On Intervals, Sensitivity Implies Chaos

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Abstract. In this note we investigate which properties can be derived for a continuous function f defined on an interval I if the only a priori given information is its sensitive dependence on initial conditions. Our main result is the following: If f is sensitive, then f is chaotic, in the sense of Devaney, on a nonempty interior subset of I; the set of aperiodic points is dense in I as well as the set of asintotically periodic points; moreover, f has positive topological entropy.

1. Introduction

The three conditions of Devaney's definition of chaos for mappings are: density of periodic points, topological transitivity, and sensitive dependence on initial conditions (see [3]). It is known that the first two conditions imply the third provided that the mapping is defined on a perfect space (see [1]).

In this note we will focus on functions defined on the interval. In this setting, sensitivity on initial conditions implies by itself a very interesting dynamics as we shall see.

Let I be the interval $[0, 1]$ in the real line $\mathbb{R}$. Let $f : I \rightarrow I$ be a continuous function. It is known (see [7] and [8]) that if f is transitive on I, then the discrete dynamical system induced by f is chaotic in the sense of Devaney on I. On intervals, transitivity is a strong condition.

The third condition in Devaney's definition is very important. Most of the authors agree in one point: chaotic dynamics must show sensitive dependence on initial conditions. The main result of this note is a partial answer to the following question: What can we say about the dynamics of $f : I \rightarrow I$ if we only know that f exhibits sensitive dependence on initial conditions?

Although on spaces with dimension greater than one sensitivity generally does not imply complex dynamics (we shall give an example on the torus showing this), on the interval sensitivity implies a complex behavior. Our main result is stated as follows:

Theorem 1.1. *If $f : I \to I$ exhibits sensitive dependence on initial conditions, then the next two conditions hold:*

   i) *There exists an invariant closed subset $A \subset I$ such that $f\big|_A$ is chaotic on A in the sense of Devaney. Furthermore, this chaotic set has nonempty interior.*

   ii) *The sets $\Gamma = \{x \in I \mid \omega(x, f)$ has infinite cardinality$\}$ and its complement, $\Psi = I - \Gamma$, are both dense in I.*

In section 4 we will show that sensitivity also implies that the topological entropy of $f : I \to I$ is positive. This fact, positive entropy of f, is accepted by other authors (see [2]) as a criterion to decide whether f exhibits chaotic dynamics or not.

2.   Basic Definitions

Let x be a point of I, and the orbit of x under f is the set

$$o(x, f) = \{x, f(x), f^2(x), \ldots\},$$

where f^n is the composition of f with itself n times, $f^n = f \circ f \circ \cdots \circ f$. Also, f^0 is the identity function and $f^1 = f$. We say that x is a *periodic point* of f (of *period* n) if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$ and $f^k(x) \neq x$ for all $1 \leq k < n$. If $f(x) = x$, then x is a periodic point of period one. Let us denote by $\mathrm{Per}(f)$ the set of all periodic points of f. Each interval or subinterval referred in this note has not empty interior.

Let y be a point in I; the set of all limit points of $o(y, f)$ is denoted by $\omega(y, f)$, i.e.,

$$\omega(y, f) = \left\{ z \in I \mid \text{ there exists } \{n_i\} \subset \mathbb{N}, \lim_{n_1 \to \infty} f^{n_i}(y) = z \right\}.$$

We say that $y \in I$ is an asymptotically periodic point of f if there exists $x \in \mathrm{Per}(f)$ such that $\lim_{n \to \infty} |f^n(y) - f^n(x)| = 0$. Notice that in this case $\omega(y, f) = o(x, f)$.

It is known (see [2]) that the set $\omega(y, f)$ has finite cardinality if and only if y is asymptotically periodic point. Therefore if the cardinality of $\omega(y, f)$ is not finite the behavior of $o(y, f)$ does not tend to a periodic motion, thus its dynamics is not simple. The point $y \in I$ is said to be an *aperiodic point* of f if $\omega(y, f)$ is not a finite set.

It is said that f *exhibits sensitive dependence on initial conditions* (or f is *sensitive*) on I if there exists $\delta > 0$ (called *constant of sensitivity*) such that for any x in I and for any $\varepsilon > 0$, there exist y in I with $|y - x| < \varepsilon$, and $n \geq 0$ such that $|f^n(y) - f^n(x)| > \delta$. The function f is said to be *topologically transitive* (or f is *transitive*) on I if for any pair of nonempty open sets A and B in I there exists $n > 0$ such that $f^n(A) \cap B \neq \emptyset$. Recall that, on perfect compact sets, transitivity is equivalent to the existence of a dense orbit. Let $A \subset I$ be a perfect set, we say that f is chaotic on A if A is invariant under f, i.e. $f(A) = A$, $\mathrm{Per}(f\big|_A)$ is dense in A, and $f\big|_a$ is both transitive and sensitive on A. In such a case it is said that A is a *chaotic set*.

3. The Sensitivity and a Digraph

Throughout this section we will assume that $f : I \to I$ is sensitive on I with $\delta > 0$ its constant of sensitivity.

We associate a digraph G with the function f in this way: Let

$$P = \{t_0 = 0, \, t_1, \, \ldots, \, t_m = 1\}$$

be a partition of I with $||P|| < \dfrac{\delta}{4}$. The vertices of G will be the subintervals $A_k = [t_{k-1}, t_k]$, $1 \leq k \leq m$. We put an arrow from A_i to A_j if there exists $n \in \mathbb{N}$ such that $f^n(A_i) \supset A_j$ and the length of the interval $f^n(A_i)$ is equal to or larger than δ, $l(f^n(A_i)) \geq \delta$.

Lemma 3.1. *From each vertex A_i, $1 \leq i \leq m$, there are at least three arrows to consecutive vertices A_j, A^{j+1} and A^{j+2}.*

Proof. Let i, $1 \leq i \leq m$. Since f is sensitive, there exists $n_i \in \mathbb{N}$ such that $l(f^n(A_i)) > \delta$. Since for any j we have that $(A_j) < \dfrac{\delta}{4}$ (because of $||P|| < \dfrac{\delta}{4}$) we can find A_j, A^{j+1} and A^{j+2} such that $(A_j \cup A_{j+1} \cup A_{j+2}) \subset f^{n_i}(A_i) \geq \delta$. ☑

Lemma 3.2. *If there exist an arrow from A_i to A_j and an arrow from A_j to A_k, then there exists an arrow from A_i to A_k.*

Proof. The arrows $A_i \to A_j$ and $A_j \to A_k$ give us two natural numbers n_i and n_j such that $f^{n_i}(A_i) \supset A_j$ and $f^{n_j}(A_j) \supset A_k$ with $l(f^{n_i}(A_i)) \geq \delta$ and $l(f^{n_j}(A_j)) \geq \delta$. Then $f^{n_i + n_j}(A_i) \supset f^{n_j}(A_j) \supset A_k$, and $l(f^{n_i + n_j}(A_i)) \geq l(f^{n_j}(A_j)) \geq \delta$. ☑

The degree of the vertex A_i, $dg(A_i)$, will be the number of arrows starting at A_i. If there exists an arrow from A_i to A_j we say A_j is *attainable* from A_i.

Remark. It is immediate from Lemma 3.2 that if A_j is attainable from A_j, then $dg(A_i) \geq dg(A_j)$. All the vertices which are attainable from A_j are attainable from A_i as well.

Take any i, $1 \leq i \leq m$, and consider the vertex A_i. Among all attainable vertices from A_i, choose A_j such that $dg(A_j) = \min\{dg(A_k) \mid A_k \text{ is attainable from } A_i\}$. Let us call $g = dg(A_j)$ and $A_{j_1}, \ldots, A_{j_g}$ all the attainable vertices from A_j.

Assume these vertices are in this order: If $k < 1$ and $x \in A_{jk}$ and $y \in A_{j_l}$, then $x \leq y$. Notice that each pair of these vertices (subintervals) have at most one common point, and if $k < l - 1$, then $A_{j_k} \cap A_{j_l} = \emptyset$. Since any A_{j_k}, $1 \leq k \leq g$, is attainable from A_i, $dg(A_{j_k}) \geq dg(A_j) = g$. On the other hand, by Lemma 3.2, $dg(A_{j_k}) \leq dg(A_j) = g$. Thus, for any A_{j_k} we have $dg(A_{j_k}) = g$. Furthermore, by the same lemma, the g vertices that are attainable from any A_{j_k} must be $A_{j_1}, \ldots, A_{j_g}$. Now, it is immediate that for any k, $1 \leq k \leq g$, we have:

$$\bigcup_{n=1}^{\infty} f^n(A_{j_1}) = \bigcup_{n=1}^{\infty} f^n(A_{j_k}).$$

Let A be the closure of the previous union: $A = cl(\bigcup_{n=1}^{\infty} f^n(A_{j_1}))$.

Lemma 3.3. *The set A satisfies the following three conditions:*

  i) *The interior of A is not empty,* $\operatorname{int}(A) \neq \emptyset$.

  ii) *For any $x \in A$ and any $\varepsilon > 0$, there exists $[c,d] \subset I$, such that $[c,d] \subset (x-\varepsilon, x+\varepsilon)$ and $[c,d] \subset A$. Note this condition implies that A is perfect.*

  iii) $f(A) = A$.

Proof.

  i) From these arrows $A_{j_1} \to A_{j_k}$, $k = 1, 2, \ldots, g$, it follows that

  $$(A_{j_1} \cup A_{j_2} \cup A_{j_g}) \subset A.$$

  Therefore, $\operatorname{int}(A) \neq \emptyset$.

  ii) Because of the sensitivity, f satisfies this claim: If B is any subset of I with $\operatorname{int}(B) \neq \emptyset$, then $\operatorname{int}(f(B)) \neq \emptyset$.

  Let $x \in A$ and let $\varepsilon > 0$. There exists $y \in \bigcup_{n=1}^{\infty} f^n(A_{j_1})$ such that $|x - y| < \varepsilon$. Hence there exist $z \in A_{j_1}$ and $n_1 \in \mathbb{N}$ such that $f^{n_1}(z) = y$. Since f^{n_1} is a continuous map there exists $\gamma > 0$ such that $[z - \gamma, z + \gamma] \cap A_{j_1}$ is a closed subinterval with

  $$f^{n_1}([z - \gamma, z + \gamma] \cap A_{j_1}) \subset ((x - \varepsilon, x + \varepsilon) \cap A).$$

  Thus, there exists $[c, d] \subset ((x - \varepsilon, x + \varepsilon) \cap A)$.

  iii) Let us prove this part in two steps.

  First. If B is a subset of I invariant under f, then $f(cl(B)) = cl(B)$.

  Since $B = f(B) \subset f(cl(B))$ and $f(cl(B))$ is a closed set in I, then $cl(B) \subset f(cl(B))$.

  Now, let $y \in f(cl(B))$. There exist $x \in cl(B)$ and a sequence $\{x_1, x_2, \cdots\}$ in B such that $f(x) = y$ and $\lim_{\to \infty} x_n = x$. Since f is continuous and $B = f(B)$, we obtain $\lim_{\to \infty} f(x_n) = y$ and $\{f(x_n)\} \subset B$. Thus, $y \in cl(B)$.

  Second. Let us prove that $f\left(\bigcup_{n=1}^{\infty} f^n(A_{j_1})\right) = \bigcup_{n=1}^{\infty} f^n(A_{j_1})$.

  It is immediate that

  $$f\left(\bigcup_{n=1}^{\infty} f^n(A_{j_1})\right) = \bigcup_{n=2}^{\infty} f^n(A_{j_1}) \subset \bigcup_{n=1}^{\infty} f^n(A_{j_1}).$$

  Furthermore, because of the arrow $A_{j_1} \to A_{j_1}$ it follows that

  $$A_{j_1} \subset \bigcup_{n=2}^{\infty} f^n(A_{j_1}).$$

Therefore,

$$\bigcup_{n=1}^{\infty} f^{n+1}(A_{j_1}) \subset \bigcup_{n=2}^{\infty} f^n(A_{j_1}).$$

This completes the proof of Lemma 3.3. ☑

Theorem 3.4. $f\big|_A : A \to A$ *is chaotic on* A.

Proof. It is enough to show that the set $\mathrm{Per}(f\big|_A)$ is dense in A, and $f\big|_A$ is transitive on A.

Step 1. We will prove the density of $\mathrm{Per}(f\big|_A)$.

Let $(a, b) \subset I$ such that $(a, b) \cap A \neq \emptyset$. By part ii) of Lemma 3.3, there exist two closed subintervals, $[c, d]$ and $[s, t]$, and a natural number n_1 such that

$$[c, d] \subset A_{j_i}, \quad [s, t] \subset ((a, b) \cap A), \quad \text{and} \quad f^{n_1}([c, d]) = [s, t].$$

Since f is sensitive there exist $n_2 \in \mathbb{N}$ such that $f^{n_2}([s, t])$ contains an interval A_{j_k} for some $1 \leq k \leq g$. The arrow $A_{j_k} \to A_{j_1}$ give us $n_3 \in \mathbb{N}$ such that $f^{n_3}(A_{j_k}) \supset A_{j_1}$. Hence $[c, d] \subset f^n([c, d])$, where $n = n_1 + n_2 + n_3$. Thus f^n has a fixed point in $[c, d]$, and therefore f has a periodic point in $[c, d]$. Since $f^{n_1}([c, d]) = [s, t] \subset ((a, b) \cap A)$, $f|_A$ has a periodic point in $(a, b) \cap A$.

Step 2. Let us now prove that $f|_A$ is transitive in A.

Let B and C be two open sets in I such that $B \cap A \neq \emptyset$ and $C \cap A \neq \emptyset$. By Lemma 3.3, there exist three closed subintervals, U, V and W, and a natural number m_1 such that

$$U \subset A_{j_1}, \quad V \subset (B \cap A), \quad W \subset (C \cap A) \quad \text{and} \quad f^{m_1}(U) = W.$$

Due to the sensitivity and following the argument used in step 1, there exists $m_2 \in \mathbb{N}$ such that $A_{j_1} \subset f^{m_2}(V)$. Thus, taking $m = m_1 + m_2$ we have that $W \subset f^m(V)$, and therefore

$$f^m(B \cap A) \cap (C \cap A) \neq \emptyset. \qquad\qquad ☑$$

Proposition 3.5. *Take any map in the interval* $g : I \to I$, *sensitive or not, and assume* B *is perfect and a chaotic set for* g. *Then either* B *is a Cantor set or* B *is a finite union of closed intervals.*

Proof. Recall that any nonempty compact perfect totally disconnected metric space is a Cantor set (see [4]). The set B is already compact and perfect. And it is a metric space as well.

If every component of B has empty interior, B is totally disconnected. Hence B is a Cantor set.

Let C be a component of B with nonempty interior. Notice that C is a closed subinterval of I. Since $g|_B$ is chaotic, there exists a periodic point of g in C. Let us assume that the period of that point is m. Also there exists another point in C, say y, with a dense orbit in B under $g|_B$. That is to say, $B = cl(o(y, g))$.

Now consider the following three conditions:

I) Since $g^m(C) \cap C \neq \emptyset$, $g^m(C) \subset C$.

II) It follows that $\cup_{n=0}^{\infty} g^n(C) \subset C \cup g(C) \cup \cdots \cup g^{m-1}(C)$, and

III) $B = cl(o(y,g)) \subset (C \cup g(C) \cup \cdots \cup g^{m-1}(C)) \subset B$.

Thus $B = C \cup g(C) \cup \cdots \cup g^{m-1}(C)$.   ☑

Returning to f, let us now prove that the set of asymptotically periodic points and the set of aperiodic points are both dense in I.

Proposition 3.6. *Let* $\Gamma = \{x \in I \mid \omega(x,f) \text{ has infinite cardinality}\}$ *and* $\Psi = I - \Gamma$. *Then the sets* Γ *and* Ψ *are both dense in* I.

Proof. Let $x \in I$ and $\varepsilon > 0$. We take a partition P as above adding the next condition: $\| P \|$ is small enough such that there exists $A_i = [t_{i-1}, t_i] \subset (x-\varepsilon, x+\varepsilon)$.

Due to these arrows: $A_i \to A_j$ and $A_j \to A_{j_1}$, and to the definition of set A,

$$A = cl \left(\bigcup_{n=1}^{\infty} f^n(A_{j_i}) \right),$$

there exists a natural number m such that $f^m(A_i) \cap A$ has not empty interior. Hence there exist $a \in I, |x - a| < \varepsilon$, and $b \in I, |x - b| < \varepsilon$, such that $f^m(a) \in \text{Per}(f^n|_A)$ and the orbit of $f^m(b) \in A$ is dense in A (since $f|_A$ is transitive on A). This implies that the set $\omega(a,f)$ has finite cardinality and the set $\omega(b,f)$ has infinite cardinality.   ☑

4.   Sensotivity and Topological Entropy

There is a very important concept related to complex behavior generated by mappings: *topological entropy* (see [9] for definition and main properties).

The entropy of a mapping can be zero, positive or infinite. The dynamics generated by functions on the interval with zero entropy, $ent(f) = 0$, have been studied. In particular it is known (see [6]) that if $f|_D : D \to D$ is transitive on D, $D \subset I$, and $ent(f) = 0$, then either D is a periodic orbit or D is a Cantor set where $f|_D$ is minimal (that is, every point of D has dense orbit in D). Thus, in such a second case there are no periodic points of f in D. Hence zero entropy implies no chaotic dynamics on any subset of I. Therefore, by Theorem 3.4, we have: If f is sensitive on I, then the topological entropy of f is positive or infinite.

Lemma 4.1. *Let* $n \in \mathbb{N}$. *Then* $ent(f^n) = n \cdot ent(f)$.

Lemma 4.2. *If there exist two disjoint closed subintervals of* I, E *and* F, *such that* $E \cup F \subset f(E) \cap f(F)$, *then* $ent(f) \geq \log(2)$.

From now on, in this section, let us assume $f : I \to I$ is sensitive and δ is its constant of sensitivity. Also, let us have available the results and notation that we produced in the previous section.

Lemma 4.3. *Let V and W be two subsets of I. If there exist two natural numbers p and q such that $V \cup W \subset f^p(V)$ and $V \cup W \subset f^q(W)$, then there exists a natural number n such that $V \cup W \subset f^n(V) \cap f^n(W)$.*

Proof. Consider the following inclusions:

$$(V \cup W) \subset f^p(V) \subset f^p(V \cup W) \subset f^p(f^q(W)) = f^{p+q}(W),$$
$$(V \cup W) \subset f^q(W) \subset f^q(V \cup W) \subset f^q(f^p(V)) = f^{q+p}(V).$$

Taking $n = p + q$ the proof is complete. ☑

Theorem 4.4. *The topological entropy of f is positive.*

Proof. Let us consider the subintervals A_{j_1} and A_{j_3}. We know that the digraph G has the following arrows: $A_{j_1} \to A_{j_3}$ and $A_{j_3} \to A_{j_1}$. Hence there exist $p, q \in \mathbb{N}$ such that $A_{j_1} \subset f^p(A_{j_1})$ and $A_{j_1} \subset f^q(A_{j_3})$. By Lemma 3.1, $A_{j_3} \subset f^p(A_{j_1})$ and $A_{j_3} \subset f^q(A_{j_3})$ as well.

$$A_{j_1} \cup A_{j_3} \subset f^p(A_{j_1}) \quad \text{and} \quad A_{j_1} \cup A_{j_3} \subset f^q(A_{j_3}).$$

Due to the previous lemma, there exists $n \in \mathbb{N}$ such that

$$A_{j_1} \cup A_{j_3} \subset f^n(A_{j_1}) \cap f^n(A_{j_3}).$$

$A_{j_1} \cap A_{j_3} = \emptyset$, $ent(f^n) \geq \log(2)$. Therefore, $ent(f) > 0$. ☑

5. Some Remarks About Sensitivity

Let us finish this note with two remarks on sensitivity.

1.- Consider the next family of continuous functions defined on $I = [0, 1]$. We say $f : I \to I$ is of type $\mathcal{L}$ if the following two conditions hold:

i) There exist a partition of I, $x_0 = 0 < x_1 < \ldots < x_p = 1$, such that the graph of f is a polygonal curve with vertices on $(0, f(0))$, $(x_1, f(x_1))$, ..., $(1, f(1))$. That is, f is a piecewise monotone linear function, and

ii) The slope of any straight line segment of that polygonal curve has absolute value larger than one.

The family $\mathcal{L}$ is studied in [5]. For the sake of completness in the sequel we prove that if f is of type $\mathcal{L}$, then f is sensitive on I.

Notice that if f and g are of type $\mathcal{L}$, then $f \circ g$ is of type $\mathcal{L}$ as well. Thus for any $n \in \mathbb{N}$, f^n is of type $\mathcal{L}$ provided f is. Let f be of type $\mathcal{L}$. Let $m_1, \ldots, m_p$ be the slopes of the straight line segments in the graph of f. We denote by ε_p the $\min\{|m_i|, 1 \leq i \leq p\}$. Notice that $\varepsilon_p > 1$.

Lemma 5.1. *Let f be of type $\mathcal{L}$. Then $\varepsilon_{f^2} \geq (\varepsilon_f)^2$. Furthermore, for any $n \geq 2$, $\varepsilon_{f^n} \geq (\varepsilon_f)^n$.*

Proof. Let $x \in I$ be such that $x \neq x_1$ and $f(x) \neq x_i$, for any $i \leq i \leq p$. By the Chain Rule we have that

$$\left| (f^2)'(x) \right| = |f'(f(x))| \, |f'(x)| \geq (\varepsilon_f)^2,$$

hence $\varepsilon_{f^2} \geq (\varepsilon_f)^2$.

By the same argument and induction it is immediate that $\varepsilon_{f^n} \geq (\varepsilon_f)^n$ for any $n \in \mathbb{N}$. ☑

Lemma 5.2. *Let f be of type $\mathcal{L}$ with $\varepsilon_f > 2$. Let δ_f be the minimum of the following distances: $x_i - x_{i-1}$, $1 \leq i \leq p$. Then for any subinterval $J \subset I$, there exists $k \geq 0$ such that the length of $f^k(J)$ is larger than δ_f, $l(f^k(j)) > \delta_f$.*

Proof. Assume, on the contrary, that there exists J, subinterval of I, such that $l(f^k) \leq \delta_f$ for any k. Thus for any k the interval $f^k(J)$ has, in its interior, at most one point x_i. The length of the intervals $f^k(J)$ satisfy the following inequalities:

$$l(f(J)) \geq \frac{\varepsilon_f}{2} l(j), \quad l\left(f^2(J)\right) \geq \left(\frac{\varepsilon_f}{2}\right)^2 l(J), \quad \ldots, \quad l\left(f^k(J)\right) \geq \left(\frac{\varepsilon_f}{2}\right)^k l(J).$$

Since $\dfrac{\varepsilon_f}{2} > 1$ we have $\left(\dfrac{\varepsilon_f}{2}\right)^k \to \infty$ as $k \to \infty$. Thus $l(f^k(J)) \to \infty$. But, this is impossible. ☑

Proposition 5.3. *If f is of type $\mathcal{L}$, then f is sensitive on I.*

Proof. By Lemma 5.2, there exists n such that $\varepsilon_{f^n} > 2$. Hence, for any interval $J \subset I$ there exists $k \in \mathbb{N}$ such that the length of $(f^n)^k(J)$ is larger than $\delta_{f^n} = \min\{x_i - x_{i-1}, 1 \leq i \leq p\} > 0$, where $x_0, x_1, \ldots, x_p$ is the partition induced by f^n on I. Thus there exists x and y in J such that $|f^{nk}(x) - f^{nk}(y)| > \delta_{f^n}$. Taking $\delta = \frac{\delta_{f^n}}{2}$ we see that f is sensitive on I with δ as its constant of sensitivity.

In simple words our claim can be stated in this way: Take a pencil, draw a polygonal curve without leaving the square $[0,1] \times [0,1] \subset \mathbb{R}^2$, moving from left to right, starting at the point $(0, a)$ and ending at the point $(1, b)$, in such a way that in any straight line segment the slope is larger than one in absolute value. At the end you will obtain the graph of a function sensitive on I, chaotic on a nonempty interior subset of I and with positive entropy. ☑

2.- The next example shows that sensitive dependence to initial conditions does not imply chaotic dynamics if our dynamical system is defined on the torus. Let S^1 be the circle, $S^1 = \mathbb{R}/\mathbb{Z}$, and let M be the torus, $M = S^1 \times S^1$. Let us consider $f : M \to M$ defined by $f(x, y) = (x + y \mod (1), y)$. Notice that two very close points with same first coordinate but different second one, (x_0, y_0) and (x_0, y_1), eventually will separate under the iterates of f. Thus f is sensitive on M.

Since the eigenvalues of the matrix associated with f, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, are $\lambda_{1,2} = 1$, the entropy of f is zero (see [9]). Also, there is not subset in M where f could be chaotic. The reason is that any invariant closed set A where f is transitive is a periodic orbit or is a circle, and in such a case, $f|_A : A \to A$ is an irrational rotation. Hence, in either case, $f|_A$ is not chaotic.

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