $f \in U$ and $n \in \mathbb{N}$ such that $F^n(f) \in W$. Finally, F is *chaotic* (in the sense of Devaney) if the set of periodic points of F is dense in $H(\mathbb{C})$ and F is transitive (see [4] and [1].) Note that if there exists $f \in H(\mathbb{C})$ such that the $o(f, F)$ is a dense set in $H(\mathbb{C})$, then F is transitive.

Let $D : H(\mathbb{C}) \to H(\mathbb{C})$ be the linear mapping defined by $D(f) = f'$. As we said, it is known that D is transitive. From now on we focus on the set of entire functions that are periodic under D. Note that $\mathrm{Per}(D)$ is a vector subspace of $H(\mathbb{C})$. Due to Proposition 2.3, in order to show that the set $\mathrm{Per}(D)$ is dense in $H(\mathbb{C})$ it is enough to prove the following: For each $P \in A$ and each $\varepsilon > 0$, there exists $f \in \mathrm{Per}(D)$ such that $\rho(P, f) < \varepsilon$.

The constant function $f(z) = 1$ is not a periodic point under D. But, as a consequence of the next proposition, the sequence

$$\begin{aligned}
\varphi_1(z) &= \exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
\varphi_2(z) &= \cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \\
\varphi_3(z) &= 1 + \frac{z^3}{3!} + \frac{z^6}{6!} + \frac{z^9}{9!} + \cdots \\
&\vdots \\
\varphi_k(z) &= 1 + \frac{z^k}{k!} + \frac{z^{2k}}{(2k)!} + \frac{z^{3k}}{(3k)!} + \cdots
\end{aligned}$$

converges to $f(z) = 1$ in $(H(\mathbb{C}), \rho)$. Notice that for each k, φ_k is an entire function which is a periodic point under D.

Proposition 3.1. *Let $M \geq 0$, $\varepsilon > 0$ and $f \in H(\mathbb{C})$ given by $f(z) = \frac{z^M}{M!}$. Then there exists $\varphi \in \mathrm{Per}(D)$ such that $\rho(f, \varphi) < \varepsilon$.*

Proof. Let $\delta > 0$ and $N \in \mathbb{N}$ such that for g and h in $H(\mathbb{C})$, $\rho_N(g, h) < \delta$ implies $\rho(g, h) < \varepsilon$. Let $k \in \mathbb{N}$ such that

$$\sum_{i=k}^{\infty} \frac{N^i}{i!} < \delta.$$

Let us consider the function

$$\varphi(z)=\frac{z^M}{M!}+\frac{z^{M+k}}{(M+k)!}+\frac{z^{M+2k}}{(M+2k)!}+\cdots=\sum_{j=0}^{\infty}\frac{z^{M+jk}}{(M+jk)!}.$$

Since for each i the coefficient of z^i in the expression of $\varphi(z)$, say b_i, satisfies $0\le b_i\le\frac{1}{i!}$, the radius of convergence of

$$\sum_{j=0}^{\infty}\frac{z^{M+jk}}{(M+jk)!}$$

is infinite. Therefore $\varphi(z)$ is an entire function. Also it is easy to see that φ is periodic under D.

Now, let $z\in\mathbb{C}$ such that $|z|\le N$. Then

$$\left|\varphi(z)-\frac{z^M}{M!}\right|=\left|\sum_{j=1}^{\infty}\frac{z^{M+jk}}{(M+jk)!}\right|\le\left|\sum_{j=1}^{\infty}\frac{w^{M+jk}}{(M+jk)!}\right|$$

for some w with $|w|=N$.

It follows that

$$\left|\varphi(z)-\frac{z^M}{M!}\right|\le\sum_{j=1}^{\infty}\frac{N^{M+jk}}{(M+jk)!}\le\sum_{i=k}^{\infty}\frac{N^i}{i!}<\delta.$$

Therefore $\rho_N(f,\varphi)<\delta$ and $\rho(f,\varphi)<\varepsilon$. ☑

Corollary 3.2. *Let $M\ge 0$, $\varepsilon>0$, $c\in\mathbb{C}$ and $f\in H(\mathbb{C})$ given by $f(z)=cz^M$. Then there exists $\varphi\in\operatorname{Per}(D)$ such that $\rho(f,\varphi)<\varepsilon$.*

Proof. It readily follows from Proposition 3.1 and Lemma 2.1. ☑

Corollary 3.3. *For each $P\in A$ and each $\varepsilon>0$, there exists $\varphi\in\operatorname{Per}(D)$ such that $\rho(P,\varphi)<\varepsilon$.*

Proof. Assume that

$$P(z)=p_0+p_1z+p_2z^2+\cdots+p_kz^k.$$

Due to Corollary 3.2, there exist $k+1$ entire functions, periodic under D, $\varphi_0,\varphi_1,\ldots,\varphi_k$, such that for each $0\le i\le k$, $\rho(P_i,\varphi_i)<\dfrac{\varepsilon}{k+1}$, where $P_i(z)=p_iz^i$. Let $\varphi=\sum_{i=0}^{k}\varphi_i$. By Lemma 2.1, $\rho(P,\varphi)<\varepsilon$. Since φ is a periodic point under D, the proof is complete. ☑

The function given by C. Blair and L. A. Rubel in [2] implies that D is transitive. This result and Corollary 3.3 supply the proof of the next theorem.

Theorem 3.4. *$D:H(\mathbb{C})\to H(\mathbb{C})$ is chaotic in the sense of Devaney.*

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Héctor Méndez-Lango
Departamento de Matemáticas
Facultad de Ciencias, UNAM
Ciudad Universitaria, C.P. 04510, D. F. MEXICO.
e-mail: hml@fciencias.unam.mx.