The problem of the first return attached to a pseudodifferential operator in dimension 3

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Abstract. In this article we study the problem of first return associated to an elliptic pseudodifferential operator with non-radial symbol of dimension 3 over the $p$-adics.

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El problema del primer retorno asociado a un operador seudodiferencial en dimensión 3

Resumen. En este artículo estudiamos el problema del primer retorno asociado a un operador seudodiferencial elíptico con símbolo no radial de dimensión 3 sobre el cuerpo de los números $p$-ádicos.

Palabras clave: Caminatas aleatorias, ultradifusión, números $p$-ádicos, análisis no arquimediano.

1. Introduction

Avetisov et al. have constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes (see [2],[3]). From a mathematical point of view, in these models the time-evolution of a complex system is described by a $p$-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

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The problem of the first return in dimension 1 was studied in [4], and in arbitrary dimension in [7]. In both articles, pseudodifferential operators with radial symbols were considered. More recently, Chacón-Cortés [6] considers pseudodifferential operators over $\mathbb{Q}_p^4$ with non-radial symbol; he studies the problem of first return for a random walk $X(t, w)$ whose density distribution satisfies certain diffusion equation.

In [5], the authors study elliptic pseudodifferential operators in dimension 3 and find a function, $Z(x, t) : x \in \mathbb{Q}_p^3, t \in \mathbb{R}^+$, that satisfies the following equation

$$
\frac{\partial u(x, t)}{\partial t} = -\int_{\mathbb{Q}_p^3} K_{-\alpha}(y)(u(x - y, t) - u(x, t))d^3y,
$$

(1)

where $K_{-\alpha}(x)$ is the Riesz kernel associated to the elliptic quadratic form $f^0(\xi) = p\xi_1^2 + \xi_2^2 - \epsilon\xi_3^2$.

Using the same techniques as in [6] we prove that the random walk $X(t, w)$ whose density distribution satisfies the equation (1) is recurrent if $\alpha \geq \frac{3}{2}$ and transient when $\alpha < \frac{3}{2}$.

This result is analog to the one showed in [7], in the sense that $2\alpha$ represent the degree of the symbol, and in this case the process is recurrent if $2\alpha$ is greater that the dimension, 3.

The article is organized as follows. In Section 2 we write some facts about $p$-adics. In Section 3 we define the symbol for the pseudodifferential operator and its Fourier transform. In Section 4 we study the Cauchy problem and give some properties of its fundamental solution, and define a Markov process over $\mathbb{Q}_p^3$. In Section 5 we determine the probability density function for a path of $X(t, \omega)$ goes back to $\mathbb{Z}_p^3$, and we show that the process is recurrent when $\alpha \geq \frac{3}{2}$, and otherwise is transient (see Theorem 5.7).

2. Preliminars

For the sake of completeness we include some preliminars. For more details the reader may consult [1,9,10].

2.1. The field of $p$-adic numbers

Along this article $p$ will denote a prime number different from 2. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm $|\cdot|_p$, which is defined as

$$
|x|_p = \begin{cases} 
0 & \text{if } x = 0, \\
 p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b}, 
\end{cases}
$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$-adic order of $x$. We extend the $p$-adic norm to $\mathbb{Q}_p^n$ by taking

$$
||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.
$$

We define $\text{ord}(x) = \min_{1 \leq i \leq n}\{\text{ord}(x_i)\}$; then $||x||_p = p^{-\text{ord}(x)}$. Any $p$-adic number $x \neq 0$ has a unique expansion $x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \ldots, p - 1\}$ and $x_0 \neq 0$. 

[Revista Integración]
Let $x$ be called the fractional part of $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0, \\ p^{\text{ord}(x)} \sum_{j=0}^{\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B^n_\gamma(a) = \{ x \in \mathbb{Q}_p^n : ||x - a||_p \leq p^\gamma \}$ the ball of radius $p^n$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B^n_\gamma(0) := B^n_0$. Note that $B^n_\gamma(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$, where $B_\gamma(a_i) := \{ x \in \mathbb{Q}_p : |x_i - a_i|_p \leq p^\gamma \}$ is the one-dimensional ball of radius $p^\gamma$ with center at $a_i \in \mathbb{Q}_p$. The ball $B^n_0(0)$ is equal to the product of $n$ copies of $B_0(0) := \mathbb{Z}_p$, the ring of $p$-adic integers.

### 2.2. The Bruhat-Schwartz space

A complex-valued function $\varphi$ defined on $\mathbb{Q}_p^n$ is called locally constant if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B^l_{l(x)}.$$  \hspace{1cm} (2)

A function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $S(\mathbb{Q}_p^n)$. For $\varphi \in S(\mathbb{Q}_p^n)$, the largest of such numbers $l = l(\varphi)$ satisfying (2) is called the exponent of local constancy of $\varphi$.

Let $S'(\mathbb{Q}_p^n)$ denote the set of all functionals (distributions) on $S(\mathbb{Q}_p^n)$. All functionals on $S(\mathbb{Q}_p^n)$ are continuous.

Set $\chi(y) = \exp(2\pi i \langle y \rangle_p)$ for $y \in \mathbb{Q}_p$. The map $\chi(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $\mathbb{Q}_p$ into $S$ (the unit circle) satisfying $\chi(y_0 + y_1) = \chi(y_0)\chi(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$.

### 2.3. Fourier transform

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we set $\xi : x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in S(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi(\xi : x)\varphi(x) \, d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition $\text{vol}(B^n_0) = 1$. The Fourier transform is a linear isomorphism from $S(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_x(\varphi)$ and $\hat{\varphi}$ for the Fourier transform of $\varphi$.

The Fourier transform $\mathcal{F}[f]$ of a distribution $f \in S'(\mathbb{Q}_p^n)$ is defined by

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]) \quad \text{for all } \varphi \in S(\mathbb{Q}_p^n).$$

The Fourier transform $f \rightarrow \mathcal{F}[f]$ is a linear isomorphism from $S'(\mathbb{Q}_p^n)$ onto $S'(\mathbb{Q}_p^n)$. Furthermore, $f = \mathcal{F}[\mathcal{F}[f](-\xi)]$.
2.4. The space $\mathfrak{M}_\lambda$

We denote by $\mathfrak{M}_\lambda$, $\lambda \geq 0$, the $\mathbb{C}$-vector space of locally constant functions $\varphi(x)$ on $\mathbb{Q}_p^n$ such that $|\varphi(x)| \leq C(1 + ||x||_p^\lambda)$, where $C$ is a positive constant. If the function $\varphi$ depends also on a parameter $t$, we shall say that $\varphi \in \mathfrak{M}_\lambda$ uniformly with respect to $t$, if its constant $C$ and its exponent of local constancy do not depend on $t$.

3. Pseudodifferential operators

We take $f(\xi) = \epsilon \xi_1^2 + p\epsilon \xi_2^2 - p\xi_3^2$ and $f^\alpha(\xi) = p\xi_1^2 + \xi_2^2 - \epsilon \xi_3^2$, with $\epsilon \in \mathbb{Z}$ a quadratic non-residue module $p$. Given $\alpha > 0$, we define the pseudodifferential operator with symbol $|f(\xi)|_p^\alpha$ by

$$ S(\mathbb{Q}_p^3) \rightarrow C(\mathbb{Q}_p^3) \cap L^2(\mathbb{Q}_p^3) \\
\varphi \rightarrow (f(\partial, \alpha) \varphi)(x) := F_{\xi \rightarrow x}^{-1} \left(|f(\xi)|_p^\alpha F_{x \rightarrow \xi} \varphi \right). $$

In [5] the authors show that the Fourier transform of the symbol is given by

$$ F\left[ |f(x)|_p^{-\alpha} \right] = \frac{1 - p^{-\alpha}}{1 - p^{2\alpha - 3}} \left[ (1 + p^{\alpha - 1})I_{V \cup V}(x) + p^{\alpha - \frac{3}{2}}(p^{2 - \alpha} + 1)I_{V_p}(x) \right]|f^\alpha(x)|^{\alpha - \frac{3}{2}}, $$

where

$$ I_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A,
\end{cases} $$

and for $\delta = 1, \epsilon, p, pe$ put $V_\delta := \{x \in \mathbb{Q}_p^3 \mid f^\alpha(x) \in \delta[\mathbb{Q}_p^3]^2 \}$. Observe that $V_{pe} = \emptyset$, otherwise the elliptic form $px_1^2 + x_2^2 - \epsilon x_3^2 - px_3^2 = 0$ would have non-trivial solution in $\mathbb{Q}_p^4$.

If we consider

$$ K_\alpha(x) := \frac{1 - p^{-\alpha}}{1 - p^{2\alpha - 3}} \left[ (1 + p^{\alpha - 1})I_{V \cup V}(x) + p^{\alpha - \frac{3}{2}}(p^{2 - \alpha} + 1)I_{V_p}(x) \right]|f^\alpha(x)|^{\alpha - \frac{3}{2}}, $$

then equation (3) can be written as

$$ \hat{K}_\alpha(x) = |f(x)|^{-\alpha}, \quad \alpha \neq \frac{3}{2} + \frac{2\pi \sqrt{-1}}{\ln p} \mathbb{Z}, \quad (4) $$

and as a distribution on $\sigma(\mathbb{Q}_p^3)$, $K_\alpha(x)$ possesses a meromorphic continuation to all $\alpha \neq \frac{3}{2} + \frac{2\pi \sqrt{-1}}{\ln p} \mathbb{Z}$ (see [5, Lemma 5]).

Since $F^{-1}\left(|f|^\alpha_p \right) = K_{-\alpha}$ we have $|f(\xi)|_p^\alpha F_{x \rightarrow \xi} \varphi \in L^1(\mathbb{Q}_p^3) \cap L^2(\mathbb{Q}_p^3)$, and the operator is well-defined. Therefore it is possible to write the operator as a convolution

$$ f(\partial, \alpha) \varphi = K_{-\alpha} \ast \varphi \quad \text{where} \quad f(\partial, \alpha) = \int_{\mathbb{Q}_p^3} K_{-\alpha}(y)(\varphi(x - y) - \varphi(x))d^3y, \quad (5) $$

[Revista Integración]
for $\varphi \in S(Q^3_p)$. Actually, the domain of the operator can be extended to the locally constant functions $u(x)$ such that
\[
\int_{||x||_p \geq 1} K_\alpha(x)|u(x)|d^3x < \infty. \tag{6}
\]

There exists an important inequality for elliptic polynomials, which is essential to do all the calculations (see [11]). In our case, for $f$ and $f^\circ$, the inequalities are given in the next lemma.

**Lemma 3.1 (\cite[Lemma 3]{6}).** Let $f, f^\circ$ be as above. Then

(i) $p^{-1} ||x||^2_p \leq |f(x)|_p \leq ||x||^2_p$, for every $x \in Q^3_p$,

(ii) $p^{-1} ||x||^2_p \leq |f^\circ(x)|_p \leq ||x||^2_p$, for every $x \in Q^3_p$.

### 4. The Cauchy problem

The Cauchy problem

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = -f(\partial, \alpha)u(x,t), & x \in Q^3_p, \quad 0 < t \leq T, \\
u(x,0) = \varphi(x),
\end{cases} \tag{7}
\]

where $\alpha > 0$, $T > 0$, $\varphi \in M_{2\lambda}$, $0 \leq \lambda < \alpha$ has a solution $u : Q^3_p \times [0,T] \to \mathbb{C}$ satisfying $u(x,t) \in M_{2\lambda}$ and

\[
u(x,t) := Z(x,t) * \varphi(x) = \int_{Q^3_p} Z(x-\eta,t) \varphi(\eta)d^3\eta, \tag{8}\]

where the heat kernel $Z(x,t)$ attached to $f(x)$ is

\[
Z(x,t) := Z(x,t; f, \alpha) = \int_{Q^3_p} \chi(-\xi \cdot x)e^{-t|f(\xi)|^2_p}d^3\xi,
\]

for $x \in Q^3_p$, $t > 0$ and $\alpha > 0$ (see [5]).

**Theorem 4.1.** The function $Z(x,t)$ has the following properties:

(i) $Z(x,t) \geq 0$ for any $t > 0$.

(ii) $\int_{Q^3_p} Z(x,t)d^3x = 1$ for any $t > 0$.

(iii) $Z(x,t) \leq Ct \left(||x||_p + t^{\frac{\alpha}{2}}\right)^{-3-2\alpha}$, where $C$ is a positive constant, for any $t > 0$ and any $x \in Q^3_p$.

(iv) $Z(x,t) * Z(x,t') = Z(x,t + t')$ for any $t, t' > 0$. 

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\( \lim_{t \to 0^+} Z(x, t) = \delta(x) \) in \( S'(\mathbb{Q}^3_p) \).

\( Z(x, t) \in C(\mathbb{Q}^3_p, \mathbb{R}) \cap L^1(\mathbb{Q}^3_p) \cap L^2(\mathbb{Q}^3_p) \) for any \( t > 0 \).

**Proof.** See Theorems 1, 2, Proposition 2 and Corollary 1 of [11].

\( \square \)

### 4.1. Markov processes over \( \mathbb{Q}^3_p \)

The space \( (\mathbb{Q}^3_p, \|\cdot\|_p) \) is a complete non-Archimedean metric space. Let \( \mathcal{B} \) be the Borel \( \sigma \)-algebra of \( \mathbb{Q}^3_p \); thus \( (\mathbb{Q}^3_p, \mathcal{B}, d^3x) \) is a measure space. By using the terminology and results of [8, Chapters 2, 3], we set

\[
p(t, x, y) := Z(x \mp y, t) \quad \text{for} \quad t > 0, \; x, y \in \mathbb{Q}^3_p,
\]

and

\[
P(t, x, B) = \begin{cases} 
\int_B p(t, y, x) d^3y & \text{for } t > 0, \; x \in \mathbb{Q}^3_p, \; B \in \mathcal{B}, \\
1_B(x) & \text{for } t = 0.
\end{cases}
\]

**Lemma 4.2.** With the above notation the following assertions hold:

(i) \( p(t, x, y) \) is a normal transition density.

(ii) \( P(t, x, B) \) is a normal transition function.

**Proof.** The result follows from Theorem 4.1 (see [8, Section 2.1] for further details).

\( \square \)

**Lemma 4.3.** The transition function \( P(t, x, B) \) satisfies the following two conditions:

\( \text{L}(B) \) For each \( u \geq 0 \) and compact \( B \),

\[
\lim_{x \to \infty} \sup_{t \leq u} P(t, x, B) = 0.
\]

\( \text{M}(B) \) For each \( \epsilon > 0 \) and compact \( B \),

\[
\lim_{t \to 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}^3_p \setminus B^\epsilon(x)) = 0.
\]

**Proof.** (i) By Theorem 4.1 (iii) and the fact that \( \|\cdot\|_p \) is an ultranorm, we have

\[
P(t, x, B) \leq Ct \int_B \left( \|x - y\|_p + t \frac{1}{p} \right)^{-3-2\alpha} d^3y
\]

\[
= t \left( \|x\|_p + t \frac{1}{p} \right)^{-3-2\alpha} \text{vol}(B) \quad \text{for } x \in \mathbb{Q}^3_p \setminus B.
\]

Therefore, \( \lim_{x \to \infty} \sup_{t \leq u} P(t, x, B) = 0. \)

\[\text{Revista Integración}\]
(ii) By using Theorem 4.1 (iii), \( \alpha > 0 \), and the fact that \( \| \cdot \|_p \) is an ultranorm, we have

\[
P(t, x, \mathbb{Q}^3_p \setminus B^3_t(x)) \leq C t \int_{\|x - y\|_p > \epsilon} \left( \|x - y\|_p + t \right)^{-\alpha} d^3 y
\]

\[
= C t \int_{\|z\|_p > \epsilon} \left( \|z\|_p + t \right)^{-\alpha} d^3 z
\]

\[
\leq C t \int_{\|z\|_p > \epsilon} \|z\|^{-\alpha} d^3 z
\]

\[
= C' (\alpha, \epsilon) t.
\]

Therefore,

\[
\lim_{t \to 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}^3_p \setminus B^3_t(x)) \leq \lim_{t \to 0^+} \sup_{x \in B} C' (\alpha, \epsilon) t = 0.
\]

Theorem 4.4. \( Z(x, t) \) is the transition density of a time and space homogeneous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.

Proof. The result follows from [8, Theorem 3.6] by using that \((\mathbb{Q}^3_p, \|x\|_p)\) is a semi-compact space, i.e., a locally compact Hausdorff space with a countable base, and \( P(t, x, B) \) is a normal transition function satisfying conditions \( L(B) \) and \( M(B) \) (cf. Lemmas 4.2 and 4.3).

5. **The first passage time**

The solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = - \int_{\mathbb{Q}^3_p} K_{-\alpha}(y) [u(x - y) - u(x, t)] d^3 y, & x \in \mathbb{Q}^3_p, \quad 0 < t \leq T, \\
u(x, 0) = \Omega(\|x\|_p),
\end{cases}
\]

is given by

\[
u(x, t) = \int_{\mathbb{Q}^3_p} \chi(-\xi \cdot x) \Omega(\|\xi\|_p) e^{-t|f(\xi)|_p} d^3 \xi.
\]

Among other properties, the solution (10) is infinitely differentiable with respect to \( t \geq 0 \), and for \( m \in \mathbb{N} \),

\[
\frac{\partial^m u}{\partial t^m}(x, t) = (-1)^m \int_{\mathbb{Q}^3_p} |f(\xi)|^m p \chi(-\xi \cdot x) \Omega(\|\xi\|_p) e^{-t|f(\xi)|_p} d^3 \xi.
\]
Lemma 5.1 ([5, Lemma 6]). For $\text{Re}(\alpha) > 0$ we have

$$- \left( \int_{\|x\|_p > 1} K_{-\alpha}(x) \, d^3x \right) = \frac{p^{-\alpha}(1 - p^{-1})(p^\alpha + p^{-1} + p^{-2})}{1 - p^{-2\alpha - 3}} \leq 1.$$ 

Definition 5.2. The random variable $\tau_{Z^3_p}(\omega) := \tau(\omega) : Q^3_p \to \mathbb{R}^+$ defined by

$$\inf \{ t > 0; X(t, \omega) \in Z^3_p \mid \text{there exists } t' \text{ such that } 0 < t' < t \text{ and } X(t', \omega) \notin Z^3_p \}$$

is called the first passage time of a path of the random process $X(t, \omega)$ entering the domain $Z^3_p$.

Note that the initial condition in (9) implies that

$$\Pr \left( \{ \omega \in Q^3_p; X(0, \omega) \in Z^3_p \} \right) = 1.$$

Definition 5.3. We say that $X(t, \omega)$ is recurrent with respect to $Z^3_p$ if

$$\Pr \left( \{ \omega \in Q^3_p; \tau(\omega) < \infty \} \right) = 1. \quad (12)$$

Otherwise we say that $X(t, \omega)$ is transient with respect to $Z^3_p$.

The meaning of (12) is that every path of $X(t, \omega)$ is sure to return to $Z^3_p$. If (12) does not hold, then there exist paths of $X(t, \omega)$ that abandon $Z^3_p$ and never go back.

By using the same arguments given by Chacón in [6] we define the survival probability as

$$S(t) := S_{Z^3_p}(t) = \int_{Z^3_p} \varphi(x, t) d^3x,$$

which is the probability that a path of $X(t, \omega)$ remains in $Z^3_p$ at the time $t$. Because there are no external forces acting on the random walk, we have

$$S'(t) = \left( \text{Probability that a path of } X(t, \omega) \text{ goes back to } Z^3_p \text{ at the time } t \right) - \left( \text{Probability that a path of } X(t, \omega) \text{ exits } Z^3_p \text{ at the time } t \right)$$

$$= g(t) - C \cdot S(t) \text{ with } 0 < C \leq 1. \quad (13)$$

In order to determine the probability density function $g(t)$ we compute $S'(t)$. 

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\[ S'(t) = \int_{\mathbb{R}^3} \frac{\partial \varphi(x,t)}{\partial t} d^3x = -\int \int K_{-\alpha}(y)[u(x-y,t) - u(x,t)] d^3y d^3x \]

\[ = -\int \int \int K_{-\alpha}(y)u(x-y,t) d^3y d^3x + \int \int K_{-\alpha}(y)u(x,t) d^3y d^3x \]

\[ = -\int \int K_{-\alpha}(y)u(y,t) d^3y + \int \int K_{-\alpha}(y) d^3y \int u(x,t) d^3x \]

\[ = -\int \int K_{-\alpha}(y)u(y,t) d^3y - \left( \frac{p^{-\alpha}(1-p^{-1})(p^\alpha + p^{-1} + p^{-2})}{1-p^{2\alpha-3}} \right) S(t). \]

Therefore,

\[ g(t) = -\int_{|y| > 1} K_{-\alpha}(y)u(y,t) d^3y, \quad (14) \]

and the constant \( C := \frac{p^{-\alpha}(1-p^{-1})(p^\alpha + p^{-1} + p^{-2})}{1-p^{2\alpha-3}} \) satisfies \( 0 < C \leq 1 \).

**Proposition 5.4.** The probability density function \( f(t) \) of the random variable \( \tau(\omega) \) satisfies the non-homogeneous Volterra equation of second kind

\[ g(t) = \int_0^\infty g(t-\tau)f(\tau)d\tau + f(t). \quad (15) \]

**Proof.** The result follows from (14) by using the argument given in the proof of Theorem 1 in [4]. \[ \checkmark \]

**Lemma 5.5.** For \( f(x) = ex_1^2 + px_2^2 - px_3^2 \) and \( \text{Re}(s) > 0 \) the following formulas hold:

\( (i) \)

\[ \int_{|x| = 1} \frac{1}{s + p^{-2\gamma\alpha}}|f(y)|_p d^3y = \frac{1 - p^{-1}}{s + p^{-2\gamma\alpha}} + \frac{p^{-1}(1 - p^{-2})}{s + p^{-2\gamma\alpha} - \alpha}. \]

\( (ii) \) If \( ||\xi||_p \geq p \), then there exist constants \( C_1 \) and \( C_2 \) such that

\[ \int_{|x| = 1} \frac{\chi(y \cdot \xi)}{s + p^{-2\gamma\alpha}}|f(y)|_p d^3y = \begin{cases} 
C_1 & \text{if } ||\xi||_p = p, \\
C_2 & \text{if } ||\xi||_p > p.
\end{cases} \]

**Proof.** By using the same technique as in [6, Lemma 15] we write \( U := \sqcup U^{(i)} \), where \( U^{(i)} = U_1^{(i)} \times U_2^{(i)} \times U_3^{(i)} \) and

\[ U_3^{(i)} := \begin{cases} 
Z_p & \text{if } i_j = 1, \\
Z_p^* & \text{if } i_j = 0.
\end{cases} \]
Therefore, by using (16) and Fubini’s Theorem we have
\[\text{where} \]
\[\text{The Laplace transform} \ G(s) \ \text{of} \ g(t) \ \text{is given by} \ G(s) = G_1(s) + G_2(s), \]
\[\text{where} \]
\[G_1(s) = -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^\alpha)}{1-p^{-2\alpha-3}} \]
\[\times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=\nu}^{\infty} p^{-3\gamma} \left( \frac{1-p^{-1}}{s+p^{-2\gamma\alpha}} + \frac{(1-p^{-2})p^{-1}}{s+p^{-2\gamma\alpha-\alpha}} \right), \]

and
\[G_2(s) = -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^\alpha)}{1-p^{-2\alpha-3}} \]
\[\times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} p^{-3(\nu-1)} \left( \frac{C_1}{s+p^{-2(\nu-1)\alpha}} - \frac{C_2}{s+p^{-2(\nu-1)\alpha-\alpha}} \right). \]

**Proof.** We first note that, if \(\text{Re}(s) > 0\), then
\[K_{-\alpha}(x) e^{-st} e^{-t|f(\xi)|_p^\alpha} \Omega \left( \|\xi\|_p \right) \in L^1 \left( (0, \infty) \times \mathbb{Q}_p^3 \times \mathbb{Q}_p^3 \setminus \mathbb{Z}_p^3, \, dtd\xi d\xi^3x \right). \] (16)

Therefore, by using (16) and Fubini’s Theorem we have
\[G(s) = \int_0^\infty e^{-st} g(t) dt \]
\[= -\int_0^\infty e^{-st} \int_{|\xi|_p^3>1} K_{-\alpha}(x) u(x, t) d^3x dt \]
\[= -\int_0^\infty e^{-st} \int_{|\xi|_p^3>1} K_{-\alpha}(x) \int_{\mathbb{Q}_p^3} \chi(-\xi \cdot x) \Omega(|\xi||_p) e^{-t|f(\xi)|_p^\alpha} \xi d^3\xi d^3x dt \]
\[= -\int_{|\xi|_p^3>1} K_{-\alpha}(x) \int_{\mathbb{Z}_p^3} \chi(-\xi \cdot x) \int_0^\infty e^{-s+t|f(\xi)|_p^\alpha} dtd\xi d^3x \]
\[= -\int_{|\xi|_p^3>1} K_{-\alpha}(x) \int_{\mathbb{Z}_p^3} \frac{\chi(-\xi \cdot x)}{s + |f(\xi)|_p^\alpha} d^3\xi d^3x. \]

\[\square\]
After the change of variables $x = p^{-\nu} y$ and $\xi = p^{\gamma} y'$, and due to the fact that $K_{-\alpha}(p^{-\nu} y) = p^{-2\alpha\nu - 3\nu} K_{-\alpha}(y)$, we obtain

$$G(s) = -\sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{||y||_p = 1} K_{-\alpha}(y) \sum_{\gamma=0}^{\infty} p^{-3\gamma} \int_{||y'||_p = 1} \frac{\chi(-p^{-\nu+\gamma} y \cdot y')}{s + p^{-2\alpha\gamma}|f(y')|^\alpha} d^3 y' d^3 y.$$

In order to calculate the interior integral we split the set $||y'||_p = 1$ into two parts, when $||p^{-\nu+\gamma} y \cdot y'||_p \leq 1$, and when $||p^{-\nu+\gamma} y \cdot y'||_p > 1$. The first case occurs when $\gamma \geq \nu$, and then $\chi(-p^{-\nu+\gamma} y \cdot y') = 1$. The second case occurs when $\gamma = 0, \ldots, \nu - 1$. By Lemma 5.5 $G(s)$ takes the form $G(s) = G_1(s) + G_2(s)$, where

$$G_1(s) := -\sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{||y||_p = 1} K_{-\alpha}(y) \sum_{\gamma=0}^{\infty} p^{-3\gamma} \int_{||y'||_p = 1} \frac{1}{s + p^{-2\alpha\gamma}|f(y')|^\alpha} d^3 y' d^3 y$$

$$= \frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^\alpha)}{1-p^{-2\alpha-3}} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=0}^{\infty} p^{-3\gamma} \left( \frac{1-p^{-1}}{s + p^{-2\gamma\alpha}} + \frac{(1-p^{-2})p^{-1}}{s + p^{-2\gamma\alpha - \alpha}} \right),$$

and

$$G_2(s) := -\sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{||y||_p = 1} K_{-\alpha}(y) \sum_{\gamma=0}^{\nu-1} p^{-3\gamma} \int_{||y'||_p = 1} \frac{\chi(-p^{-\nu+\gamma} y \cdot y')}{s + p^{-2\alpha\gamma}|f(y')|^\alpha} d^3 y' d^3 y$$

$$= \frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^\alpha)}{1-p^{-2\alpha-3}} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} p^{-\nu\alpha-1} \left( \frac{G_1}{s + p^{-2\gamma\alpha - 1} - \alpha} - \frac{G_2}{s + p^{-2\gamma\alpha - 1} - \alpha} \right).$$

**Theorem 5.7.** (i) If $\alpha \geq \frac{3}{2}$, then $X(t, \omega; W)$ is recurrent with respect to $\mathbb{Z}_p^3$.

(ii) If $\alpha < \frac{3}{2}$, then $X(t, \omega; W)$ is transient with respect to $\mathbb{Z}_p^3$.

**Proof.** By Proposition 5.4, the Laplace transform $F(s)$ of $f(t)$ equals $\frac{G(s)}{1+G(s)}$, where $G(s)$ is the Laplace transform of $g(t)$, and thus

$$F(0) = \int_0^\infty f(t) dt = 1 - \frac{1}{1+G(0)}.$$

Hence, in order to prove that $X(t, \omega; W)$ is recurrent is sufficient to show that $G(0) = \lim_{s \to 0} G(s) = \infty$, and to prove that it is transient, that $G(0) = \lim_{s \to 0} G(s) < \infty$.

(i) Take $s \in \mathbb{R}$, $s > 0$ and set $s = p^{-2\alpha\nu} = p^{-2\gamma\alpha}$; note that $s \to 0^+ \iff v \to \infty$ ($v = \gamma$). Now, taking only the first term of $G_1(s)$ we have

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\[ G(s) > \frac{-p^{-\alpha}(1 - p^{2\alpha})(1 - p^{-1})(p^{-1} + p^{-2} + p^{\alpha})}{1 - p^{-2\alpha - 3}} \times \sum_{\gamma=1}^{\infty} p^{-3\gamma} \left( \frac{1 - p^{-1}}{s + p^{-2\gamma\alpha}} + \frac{(1 - p^{-2})p^{-1}}{s + p^{-2\gamma\alpha - \alpha}} \right) + G_2(s). \]

We get \( G_2(p^{-2\nu\alpha}) < \infty \), but the first sum diverges if \( \alpha \geq \frac{3}{2} \). Then,

\[ \lim_{s \to 0^+} G(s) = \infty. \]

(ii) Now

\[ |G(s)| \leq \frac{-p^{-\alpha}(1 - p^{2\alpha})(1 - p^{-1})(p^{-1} + p^{-2} + p^{\alpha})}{1 - p^{-2\alpha - 3}} \times \sum_{\nu=1}^{\infty} \sum_{\gamma=\nu}^{\infty} \sum_{\gamma=\nu}^{\infty} p^{-3\gamma} \left( \frac{1 - p^{-1}}{p^{-2\gamma\alpha}} + \frac{(1 - p^{-2})p^{-1}}{p^{-2\gamma\alpha - \alpha}} \right) + G_2(0). \]

One sees easily that \( G_2(0) \) converges, and that the double series converges if \( \alpha > \frac{3}{2} \). Therefore \( \lim_{s \to 0^+} G(s) < \infty \).

References


[Revista Integración]
The problem of the first return attached to a pseudodifferential operator in dimension 3


