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L^q estimates of functions in the kernel of an elliptic operator and applications

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Abstract. In this work, we will find a family of small functions η_y in the Kernel of an operator defined in the intersection of the Sobolev space $H^{2,q}(S^n)$ with the orthogonal complement in $H^{1,2}(S^n)$ of the first eigenspace of the laplacian on S^n, parameterized with a variable y belonging to a small ball contained in B^{n+1}. We will find L^q estimates of these functions and we will use those estimates to find a subcritical solution to the scalar curvature problem on S^n, and a solution $u_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1}) = |F'_{y_1}|^{\frac{n-2}{2}}(1+\eta_{y_1})\circ F_{y_1}$ of a nonlinear elliptical problem related to that problem, where $F_{y_1} : S^n \to S^n$ is a centered dilation.
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Estimativos L^q de funciones en el núcleo de un operador elíptico y aplicaciones

Resumen. En este trabajo, vamos a encontrar una familia de pequeñas funciones η_y en el kernel de un operador definido en la intersección del espacio de Sóbolev $H^{2,q}(S^n)$ con el complemento ortogonal en $H^{1,2}(S^n)$ del primer espacio propio del laplaciano sobre S^n, parametrizado con una variable y que pertenece a una pequeña bola contenida en B^{n+1}. Encontraremos estimativos L^q de estas funciones, las cuales utilizaremos para encontrar una solución subcrítica al problema de curvatura escalar sobre S^n y una solución $u_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1}) = |F'_{y_1}|^{\frac{n-2}{2}}(1+\eta_{y_1})\circ F_{y_1}$ de un problema elíptico no lineal relacionado con este problema, donde $F_{y_1} : S^n \to S^n$ es una dilatación centrada.
Palabras clave: Espacios de Sóbolev, deformaciones conformes, ecuaciones elípticas.

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1. Introduction

Let (S^n, δ_{ij}) be the unitary sphere with the standard metric. A natural question in Riemannian geometry is: given a function $K : S^n \to \mathbb{R}$, is there a metric g conformally related to the standard metric δ_{ij} such that K is the scalar curvature of S^n with respect to the metric g? This is equivalent to the problem of finding a positive smooth function $u : S^n \to \mathbb{R}$ which satisfies the equation

$$\Delta u - \frac{n(n-2)}{4}u + \frac{n-2}{4(n-1)}Ku^{\frac{n+2}{n-2}} = 0. \tag{1}$$

If we set $g = u^{\frac{4}{n-2}}\delta_{ij}$, where u is a solution of this problem, then the function K is the scalar curvature of S^n with respect to the metric g.

The problem of conformal deformation of metrics in S^n have been extensively studied by many authors (for example, see [1], [2], [3], [5], [6], [7], [8], [9] and the references therein). An important feature of this problem is that it is a conformal invariant one. More precisely, if u is a solution of equation (1) then for any conformal map $F : S^n \to S^n$ the function $\alpha_F(u) = |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$ is a solution to problem (1) with scalar curvature $K \circ F$.

The problem of conformal deformation of metrics in S^n can be approached using the so called Yamabe method, which consists in studying first the subcritical problem in the equation (1):

$$\Delta u_p - \frac{n(n-2)}{4}u_p + \frac{n-2}{4(n-1)}Ku_p^p = 0, \tag{2}$$

with $p \in \left(1, \frac{n+2}{n-2}\right)$, and then consider the limit of the solutions when $p \uparrow \frac{n+2}{n-2}$.

Let $E(u)$ be the energy norm associated with the linear part of (2), and let $\mathcal{S}$ be the set of non-negative functions $u \in W^{2,q}(S^n)$, $(q > \frac{n}{2})$ such that $E(u) = E(1)$. Let us consider the open unit ball B^{n+1} and the map $\Phi : B^{n+1} \to \mathcal{S}$ defined by

$$\Phi(y) = \alpha_y := \alpha_{F_y}(1) = |(F_y^{-1})'|^{\frac{n-2}{2}},$$

where $F_y : S^n \to S^n$ is the restriction to S^n of a special conformal map $F_y : \overline{B^{n+1}} \to \overline{B^{n+1}}$ that satisfies $F_y(0) = y$ and fix the points $\pm\frac{y}{|y|}$; this function maps 0 to y and commutes with rotations about the line joining the origin and the point y. This map is referred to as a centered dilation.

For $p \in \left(1, \frac{n+2}{n-2}\right)$ and $u \in \mathcal{S}$, let $J_p(u)$ defined by $J_p(u) = \int_{S^n} Ku^{p+1}d\sigma$. If u is a critical point of $J_p(\cdot)$ on $\mathcal{S}$, then a multiple of u satisfies problem (2). Let us define the function $\overline{J}_p = J_p \circ \Phi$. In this paper, we will consider the equation

$$Lu + \frac{n(n-2)}{4}vol(S^n)(\overline{J}_p(y))^{-1}Ku^p = 0, \tag{3}$$

where $K : S^n \to \mathbb{R}$ is a nondegenerate function (Morse function) with $\Delta K \neq 0$ in its critical points, and $Lu = \Delta u - \frac{n(n-2)}{4}u$.

Let $F : S^n \to S^n$ be a conformal transformation and $v = \alpha_F(u) : |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$. A straightforward calculation shows that u is solution of (3) if and only if the function $\eta = v - 1$ is a solution of an equation of the form

$$\mathcal{L}(\eta) + \mathcal{Q}(\eta) = \frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}, \tag{4}$$

where $a = vol(S^n)(\overline{J}_p(y))^{-1} K \circ F^{-1} |(F^{-1})'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta}$, $\mathcal{L}(\eta) = \Delta\eta + n\eta$, $\mathcal{Q}(\eta)$ is a term which is quadratically small in η, and $\delta = \frac{n+2}{n-2} - p$. The linear operator $\mathcal{L}$ has an $(n+1)$ dimensional kernel consisting of the first order spherical harmonics. This obstruction to invert the linear operator $\mathcal{L}$ may be removed by replacing equation (4) by the projected equation $T(y, \eta) = 0$, where

$$T(y,\eta) = \mathcal{L}(\eta) + \mathbf{P}(\mathcal{Q}(\eta)) - \mathbf{P}\left(\frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}\right), \tag{5}$$

and $\mathbf{P}$ denotes the $\mathbb{L}^2$-orthogonal projection onto the orthogonal complement W of the first eigenspace of the laplacian on S^n.

This work is motivated by the work of Schoen and Zhang in [8] on the prescribed scalar curvature problem on the n-dimensional sphere, $n \geq 3$, and by the work of Escobar and García in [3] on the prescribed mean curvature on the n-dimensional unit ball, $n \geq 3$. In fact our method parallels those of [8] and [3]. In this paper we will find in Section 3, using the inverse function Theorem, small solutions η_y of equation (5), where y is close to a critical point of $\overline{J}_p$. In Section 4, we will find L^q and integral estimates of η_y and its first two derivatives.

In the last section, setting $u_y = \alpha_{F_y}(1+\eta_y)$, we perturb the function u_y and consider the function $\widetilde{u}_y = \Lambda_y u_y$ in order to achieve that $E(\widetilde{u}_y) = E(1)$. Next we define the map $\widetilde{J}_p(y) = J_p(\widetilde{u}_y)$ and we show that the functions $\overline{J}_p(y)$ and $\widetilde{J}_p(y)$ are close in the C^2 norm, using the estimates of the functions η_y. The fact that the functions $\overline{J}_p(y)$ and $\widetilde{J}_p(y)$ are close implies that $\widetilde{J}_p(y)$ has a unique critical point y_1 close to the critical point y_0 of $\overline{J}_p(y)$. This implies that $\widetilde{u}_{y_1}$ is a solution of the equation

$$Lu + \frac{n(n-2)}{4} K vol(S^n)(J_p(u))^{-1} u^p = 0. \tag{6}$$

Multiplying the function $\widetilde{u}_{y_1}$ by suitable constants, we find a solution of problem (2) and prove that $u_{y_1} = \alpha_{F_{y_1}}(1+\eta_{y_1})$ is a solution of problem (3), respectively.

2. Preliminaries

Let $y \in B^{n+1}$. Up to a rotation we will assume that $y = (0, \ldots, 0, y_{n+1})$, $y_{n+1} \geq 0$. In this case the centered dilation function F_y is given by $F_y(x) = \Sigma^{-1} \circ D_\mu \circ \Sigma(x)$, where the function

$$\Sigma(x) = \frac{2\overline{x}}{1 + x_{n+1}}$$

is the stereographic projection from the south pole of the sphere, the function

$$\Sigma^{-1}(\overline{x}) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + 4}, \frac{4 - |\overline{x}|^2}{|\overline{x}|^2 + 4}\right)$$

is the inverse of the stereographic projection, and the function $D_\mu : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $D_\mu(\overline{x}) = \mu\overline{x}$, where $x = (\overline{x}, x_{n+1}) \in S^n$ with $\overline{x} = (x_1, \dots, x_n)$ and $\mu = \frac{1-|y|}{1+|y|}$.

Since $F_y = \Sigma^{-1} \circ D_\mu \circ \Sigma$, then $F_y(x) = B^{-1}(4\mu A\overline{x}, (A^2 - 4\mu^2|\overline{x}|^2)$ and $F_y(0) = y$, where

$$A = 2(1 + x_{n+1}) \quad \text{and} \quad B = 4\mu^2|\overline{x}|^2 + 4(1 + x_{n+1})^2.$$

Note that $F_y^{-1} = F_{-y}$.

If $y \in B_{\beta(1-|y_0|)}(y_0)$ for some $0 < \beta < 1$, then we have

$$(1 - \beta)(1 - |y_0|) \leq 1 - |y| \leq (1 + \beta)(1 - |y_0|). \tag{7}$$

The number μ satisfies the inequalities

$$\mu \leq C(1 - |y_0|) \tag{8}$$

and

$$\frac{1}{\mu} \leq \frac{C}{1 - |y_0|}. \tag{9}$$

Consider the map $\Phi : B^{n+1} \to \mathcal{S}$ defined by $\Phi(y) = \alpha_y := \alpha_{F_y}(1) = |(F_y^{-1})'|^{\frac{n-2}{2}}$, where $F_y : S^n \to S^n$ is the conformal map that satisfies $F_y(0) = y$, and fix the points $\pm\frac{y}{|y|}$. For $p \in \left(1, \frac{n+2}{n-2}\right]$ and $u \in \mathcal{S}$, let $J_p(u)$ be defined by

$$J_p(u) = \int_{S^n} K u^{p+1} d\sigma.$$

If u is a critical point of $J_p(\cdot)$ on $\mathcal{S}$, $p \in \left(1, \frac{n+2}{n-2}\right)$, then a multiple of u satisfies problem (2). Let us define $\overline{J}_p = J_p \circ \Phi$. The functions $\overline{J}_p$ are eigenfunctions of the laplacian on B^{n+1} with the hyperbolic metric. In fact,

$$\triangle\overline{J}_p + \lambda_p\overline{J}_p = 0; \quad \lambda_p = \left(\frac{n-2}{2}\right)^2 (p+1)\delta,$$

where $\delta = \frac{n+2}{n-2} - p$.

Let us define the function $v_p(y) = \int_{S^n} (\alpha_y(\xi))^{p+1} d\sigma(\xi)$, so that $v_p(y) = vol(S^n)$ for $p = \frac{n+2}{n-2}$. The function v_p is positive and radially symmetric. Let us define the function $\widehat{J}_P = v_p^{-1}\overline{J}_p$. For $n \geq 3$ the functions $\widehat{J}_P$ are uniformly bounded in the $C^2(B^{n+1})$ norm and they agree with K on S^n. Using that all critical points of the function K are non-degenerate and $\triangle K \neq 0$ at each critical point, the following facts are proven in Proposition 2.1 in [8]. Since $\widehat{J}_P$ is C^2 in the closed ball, then $\frac{\partial \widehat{J}_P}{\partial r} = 0$ in the boundary of the ball. From here it can be seen that the critical points of $\widehat{J}_P$ near ∂B^{n+1} actually lie on ∂B^{n+1} and are the critical points of K. If y_0 is a critical point of the function $\overline{J}_p$ near ∂B^{n+1}, then $|\frac{\partial v_p}{\partial r}(y_0)| \leq C v_p(y_0)(1 - |y_0|)$. It is also proven that there exist constants $C_1, C_2 > 0$ such that

$$C_1\delta \leq (1 - |y_0|)^2 \leq C_2\delta, \tag{10}$$

and consequently,

$$C_1\delta \leq \mu^2 \leq C_2\delta. \tag{11}$$

The estimates of the following proposition (see [4]) are very useful in this work.

Proposition 2.1. *Let y_0 be a point near ∂B^{n+1} which is the critical point of the function $\overline{J}_p$ and let $y \in B_{\beta(1-|y_0|)}(y_0)$. Then,*

1. $\left|\nabla K\left(\frac{y_0}{|y_0|}\right)\right| \le C\mu^{1-w}$, *where w is any small positive number less than one.*
2. *If $f = \boldsymbol{P}\left(K - K\left(\frac{y}{|y|}\right)\right)$, $\|f \circ F_y\|_{0,q} \le C\mu^{2-w}$, with $0 < w < 1$.*
3. *If $\frac{n}{2} < q < n$, $\|\nabla_y(K \circ F_y)\|_{0,q} \le C\mu^{1-w}$, where $0 < w < 1$.*
4. *For $\frac{n}{2} < q < n$ and $1 - \frac{n}{2q} < r < \frac{1}{2}$, $\|\nabla_y \nabla_y (K \circ F_y)\|_{0,q} \le \mu^{-2r}$.*

The following propositions, which are useful to find a solution of problem (2), are respectively the Corollary 2.2 and Lemma 2.3 in [8].

Proposition 2.2. *There is a number $\beta < 1$ such that, if we denote by y_0 one of the critical points of $\overline{J}_p$ near ∂B^{n+1}, then the following bound holds for $y \in B_{\beta(1-|y_0|)}(y_0)$:*

$$(1-|y_0|)^{-1}\|\nabla \overline{J}_p\| + \|\nabla\nabla \overline{J}_p\| \le c, \qquad |\det(\mathrm{Hess}(\overline{J}_p)| \ge c^{-1}.$$

For $y \in B_{\beta(1-|y_0|)}(y_0)$ we have $\|\nabla \overline{J}_p\| \ge c^{-1}(1-|y_0|)$.

Proposition 2.3. *Suppose f, g are $\mathcal{C}^2$ functions in the closed unit ball $\overline{B}^{n+1}$ in $\mathbb{R}^{n+1}$. Suppose there is a positive constant c such that*

$$\|\nabla f\| + \|\nabla\nabla f\| \le c, \qquad |\det(\mathrm{Hess}(f)| \ge c^{-1} \quad \text{and} \quad \inf_{\partial B_1} \|\nabla f\| \ge c^{-1}.$$

Assume f has a unique critical point y_0 in B^{n+1}, and g is close to f in the sense that

$$\|\nabla(f-g)\| + \|\nabla\nabla(f-g)\| \le \epsilon.$$

If ϵ is sufficiently small, then g has a unique critical point y_1 in B^{n+1}.

3. The projected equation

To begin with, we will do several transformations of equation (2). One of those transformations involves the definition of an operator

$$\mathcal{T} : \mathcal{B}^{2,q} \to \mathcal{B}^{0,q}, \quad \text{where} \quad \mathcal{B}^{j,q} = C^2(B_{\beta(1-|y_0|)}(y_0), H^{j,q}(S^n) \cap W), \quad j = 0, 2,$$

by setting $\mathcal{T}(\eta)(y) = T(y, \eta)$; this operator and the inverse function Theorem allow us to find a solution to problem (5).

After multiplying a solution u of equation (2) by a suitable constant, we can rewrite that equation as

$$Lu + \frac{n(n-2)}{4} K vol(S^n)(J_p(u))^{-1} u^p = 0, \tag{12}$$

where $Lu = \Delta u - \frac{n(n-2)}{4} u$. Let y_0 be a critical point of $\overline{J}_p$ which is one of the critical points of $\overline{J}_p$ near ∂B^{n+1} given by Proposition 2.1 in [8]. Let $y \in B_{\beta(1-|y_0|)}(y_0)$, with $0 < \beta < 1$. To find a solution of equation (12), we will consider first the equation

$$Lu + \frac{n(n-2)}{4} vol(S^n)(\overline{J}_p(y))^{-1} K u^p = 0, \tag{13}$$

where we have replaced $J_p(u)$ by $\overline{J}_p(y)$.

A straightforward calculation shows that if u is solution of (13), $F : S^n \to S^n$ is a conformal transformation and $v = \alpha_F(u) : |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$, then v is a solution of the problem

$$Lv + \frac{(n-2)n}{4} vol(S^n)(\overline{J}_p(y))^{-1} K \circ F^{-1} |(F^{-1})'|^{\frac{n-2}{2}\delta} v^p = 0. \tag{14}$$

Setting $v = 1+\eta$, and defining $\mathcal{L}(\eta) = \Delta\eta + n\eta$, $\mathcal{Q}(\eta) = \frac{n(n-2)}{4}\left((1+\eta)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta\right)$, and $a = vol(S^n)(\overline{J}_p(y))^{-1} K \circ F^{-1} |(F^{-1})'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta}$, if v is a solution of equation (14), then η is a solution of problem

$$\mathcal{L}(\eta) + \mathcal{Q}(\eta) = \frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}. \tag{15}$$

Let $\{\xi_1, \xi_2, \dots \xi_{n+1}\}$ a generator set of the first eigenfunctions of the laplacian of S^n, that is,

$$\mathcal{L}(\xi_i) = \Delta\xi_i + n\xi_i = 0, \quad i = 1, 2 \dots, n+1.$$

This obstruction to invert the linear operator $\mathcal{L}$ may be removed by replacing equation (15) by the projected equation $T(y,\eta) = 0$, where

$$T(y,\eta) = \mathcal{L}(\eta) + \mathbf{P}(\mathcal{Q}(\eta)) - \mathbf{P}\left(\frac{(n-2)n}{4}(1-a)(1+\eta)^{\frac{n+2}{n-2}}\right), \tag{16}$$

and $\mathbf{P}$ denotes the $\mathbb{L}^2$-orthogonal projection onto the orthogonal complement W of the first eigenspace of S^n, where we have used that $(\mathcal{L}(\eta), \xi_i) = 0$ implies $\mathbf{P}(\mathcal{L}(\eta)) = \mathcal{L}(\eta)$.

In order to keep track of the dependence on y, as in [8], we define a map

$$\mathcal{T} : \mathcal{B}^{2,q} \to \mathcal{B}^{0,q}, \quad \text{where} \quad \mathcal{B}^{j,q} = C^2(B_{\beta(1-|y_0|)}(y_0), H^{j,q}(S^n) \cap W) \quad j = 0, 2,$$

by setting $\mathcal{T}(\eta)(y) = T(y,\eta)$, where η is the map $\eta(y) = \eta_y$. We choose a norm on $\mathcal{B}^{j,q}$ which reflects the scales which appear in the problem:

$$||\eta||_{\mathcal{B}^{j,q}} = \sup_y\{||\eta_y||_{j,q} + (1-|y_0|)||\nabla_y\eta_y||_{j,q} + (1-|y_0|)^2||\nabla_y\nabla_y\eta_y||_{j,q}\}, \quad j = 0, 2.$$

Hence,

$$||\mathcal{T}(\eta)||_{\mathcal{B}^{0,q}} = \sup_y\{||T(y,\eta)||_{0,q} + (1-|y_0|)||\nabla_y T(y,\eta)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T(y,\eta)||_{0,q}\}.$$

One of the main objectives of this work is to prove the existence of solutions of the projected equation (16). To reach it we will prove a similar result to Lemma 2.5 in [8].

Theorem 3.1. *For $p \to \frac{n+2}{n-2}$ and $q \in (n/2, n)$, the function $\mathcal{T}$ is C^1 and satisfies the following bounds:*

1. *$||\mathcal{T}(0)|| \le C\epsilon(p)\mu^\sigma$, where $\epsilon(p) \to 0$ when $p \to \frac{n+2}{n-2}$ and $\sigma < 2$.*
2. *$||\mathcal{T}'(0)|| \le C$.*

3. $||\mathcal{T}'(\eta_1) - \mathcal{T}'(\eta_0)|| \leq C||\eta_1 - \eta_0||$, $\|\eta_1\| \leq \frac{1}{4}$, $\|\eta_0\| \leq \frac{1}{4}$.

Moreover, $||(\mathcal{T}'(0))^{-1}|| \leq C$*, where the constant* C *is independent on* p*. There exists* $\eta \in \mathcal{B}^{2,q}$ *with* $\|\eta\| \leq C\epsilon(p)\mu^\sigma$ *and* $\mathcal{T}(\eta) = 0$*. Furthermore* η *is the unique small solution of* $\mathcal{T}(\eta) = 0$*.*

Proof. The bound for

$$||\mathcal{T}(0)||_{\mathcal{B}^{0,q}} = \{\sup_y ||T(y,0)||_{0,q} + (1-|y_0|)||\nabla_y T(y,0)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T(y,0)||_{0,q}\}$$

follows from the following three lemmas.

Lemma 3.2. *For any* $q \in \left(\frac{n}{2}, n\right)$, $\|T(y,0)\|_{0,q} \leq C\mu^{2-2w}$*, where* $0 < w < 1$*.*

Proof. For $\eta = 0$ we have that

$$\begin{aligned} T(y,0) &= -\mathbf{P}\left(\frac{(n-2)n}{4}(1-a_0)\right) \\ &= \frac{(n-2)n}{4} vol(S^n)(\overline{J_p}(y))^{-1}\mathbf{P}\left(K\circ F_y|F_y'|^{\frac{n-2}{2}\delta} - (vol(S^n))^{-1}\overline{J_p}(y)\right), \end{aligned}$$

where $a_0 = a(\xi, y, 0) = vol(S^n)(\overline{J_p}(y))^{-1}K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}$, and $|F_y'| = \frac{1-|y|^2}{|y+\xi|^2}$, $\xi \in S^n$. It is easy to see that

$$|T(y,0)| \leq C\left[\left||F_y'|^{\frac{n-2}{2}\delta} - 1\right| + \left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| + \left|K\circ F_y - K\left(\frac{y}{|y|}\right)\right|\right].$$

To finish the lemma, in the following claims we will show that the terms in the right hand side of the previous inequality have the required bound.

Claim 1. $||F_y'|^{\frac{n-2}{2}\delta} - 1| \leq C\mu^{2-2w}$, with $0 < w < 1$ and $y \in B_{\beta(1-|y_0|)}(y_0)$.

Proof. Let us observe that $|F_y'|^{\frac{n-2}{2}\delta}$ is of the form δ^δ. Taking $0 < w < 1$ and using the L'Hôpital rule we get

$$\lim_{\delta\to 0}\frac{\delta^\delta - 1}{\delta^{1-w}} = 0.$$

Then, for δ small enough, $|\delta^\delta - 1| \leq C\delta^{1-w} \leq C\mu^{2-2w}$, and consequently,

$$||F_y'|^{\frac{n-2}{2}\delta} - 1| \leq C\mu^{2-2w}.$$ ☑

Claim 2. $\left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| \leq C\mu^{2-2w}$, where $0 < w < 1$.

Proof. First observe that

$$\left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| \leq \left|\frac{\overline{J_p}(y)}{vol(S^n)} - \frac{\overline{J_p}(y)}{v_p(y)}\right| + \left|\frac{\overline{J_p}(y)}{v_p(y)} - K\left(\frac{y}{|y|}\right)\right|.$$

Using Claim (1), we get

$$\left|\frac{\overline{J_p}(y)}{vol(S^n)} - \frac{\overline{J_p}(y)}{v_p(y)}\right| \leq C_1 \left|\frac{v_p(y) - vol(S^n)}{v_p(y)vol(S^n)}\right| \leq M_1\mu^{2-2w}.$$

To find the bound of the second term in the right hand side, we consider the function $\widehat{J_p} = \frac{\overline{J_p}}{v_p}$. By Taylor's Theorem, there exists ζ between y and $\frac{y}{|y|}$ such that

$$\widehat{J_p}(y) = \widehat{J_p}\left(\frac{y}{|y|}\right) + \frac{\partial \widehat{J_p}}{\partial r}\left(\frac{y}{|y|}\right)\left(y - \frac{y}{|y|}\right) + \frac{\partial^2 \widehat{J_p}}{\partial r^2}(\zeta)\left(y - \frac{y}{|y|}\right)^2.$$

Since $\frac{\partial \widehat{J_p}}{\partial r}\left(\frac{y}{|y|}\right) = 0$ and $\widehat{J_p}|_{S^n} = K$, then

$$\left|\frac{\overline{J_p}(y)}{v_p(y)} - K\left(\frac{y}{|y|}\right)\right| = \left|\widehat{J_p}(y) - \widehat{J_p}\left(\frac{y}{|y|}\right)\right| \leq \left|\frac{\partial^2 \widehat{J_p}}{\partial r^2}(\zeta)\right|\left|y - \frac{y}{|y|}\right|^2 \leq C\mu^2.$$

Therefore,

$$\left|(vol(S^n))^{-1}\overline{J_p}(y) - K\left(\frac{y}{|y|}\right)\right| \leq C\mu^{2-2w} + C\mu^2 \leq C\mu^{2-2w}. \qquad ☑$$

The inequality $|T(y,0)| \leq C\mu^{2-2w}$ follows from Claims 1 and 2 and Proposition 2.1. Consequently,

$$||T(y,0)||_{0,q} = \left(\int_{S^n} |T(y,0)|^q d\sigma_g\right)^{1/q} \leq C\mu^{2-2w}. \qquad ☑$$

Now, we will do the estimates of the first derivative of $T(y,0)$ in the y variable.

Lemma 3.3. *For any* $q \in (\frac{n}{2}, n)$, $\|\nabla_y T(y,0)\|_{0,q} \leq C\mu^{1-w}$, *with* $0 < w < 1$.

Proof. A calculation shows that

$$\left|\frac{\partial T(y,0)}{\partial y_i}\right| \leq C\left[\left|\frac{\partial |F'_y|^{\frac{n-2}{2}\delta}}{\partial y_i}\right| + \left|\frac{\partial K \circ F_y}{\partial y_i}\right| + \left|\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right|\right].$$

The proof of the following claims conclude the proof of the lemma.

Claim 3.

$$\left\|\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right\|_{0,q} \leq C\mu.$$

Proof. Since $\frac{\partial \hat{J}_P}{\partial r} = 0$ in ∂B^{n+1}, the mean value Theorem implies $\left|\frac{\partial \hat{J}_p}{\partial r}(y)\right| \leq C(1 - |y_0|)$. Hence, $\left|\frac{\partial \hat{J}_p}{\partial y_i}\right| \leq C(1 - |y_0|)$. From $\hat{J}_p(y) = \frac{\overline{J_p}(y)}{v_p(y)}$ and $\frac{\partial(\overline{J_p}(y))}{\partial y_i} = v_p(y)\frac{\partial(\hat{J}_p(y))}{\partial y_i} + \hat{J}_p(y)\frac{\partial(v_p(y))}{\partial y_i}$, we get

$$\left|\frac{\partial(\overline{J_p}(y))}{\partial y_i}\right| \leq C\left|\frac{\partial(\hat{J}_p(y))}{\partial y_i}\right| + C\left|\frac{\partial(v_p(y))}{\partial y_i}\right| \leq C(1 - |y_0|).$$

Therefore

$$\left|\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right| \le C\left|\frac{\partial(\overline{J_p}(y))}{\partial y_i}\right| \le C(1-|y_0|) \le C\mu. \quad ☑$$

Claim 4.

$$\|\nabla_y |F_y'|^{\frac{n-2}{2}\delta}\|_{0,q} \le C\mu.$$

Proof. Since $|F_y'|(\xi) = \frac{1-|y|^2}{|y+\xi|^2}$, a straightforward calculation shows that

$$\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i} = -(n-2)\delta |F_y'|^{\frac{n-2}{2}\delta}\left(\frac{y_i}{1-|y|^2} + \frac{y_i+\xi_i}{|y+\xi|^2}\right), \text{ and therefore } \left|\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\right| \le C\mu. \quad ☑$$

Proposition 2.1 and Claims 3 and 4 yields to $|\nabla_y T(y,0)| \le C\mu^{1-w}$, and therefore,

$$||\nabla_y T(y,0)||_{0,q} = \left(\int_{S^n} |\nabla_y T(y,0)|^q d\sigma\right)^{1/q} \le C\mu^{1-w},$$

where w is a positive number less than one. ☑

Now, we will estimate the second derivatives of $T(y,0)$ with respect to the y variable.

Lemma 3.4. *For any $q \in (\frac{n}{2}, n)$ and $1-\frac{n}{2q} < r < \frac{1}{2}$, we have $\|\nabla_y \nabla_y T(y,0)\|_{0,q} \le C\mu^{-2r}$.*

Proof. Differentiating $T(y,0)$ twice with respect to the y variable we get

$$\frac{\partial^2 T(y,0)}{\partial y_j \partial y_i} = vol(S^n)\frac{n(n-2)}{4}\frac{\partial}{\partial y_j}\mathbf{P}\left[A+B+D\right], \quad \text{where} \quad A = (\overline{J_p}(y))^{-1} K\circ F_y \frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i},$$

$$B = (\overline{J_p}(y))^{-1}|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial K\circ F_y}{\partial y_i} \quad \text{and} \quad D = K\circ F_y |F_y'|^{\frac{n-2}{2}\delta}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}.$$

Let us estimate the first derivatives of A, B and D. Since

$$\frac{\partial A}{\partial y_j} = K\circ F_y \frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_j} + (\overline{J_p}(y))^{-1}\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\frac{\partial K\circ F_y}{\partial y_j} + (\overline{J_p}(y))^{-1} K\circ F_y \frac{\partial^2 |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_j \partial y_i},$$

and

$$\frac{\partial}{\partial y_j}\left(\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_i}\right) = -\frac{(n-2)^2}{2}\delta^2 |F_y'|^{\frac{n-2}{2}\delta}\left(\frac{|y+\xi|^2}{1-|y|^2}\right)\frac{\partial}{\partial y_j}\left(\frac{1-|y|^2}{|y+\xi|^2}\right)\left(\frac{y_i}{1-|y|^2} + \frac{y_i+\xi_i}{|y+\xi|^2}\right),$$

Claims 3 and 4 and Proposition 2.1 yield to $\left\|\frac{\partial A}{\partial y_i}\right\|_{0,q} \le C$.

Now,

$$\begin{aligned}\frac{\partial B}{\partial y_j} &= \frac{\partial}{\partial y_j}\left((\overline{J_p}(y))^{-1}|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial K\circ F_y}{\partial y_i}\right) = |F_y'|^{\frac{n-2}{2}\delta}\frac{\partial K\circ F_y}{\partial y_i}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_j}\\ &\quad + (\overline{J_p}(y))^{-1}\frac{\partial K\circ F_y}{\partial y_i}\frac{\partial |F_y'|^{\frac{n-2}{2}\delta}}{\partial y_j} + (\overline{J_p}(y))^{-1}|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial^2 K\circ F_y}{\partial y_j \partial y_i}.\end{aligned}$$

Hence, the inequality $\left\|\frac{\partial B}{\partial y_i}\right\|_{0,q} \le C\mu^{-2r}$ follows from the inequalities in Proposition 2.1 and Lemma 3.3. Finally, since

$$\begin{aligned}\frac{\partial D}{\partial y_j} &= \frac{\partial}{\partial y_j}\left(K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\right) = |F_y'|^{\frac{n-2}{2}\delta}\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\frac{\partial K\circ F_y}{\partial y_j}\\ &\quad + K\circ F_y\frac{\partial(\overline{J_p}(y))^{-1}}{\partial y_i}\frac{\partial|F_y'|^{\frac{n-2}{2}\delta}}{\partial y_j} + K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\frac{\partial^2(\overline{J_p}(y))^{-1}}{\partial y_j\partial y_i},\end{aligned}$$

from Claims 3 and 4 and Proposition 2.1, we get $\left\|\frac{\partial D}{\partial y_i}\right\|_{0,q} \le C$. The previous inequalities yield $||\nabla_y\nabla_y T(y,0)||_{0,q} \le C\mu^{-2r}$, as desired. ☑

Using the previous lemmas, we reach the bound

$$\begin{aligned}||\mathcal{T}(0)||_{\mathcal{B}^{0,q}} &= \sup_y\{||T(y,0)||_{0,q} + (1-|y_0|)||\nabla_y T(y,0)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T(y,0)||_{0,q}\}\\ &\le C\mu^{2-2w} + C\mu^{2-2r} \le C\epsilon(p)\mu^{\sigma},\end{aligned}$$

where $\sigma < 2$ and $\epsilon(p) = \mu^{\sigma'}$, with σ' a small positive number.
Now we will estimate $||\mathcal{T}'(0)|| = \sup_{||\phi||_{B^{2,q}}\le 1} ||\mathcal{T}'(0)\phi||_{0,q}$, where $||\mathcal{T}'(0)\phi||_{0,q}$ is given by

$$\sup_y\{||T'(y,0)(\phi)||_{0,q} + (1-|y_0|)||\nabla_y T'(y,0)(\phi)||_{0,q} + (1-|y_0|)^2||\nabla_y\nabla_y T'(y,0)(\phi)||_{0,q}\}.$$

For this, consider $\phi \in B^{2,q}$ satisfying $||\phi||_{B^{2,q}} \le 1$. Let $y \in B_{\alpha(1-|y_0|)}(y_0)$. Since

$$\begin{aligned}T(y,\eta) &= \mathcal{L}(\eta) + \mathbf{P}(Q(\eta))\\ &\quad -\mathbf{P}\Big(\frac{n(n-2)}{4}(1 - vol(S^n)(\overline{J_p}(y))^{-1}K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta})(1+\eta)^{\frac{n+2}{n-2}}\Big),\end{aligned}$$

we have that

$$T_y'(0)(\phi) = \mathcal{L}(\phi) - \mathbf{P}\left(\frac{n(n+2)}{4}\phi(1-a_0) + \frac{n(n-2)}{4}\delta\phi a_0\right),$$

where $a_0 = vol(S^n)(\overline{J_p}(y))^{-1}K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}$. Since $q > \frac{n}{2}$, from the Sobolev embedding Theorem we get $\|\phi\|_{L^\infty} \le C\|\phi\|_{2,q} \le C\|\phi\|_{B^{2,q}} \le C$. Therefore $|\mathcal{L}(\phi)| \le C$.
From this inequality and the estimates of Lemma 3.2, we obtain $|T'(y,0)(\phi)| \le C$, and $\|T'(y,0)(\phi)\|_{0,q} \le C$. Working similarly, and using the fact that $\phi, \nabla_y\phi, \nabla_y\nabla_y\phi$ belong to $\mathcal{H}^{2,q}(S^n)$ for $q > \frac{n}{2}$, we get $||\mathcal{T}'(0)|| \le C$.
Now, we will show that the derivative of $\mathcal{T}'$ is Lipschitz; that is,

$$||\mathcal{T}'(\eta_1) - \mathcal{T}'(\eta_0)|| \le C||\eta_1 - \eta_0||, \quad \|\eta_1\|, \|\eta_0\| \le \frac{1}{4}.$$

For this, taking $\phi \in \mathcal{B}^{2,p}$ such that $\|\phi\|_{\mathcal{B}^{2,p}} \le 1$, we get

$$\begin{aligned}\mathcal{T}'(\eta).\phi &= \mathcal{L}(\phi) + \frac{n(n+2)}{4}\mathbf{P}\left[(1+\eta)^{\frac{4}{n-2}}\phi - \phi\right]\\ &\quad -\mathbf{P}\left[\frac{n(n+2)}{4}(1-a_\eta)(1+\eta)^{\frac{4}{n-2}}\phi - \delta\frac{n(n-2)}{4}a_\eta(1+\eta)^{\frac{4}{n-2}}\phi\right],\end{aligned}$$

where $a_\eta = a_0(1+\eta)^{-\delta}$. Since

$$
\begin{aligned}
(\mathcal{T}_y'(\eta_1) - \mathcal{T}_y'(\eta_0))\phi = &\mathbf{P}\left(\left[(1+\eta_1)^{\frac{4}{n-2}} - (1+\eta_0)^{\frac{4}{n-2}}\right]\phi\right) \\
&- \mathbf{P}\left[\left(\frac{n(n+2)}{4} + \delta\frac{n(n-2)}{4}\right)(a_{\eta_0} - a_{\eta_1})(1+\eta_1)^{\frac{4}{n-2}}\phi\right] \\
&- \mathbf{P}\left[\left(\tfrac{n(n+2)}{4}(a_{\eta_0}-1) + \delta\tfrac{n(n-2)}{4}a_{\eta_0}\right)[(1+\eta_1)^{\frac{4}{n-2}} - (1+\eta_0)^{\frac{4}{n-2}}]\phi\right],
\end{aligned}
$$

using that $\|\eta_1\|, \|\eta_0\| \leq \frac{1}{4}$ and the mean value Theorem, we get

$$
\begin{aligned}
|(\mathcal{T}_y'(\eta_1) - \mathcal{T}_y'(\eta_0))\phi| \leq &C|\eta_1-\eta_0||\phi| + C|a_0|\delta|\eta_1-\eta_0||\phi| \\
&+ C(|a_{\eta_0}-1| + |a_{\eta_0}|)|\eta_1-\eta_0||\phi|,
\end{aligned}
$$

and therefore

$$
\|(\mathcal{T}_y'(\eta_1) - \mathcal{T}_y'(\eta_0))\phi\|_{0,q} \leq C\|\eta_1-\eta_0\|_{0,q}.
$$

To finish the proof of Theorem 1, we need to show that $\mathcal{T}'(0)$ has a bounded inverse. Let $\phi \in \mathcal{B}^{2,q}(S^n)$ and $\Psi \in \mathcal{B}^{0,q}(S^n)$. Consider the problem $\mathcal{T}'(0)\phi = \Psi$. Let us recall that

$$
\|\phi\|_{\mathcal{B}^{2,q}(S^n)} = \sup_y\{\|\phi\|_{2,q} + (1-|y_0|)\|\nabla_y\phi\|_{2,q} + (1-|y_0|)^2\|\nabla_y\nabla_y\phi\|_{2,q}\}.
$$

Elliptic estimates shows that $\|\phi\|_{2,q} \leq C\|\mathcal{L}(\phi)\|_{0,q}$. Since

$$
\Psi = T_y'(0)(\phi) = \mathcal{L}(\phi) - \mathbf{P}\Big(\frac{n(n+2)}{4}\phi(1-a_0) + \frac{n(n-2)}{4}\delta\phi a_0\Big),
$$

from the estimates of Lemma 3.2 we get

$$
\begin{aligned}
\Big\|\mathbf{P}\Big(\frac{n(n+2)}{4}\phi(1-a_0) + \frac{n(n-2)}{4}\delta\phi a_0\Big)\Big\|_{0,q} &\leq C\epsilon(p)\mu^\sigma\|\phi\|_{0,q} \\
&\leq C\epsilon(p)\mu^\sigma\|\phi\|_{2,q};
\end{aligned}
$$

then,

$$
\|\phi\|_{2,q} \leq C\|\mathcal{L}(\phi)\|_{0,q} \leq k(\|\Psi\|_{0,q} + C\epsilon(p)\mu^\sigma\|\phi\|_{2,q}).
$$

Taking $\mu^\sigma\epsilon(p)$ small we get that $1 - kC\epsilon(p)\mu^\sigma > 0$ and $\|\phi\|_{2,q} \leq C\|\Psi\|_{0,q}$. Working analogously, we have that

$$
\|\nabla_y\phi\|_{2,q} \leq L\|\nabla_y\Psi\|_{0,q} + L_1\mu^{1-w}\|\Psi\|_{0,q}
$$

and

$$
\|\nabla_y\nabla_y\phi\|_{2,q} \leq C_1\|\nabla_y\nabla_y\Psi\|_{0,q} + C_2\mu^{1-w}\|\nabla_y\Psi\|_{0,q} + (C_3\mu^{-2r} + C_4\mu^{2-2w})\|\Psi\|_{0,q}.
$$

Therefore,

$$
\|\phi\|_{\mathcal{B}^{2,q}(S^n)} \leq C\sup_y\{\|\Psi\|_{0,q} + (1-|y_0|)\|\nabla_y\Psi\|_{0,q} + (1-|y_0|)^2\|\nabla_y\nabla_y\Psi\|_{0,q}\} \leq C\|\Psi\|_{\mathcal{B}^{0,q}(S^n)}.
$$

The rest of the proof follows from the inverse function Theorem. ☑

4. Integral and L^q estimates of the function η_y

In this section, given the solution η_y, $y \in B_{\beta(1-|y_0|)}$, of the projected equation, we will find L^q estimates not only of the function η_y, but also of its first and second y derivatives; in addition, we will do also integral estimates of $\nabla_y \eta_y$ and $\nabla_y \nabla_y \eta_y$.

Lemma 4.1. *For $q \in \left(\frac{n}{2}, n\right)$, $\|\eta_y\|_{0,q} \le C\epsilon(p)\mu^\sigma$, with $\sigma < 2$, where $\epsilon(p) \to 0$ as $p \to \frac{n+2}{n-2}$.*

Proof. From Theorem 3.1, $T(y, \eta_y) = 0$. Then,

$$\mathcal{L}(\eta_y) = -\frac{n(n-2)}{4}\mathbf{P}\left((1+\eta_y)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta_y\right) + \frac{n(n-2)}{4}\mathbf{P}\left((1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right).$$

Setting $a = a_0 D$, where $D = (1+\eta_y)^{-\delta}$, we have

$$|1-a| = |a-1| = |a_0 D - 1| = |a_0(D-1) + (a_0 - 1)| \le |a_0||D-1| + |a_0 - 1|.$$

From the mean value Theorem it follows that

$$|\mathcal{L}(\eta_y)| \le C|\eta_y|^2 + C\delta|a_0||\eta_y| + C|a_0 - 1|.$$

Using Hölder's inequality,the estimates of Lemma 1, Theorem 1, $q > \frac{n}{2}$ and the Sobolev embedding Theorem, we have

$$\begin{aligned}\|\mathcal{L}(\eta_y)\|_{0,q,S^n} &\le C\|\eta_y\|_\infty \|\eta_y\|_{0,q,S^n} + C\mu^2\|\eta_y\|_{0,q,S^n} + C\epsilon(p)\mu^\sigma \\ &\le C\epsilon(p)\mu^\sigma \|\eta_y\|_{2,q,S^n} + C\mu^2\|\eta_y\|_{2,q,S^n} + C\epsilon(p)\mu^\sigma.\end{aligned}$$

Since $\|\eta_y\|_{2,q,S^n} \le C\|\mathcal{L}(\eta_y)\|_{0,q,S^n}$, then $\|\eta_y\|_{0,q,S^n} \le \|\eta_y\|_{2,q,S^n} \le C\epsilon(p)\mu^\sigma$, as desired. ☑

Lemma 4.2. *For $q \in \left(\frac{n}{2}, n\right)$, $\|\nabla_y \eta_y\|_{0,q} \le C\mu^{1-w}$, with $0 < w < 1$.*

Proof. Differentiating the equation

$$0 = T(y, \eta_y) = \mathcal{L}(\eta_y) + \mathbf{P}(Q(\eta_y)) - \mathbf{P}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right),$$

we find that the terms of its derivative satisfy the inequalities

$$|\nabla_y a| \le C(|\nabla_y a_0| + \mu^2|\eta_y'|), \quad \text{where} \quad a = a_0(1+\eta_y)^{-\delta},$$

$$\begin{aligned}\left|\frac{\partial}{\partial y_i}\mathbf{P}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)\right| &\le C|1-a||\eta_y'| + C|\nabla_y a| \\ &\le (C\delta|a_0||\eta_y| + C_2|a_0 - 1|)|\eta_y'| + C_3|\nabla_y a| \\ &\le C(\mu^2|a_0||\eta_y| + |a_0 - 1| + \mu^2)|\eta_y'| + C|\nabla_y a_0|,\end{aligned}$$

and

$$\left|\frac{n(n-2)}{4}\frac{\partial}{\partial y_i}\mathbf{P}\left((1+\eta_y)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta_y\right)\right| \le C|\eta_y'||\eta_y|,$$

where we have used the estimates of Theorem 3.1 and $\delta = C\mu^2$.

Hence,

$|\mathcal{L}(\eta_y')| \le C\mu^2|a_0||\eta_y||\eta_y'| + C_2|a_0-1||\eta_y'| + C_3|\nabla_y a_0| + C\mu^2|\eta_y'| + C|\eta_y'||\eta_y|$.

Using Hölder's inequality, the estimates in Theorem 3.1 and Lemma 4.1, we arrive to

$$\begin{aligned}\|\eta_y'\|_{2,q,S^n} \le& \|\mathcal{L}(\eta_y')\|_{0,q,S^n} \le C_2\epsilon(p)\mu^{\sigma+2}\|\eta_y'\|_{0,q,S^n} + C_3\|\nabla_y a_0\|_{0,q,S^n}\\ &+ C\mu^2\|\eta_y'\|_{0,q,S^n} + C\|\eta_y'\|_{0,q,S^n}\|\eta_y\|_\infty + C_2\|a_0-1\|_{0,q}\|\eta_y'\|_\infty\\ \le& C_2\epsilon(p)\mu^{\sigma+2}\|\eta_y'\|_{0,q,S^n} + C_3\mu^{1-w} + C\mu^2\|\eta_y'\|_{0,q,S^n} + C\epsilon(p)\mu^\sigma\|\eta_y'\|_{0,q,S^n}\\ &+ C\epsilon(p)\mu^\sigma\|\eta_y'\|_{2,q,S^n},\end{aligned}$$

and therefore $\|\eta_y'\|_{2,q,S^n} \le C\mu^{1-w}$ for $0 < w < 1$. ☑

Differentiating twice the equation $T(y,\eta) = 0$ and working as in Lemma 4.2, we get

Lemma 4.3. *For* $q \in \left(\frac{n}{2}, n\right)$, $\|\nabla_y\nabla_y\eta_y\|_{0,q} \le C\mu^{-2r}$, *with* $1 - \frac{n}{2q} < r < \frac{1}{2}$.

In what follows, we will estimate the integral of the function η_y', $y \in B_{\beta(1-y_0)}(y_0)$.

Lemma 4.4. *For* $q \in \left(\frac{n}{2}, n\right)$ *and* $y \in B_{\beta(1-y_0)}(y_0)$, $\left|\int_{S^n}\nabla_y\eta_y d\sigma\right| \le C\epsilon(p)\mu^\sigma$, *with* $\sigma < 2$.

Proof. From $\mathcal{L}(\eta_y) + \mathbf{P}(Q(\eta_y)) - \mathbf{P}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right) = 0$, and $\int_{S^n}\mathbf{P}(f)d\sigma = \int_{S^n} f d\sigma$, $f \in L^2(S^n)$, we have

$$0 = \int_{S^n} T(y,\eta_y)d\sigma = \int_{S^n}\mathcal{L}(\eta_y)d\sigma + \int_{S^n}Q(\eta_y)d\sigma - \int_{S^n}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)d\sigma.$$

Using that $\mathcal{L}(\eta_y) = \Delta\eta_y + n\eta_y$, we obtain

$$\int_{S^n} n\eta_y d\sigma = -\int_{S^n}Q(\eta_y)d\sigma + \int_{S^n}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)d\sigma.$$

Setting $A = Vol(S^n)\overline{J_p}^{-1}(y)K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}$, $D = (1+\eta_y)^{-\delta}$ and $E = (1+\eta_y)^{\frac{n+2}{n-2}}$, we get

$$\int_{S^n}\left(\frac{n(n-2)}{4}(1-a)(1+\eta_y)^{\frac{n+2}{n-2}}\right)d\sigma = \frac{n(n-2)}{4}\int_{S^n}(1-AD)Ed\sigma.$$

Hence,

$$\int_{S^n} n\eta_y d\sigma = -\int_{S^n}Q(\eta_y)d\sigma + \frac{n(n-2)}{4}\int_{S^n}(1-AD)Ed\sigma,$$

and therefore,

$$\int_{S^n}\eta_y d\sigma = -\frac{1}{n}\int_{S^n}Q(\eta_y)d\sigma - \frac{n-2}{4}\int_{S^n}(AD-1)Ed\sigma.$$

Writing $(AD-1)E = (AD-1)(E-1)+A(D-1)+A-1$, and observing that $\int_{S^n} Ad\sigma = cte$, we have

$$\frac{\partial}{\partial y_i}\int_{S^n}[(AD-1)E]d\sigma = \int_{S^n}[(A'D+AD')(E-1)+(AD-1)E']d\sigma + \int_{S^n}[A'(D-1)+AD']d\sigma.$$

On the other hand,

$$\frac{\partial Q(\eta_y)}{\partial y_i} = \frac{n(n+2)}{4}\eta_y'[(1+\eta_y)^{\frac{4}{n-2}}-1].$$

Then,

$$\int_{S^n}\frac{\partial \eta_y}{\partial y_i}d\sigma = \mathcal{A}+\mathcal{B}+\mathcal{C}, \tag{17}$$

where $\mathcal{A} = -\frac{1}{n}\int_{S^n}\left[\frac{n(n+2)}{4}\eta_y'[(1+\eta_y)^{\frac{4}{n-2}}-1]\right]d\sigma$, $\mathcal{C} = -\frac{n-2}{4}\int_{S^n}[A'(D-1)+AD']d\sigma$ and $\mathcal{B} = -\frac{n-2}{4}\int_{S^n}[(A'D+AD')(E-1)+(AD-1)E']d\sigma$.

Using the estimates on η_y,η_y', the mean value Theorem and Hölder's inequality, we arrive to

$$\left|\int_{S^n}\left((1+\eta_y)^{\frac{4}{n-2}}-1\right)\eta_y'd\sigma\right| \leq C\int_{S^n}|\eta_y||\eta_y'|d\sigma \leq C\|\eta_y\|_{0,s}\|\eta_y'\|_{0,s'} \leq C\epsilon(p)\mu^{\sigma+1-w},$$

for s, s' such that $\frac{1}{s}+\frac{1}{s'}=1$. Working similarly, we get

$$\begin{aligned}\left|\int_{S^n}(A'D+AD')(E-1)\right| &\leq C\int_{S^n}|A'||\eta_y|d\sigma + C\delta\int_{S^n}|\eta_y|d\sigma\\ &\leq \|\eta_y\|_{0,s'}\|A'\|_{0,s} + C\epsilon(p)\mu^{\sigma}\\ &\leq C\epsilon(p)\mu^{\sigma+1-w} + C\epsilon(p)\mu^{\sigma}\\ &\leq C\epsilon(p)\mu^{\sigma},\end{aligned}$$

where we have used the mean value Theorem, Proposition 2.1, Lemma 4.1 and the estimates of Theorem 3.1. Using Lemma 4.2 and proceeding as before, we get

$$\left|\int_{S^n}(AD-1)E'd\sigma\right| \leq C\epsilon(p)\mu^{\sigma+1-w},$$

$$\left|\int_{S^n}A'(D-1)d\sigma\right| \leq C\epsilon(p)\mu^{\sigma+3-w}$$

and

$$\left|\int_{S^n}AD'd\sigma\right| \leq C\mu^{3-w}.$$

Consequently,

$$\left|\int_{S^n}\nabla_y\eta_y d\sigma\right| \leq C\epsilon(p)\mu^{\sigma} + C\mu^{3-w} \leq C\epsilon(p)\mu^{\sigma},$$

with $\sigma < 2$. ☑

Finally, we will estimate the integral of η''_y.

Lemma 4.5. *For* $q \in \left(\frac{n}{2}, n\right)$, $\left|\int_{S^n} \nabla_y \nabla_y \eta_y d\sigma\right| \le C\epsilon\mu^{\sigma-2r}$, *with* $r < \frac{1}{2}$.

Proof. Denoting $\frac{\partial^2 \eta_y}{\partial y_j \partial y_i}$ by η''_y, and differentiating the terms on the right hand side of equation (17) with respect to y_j, we get

$$\begin{aligned}\int_{S^n} \eta''_y d\sigma = &-\frac{n+2}{4}\int_{S^n} \eta''_y[(1+\eta_y)^{\frac{4}{n-2}} - 1]d\sigma - \frac{n+2}{n-2}\int_{S^n}(1+\eta_y)^{\frac{6-n}{n-2}}\eta'_{y_i}\eta'_{y_j}d\sigma\\ &-\frac{n-2}{4}\int_{S^n}[(A''D + 2A'D' + AD'')(E-1) + (A'D + AD')E']d\sigma\\ &-\frac{n-2}{4}\int_{S^n}[(A'D + AD')E' - (AD-1)E'' - A''(D-1) + 2A'D' + AD'']d\sigma.\end{aligned}$$

In what follows we will estimate the terms in the right hand side of this equality. Using Hölder's inequality, Proposition 2.1 and the four previous lemmas, we have:

$$\left|\frac{n+2}{4}\int_{S^n}\eta''_y[(1+\eta_y)^{\frac{4}{n-2}} - 1]d\sigma\right| \le C\int_{S^n}|\eta''_y||\eta_y|d\sigma \le C\epsilon(p)\mu^{\sigma-2r};$$

$$\left|\frac{n+2}{n-2}\int_{S^n}(1+\eta_y)^{\frac{6-n}{n-2}}\eta'_{y_i}\eta'_{y_j}d\sigma\right| \le C\int_{S^n}|\eta'_y|^2 d\sigma \le C\mu^{2-2w};$$

$$\begin{aligned}\int_{S^n}|(A''D + 2A'D' + AD'')(E-1)|d\sigma &\le C\int_{S^n}|(A''D + 2A'D' + AD'')||\eta_y|d\sigma\\ &\le C\int_{S^n}|A''||\eta_y|d\sigma + C\delta\int_{S^n}|A'||\eta'_y||\eta_y|d\sigma\\ &+ C\int_{S^n}|A|(\delta(\delta+1)|\eta_y|^2 + \delta|\eta''_y|)|\eta_y|d\sigma\\ &\le C\epsilon(p)\mu^{\sigma-2r};\end{aligned}$$

$$\left|\int_{S^n}(A'D + AD')E'd\sigma\right| \le C\int_{S^n}|A'||\eta'_y|d\sigma + C\delta\int_{S^n}|A||\eta'_y|^2 d\sigma \le C\mu^{2-2w};$$

$$\left|\int_{S^n}(AD-1)E''d\sigma\right| \le C\int_{S^n}|AD-1|(|\eta'_y|^2 + |\eta''_y|)d\sigma \le C\epsilon(p)\mu^{\sigma-2r};$$

$$\left|\int_{S^n}A''(D-1)d\sigma\right| \le C\int_{S^n}|A''||\eta_y|d\sigma \le C\epsilon(p)\mu^{\sigma-2r};$$

$$\left|\int_{S^n}2A'D'd\sigma\right| \le C\delta\int_{S^n}|A'||\eta'_y|d\sigma \le C\mu^{4-2w},$$

and

$$\left|\int_{S^n}AD''d\sigma\right| \le C\delta(\delta+1)\int_{S^n}|\eta'_y|^2 d\sigma + C\delta\int_{S^n}|\eta''_y|d\sigma \le C\mu^{2-2r}.$$

Putting together these inequalities, we obtain the desired bound for $\left|\int_{S^n}\nabla_y\nabla_y\eta_y d\sigma\right|$.

☑

5. Solutions of some nonlinear elliptic equations

In this section, using the estimates of Sections 3 and 4, we will prove that the functions $\widetilde{J}_p(y)$ and $\overline{J}_p(y)$ are close in the $\mathcal{C}^2$-norm. The fact this functions are close implies that $\widetilde{J}_p(y)$ has a unique critical point y_1 close to the critical point y_0 of $\overline{J}_p(y)$. This implies that $\widetilde{u}_{y_1}$ is a solution of equation (6).

Multiplying the function $\widetilde{u}_{y_1}$ by the constant $(J_p(\widetilde{u}_{y_1}))^{1-p}$ we will find that $u = (J_p(\widetilde{u}_{y_1}))^{1-p}\widetilde{u}_{y_1}$ is a solution of the subcritical problem (2). Recalling that η_y is a solution of the equation $T(y,\eta) = 0$, if we let $u_y = \alpha_{F_y^{-1}}(1+\eta_y)$ we will prove that $u_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1})$ is a solution of the perturbed equation (3).

Consider the quotient

$$(\Lambda_y)^{1-p} = \frac{\int_{S^n} K\alpha_y^{p+1}}{\int_{S^n} Ku_y^{p+1}},$$

and define the functions $\gamma_y = \Lambda_y(1+\eta_y)$ and $\widetilde{u}_y = \alpha_{F_y}(\gamma_y)$.

Recalling that $\mathcal{S}$ is the set of non-negative functions $u \in W^{2,q}(S^n)$, $(q > \frac{n}{2})$ such that $E(u) = E(1)$, we get the following proposition:

Proposition 5.1. *The function $\widetilde{u}_y$ belongs to the set $\mathcal{S}$.*

Proof. By Theorem 3.1, the function η_y satisfies the equation

$$\mathcal{L}(\eta) + \mathbf{P}(\mathcal{Q}(\eta)) - \frac{n(n-2)}{4}\mathbf{P}\left((1-a)(1+\eta)^{\frac{n+2}{n-2}}\right) = 0,$$

where

$$\mathcal{L}(\eta) = \Delta\eta + n\eta, \quad \mathcal{Q}(\eta) = \frac{n(n-2)}{4}\left((1+\eta)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\eta\right)$$

and

$$a = vol(S^n)(\overline{J}_p(y))^{-1}K \circ F_y|F_y'|^{\frac{n-2}{2}\delta}(1+\eta)^{-\delta}.$$

Summing the constant $n - \frac{n(n+2)}{4}$ in both side of the equation $T(y,\eta) = 0$ and simplifying, we get

$$\mathcal{L}(1+\eta) - \mathbf{P}\left[\frac{n(n+2)}{4}(1+\eta)\right] + \mathbf{P}\left[\frac{n(n-2)}{4}\tilde{a}(1+\eta)^p\right] = 0,$$

where $\tilde{a} = a(1+\eta)^\delta$. Therefore,

$$\mathcal{L}(\gamma_y) - \mathbf{P}\left[\frac{n(n+2)}{4}\gamma_y\right] + \frac{1}{(\Lambda_y)^{p-1}}\mathbf{P}\left[\frac{n(n-2)}{4}\tilde{a}(\gamma_y)^p\right] = 0.$$

Since

$$(\Lambda_y)^{1-p} = \frac{\int_{S^n} K\alpha_y^{p+1}}{\int_{S^n} Ku_y^{p+1}},$$

we have

$$\mathcal{L}(\gamma_y) - \mathbf{P}\left[\frac{n(n+2)}{4}\gamma_y\right] + \frac{n(n-2)}{4}vol(S^n)\frac{1}{\int_{S^n} Ku_y^{p+1}dz}\mathbf{P}\left(K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\gamma_y^p\right) = 0.$$

Multiplying this equation by γ and integrating, we have

$$\int_{S^n}\left(\mathcal{L}(\gamma_y)\gamma_y - \frac{n(n+2)}{4}\gamma_y^2\right)d\zeta + \frac{n(n-2)}{4}vol(S^n) = 0,$$

where we have used that $\int_{S^n}\mathbf{P}(f) = \int_{S^n} f$ for every integrable function f, and $\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\gamma_y^{p+1}d\zeta = \int_{S^n} Ku_y^{p+1}dz$.
Consequently,

$$E(\gamma_y) = \int_{S^n}|\nabla\gamma_y|^2 d\zeta + \frac{n(n-2)}{4}\int_{S^n}\gamma_y^2 d\zeta = \frac{n(n-2)}{4}vol(S^n).$$

Since $\widetilde{u}_y = \alpha_{F_y}(\gamma_y)$, the conformal invariance of the energy E implies that the function $\widetilde{u}_y \in \mathcal{S}$, as desired. ☑

Let us define the function

$$\widetilde{J}_p(y) = \int_{S^n} K\widetilde{u}_y^{p+1}d\sigma.$$

Now, we will prove that the difference of the functions $\widetilde{J}_p(y)$ and $\overline{J_p}(y) = \int_{S^n} K\alpha_y^{p+1}$ are very close in C^2 norm.

Proposition 5.2. *Let y_0 be a critical point of the function $\overline{J_p}(y)$, and let $y \in B_{\beta(1-|y_0|)}(y_0)$. Then,*

$$|\widetilde{J}_p(y) - \overline{J}_p(y)| \le C\epsilon(p)\mu^\sigma,$$

$$\left|\nabla_y(\widetilde{J}_p(y) - \overline{J}_p(y))\right| \le C\mu^{1-w}$$

and

$$\left|\nabla_y\nabla_y(\widetilde{J}_p(y) - \overline{J}_p(y))\right| \le C\epsilon(p)\mu^{1-2r},$$

where $\sigma < 2$, $0 < w < 1$, $r < \frac{1}{2}$ and $\epsilon(p)$ goes to zero as p goes to $\frac{n+2}{n-2}$.

Proof. A change of variables yields

$$\begin{aligned}\widetilde{J}_p(y) - \overline{J}_p(y) &= \int_{S^n}\left(K\circ F_y\Lambda_y^{p+1}|(F_y)'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta - K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}\right)\\ &= \int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\\ &+ (\Lambda_y^{p+1}-1)\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta.\end{aligned}$$

To estimate this difference, we will do it for the terms in the right hand side separately. The mean value Theorem and Theorem 3.1 implies

$$\begin{aligned}\left|\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right| &\le C\int_{S^n}|\eta_y|d\zeta \le C\|\eta_y\|_\infty\\ &\le C\epsilon(p)\mu^\sigma,\end{aligned}$$

and

$$\left|\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right| \le C.$$

To estimate $(\Lambda_y^{p+1}-1)$, we make a change of variables to get

$$\Lambda_y^2 = \frac{\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}}{\int_{S^n} K\circ F_y|(F_y)'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta}.$$

Since $|\Lambda_y|\le 1$ and $\Lambda_y^2-1=(\Lambda_y-1)(\Lambda_y+1)$, then

$$|\Lambda_y-1|\le C|\Lambda_y^2-1|\le C\left|\frac{I}{M}-1\right|\le C|M-I|,$$

where

$M=\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta$, and $I=\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}d\zeta$.

Then,

$$|\Lambda_y^{p+1}-1|\le C|M-I|\le C\epsilon(p)\mu^\sigma.$$

From the previous estimates we get

$$|\widetilde{J}_p(y)-\overline{J}_p(y)|\le C\epsilon(p)\mu^\sigma.$$

Now, we need to estimate the difference of the first derivatives:

$$\begin{aligned}\nabla_y\left(\widetilde{J}_p(y)-\overline{J}_p(y)\right) =&\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)\\ &+\nabla_y(\Lambda_y^{p+1}-1)\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\\ &+(\Lambda_y^{p+1}-1)\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right).\end{aligned}$$

Let us write the first term in the right hand side as

$$\begin{aligned}\left(\nabla_y\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right) =&\\ =&\int_{S^n}\nabla_y(K\circ F_y)|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)^{p+1}-1]d\zeta\\ &+\int_{S^n} K\circ F_y\nabla_y(|F_y'|^{\frac{n-2}{2}\delta})[(1+\eta_y)^{p+1}-1]d\zeta\\ &+\int_{S^n}\left[K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(p+1)(1+\eta_y)^p\eta_y']\right]d\zeta,\end{aligned}$$

where,

$$\begin{aligned}\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}(p+1)(1+\eta_y)^p\eta_y' d\zeta = &\int_{S^n}(K\circ F_y - 1)|F_y'|^{\frac{n-2}{2}\delta}[(p+1)(1+\eta_y)^p\eta_y']d\zeta\\ &+\int_{S^n}(|F_y'|^{\frac{n-2}{2}\delta}-1)(p+1)(1+\eta_y)^p\eta_y' d\zeta\\ &+\int_{S^n}[(p+1)[(1+\eta_y)^p-1]\eta_y' + (p+1)\eta_y']d\zeta,\end{aligned}$$

Since K is a Morse function, from the proof of Proposition 1.1 in [8] we have that $\|1-K\circ F_y\|_{0,q}\le C\epsilon_0\mu$, where ϵ_0 can be chosen as small as we want. From this fact, the mean value Theorem, Hölder's inequality, Proposition 2.1, Theorem 3.1 and the integral and L^p estimates of the functions η_y and $\eta_y^{'}$, we arrive to

$$\left|\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)\right| \le C\epsilon(p)\mu^{\sigma+1-w}.$$

Analogously,

$$\left|\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}(1+\eta_y)^{p+1}d\sigma\right)\right| \le C\mu^{1-w}.$$

A calculation shows that

$$|\nabla_y(\Lambda_y^{p+1}-1)|\le C|\nabla_y\Lambda_y|\le C_1|\nabla_y(M-I)|+C_2|M-I||\nabla_y M|,$$

and therefore

$$|\nabla_y(\Lambda_y^{p+1}-1)|\le C\epsilon(p)\mu^{\sigma+1-w}+C\epsilon(p)\mu^{\sigma}+C\mu^{1-w}\le C\mu^{1-w}.$$

Consequently,

$$\left|\nabla_y(\widetilde{J}_p(y)-\overline{J}_p(y))\right|\le C\epsilon(p)\mu^{\sigma+1-w}+C\mu^{1-w}\le C\mu^{1-w}.$$

Writing the difference of the second derivatives as

$$\begin{aligned}\nabla_y\nabla_y\left(\widetilde{J}_p(y)-\overline{J}_p(y)\right) =&\nabla_y\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[[(1+\eta_y)]^{p+1}-1]d\zeta\right)\\ &+\nabla_y\nabla_y(\Lambda_y^{p+1}-1)\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\\ &+2\nabla_y(\Lambda_y^{p+1}-1)\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right)\\ &+(\Lambda_y^{p+1}-1)\nabla_y\nabla_y\left(\int_{S^n} K\circ F_y|F_y'|^{\frac{n-2}{2}\delta}[(1+\eta_y)]^{p+1}d\zeta\right),\end{aligned}$$

and working as before, we obtain the desired estimate. ☑

Proposition 5.3. *The function* $\widetilde{J}_p$ *has a unique critical point* y_1 *on* $B_{\beta(1-|y_0|)}(y_0)$.

Proof. The inequalities in Proposition 5.2 imply that there exists $\epsilon > 0$, sufficiently small, such that

$$(1-|y_0|)^{-1}\left|\nabla_y(\widetilde{J}_p(y)-\overline{J}_p(y))\right| + \left|\nabla_y\nabla_y(\widetilde{J}_p(y)-\overline{J}_p(y))\right| \leq \epsilon. \tag{18}$$

For $z \in B^{n+1}$ we define

$$f(z) = (1-|y_0|)^{-2}(\overline{J}_p(y_0+\beta(1-|y_0|)z) - \overline{J}_p(y_0)),$$

$$g(z) = (1-|y_0|)^{-2}(\widetilde{J}_p(y_0+\beta(1-|y_0|)z) - \widetilde{J}_p(y_0)).$$

On one hand, by Proposition 2.2 we have

$$|\nabla f| + |\nabla\nabla f| \leq \left(\frac{|\nabla\overline{J}_p(y_0+\beta(1-|y_0|)z)|}{(1-|y_0|)} - |\nabla\nabla\overline{J}_p(y_0+\beta(1-|y_0|)z)|\right) \leq c,$$

$$\inf_{\partial B^{n+1}} |\nabla f| \geq \frac{\beta}{(1-|y_0|)}\left(\inf_{y\in\partial B_{\beta(1-|y_0|)}(y_0)} |\nabla\overline{J}_p(y)|\right) \geq c^{-1},$$

and

$$|\det \mathrm{Hess} f| = \beta^{2(n+1)}|\det \mathrm{Hess}\overline{J}_p| \geq c^{-1}.$$

On the other hand, inequality (18) implies

$$\|\nabla(f-g)\| + \|\nabla\nabla(f-g)\| \leq \epsilon.$$

Proposition 2.3 implies Proposition 5.3. ☑

If we change, in the proof of Theorem 2.4 of [8], u_{y_1} for $\widetilde{u}_{y_1} = \Lambda_{y_1}u_{y_1}$, and we follow the arguments in there, we get

Proposition 5.4. *The critical point* $\widetilde{u}_{y_1}$ *of the function* $\widetilde{J}_p$ *in Proposition 5.3 is a solution of problem* (6).

Corollary 5.5. *The function* $u = (J_p(\widetilde{u}_{y_1}))^{1-p}\widetilde{u}_{y_1}$ *is a solution of the subcritical problem* (2) *and the function* $u_{y_1} = \Lambda_{y_1}^{-1}\widetilde{u}_{y_1} = \alpha_{F_{y_1}^{-1}}(1+\eta_{y_1})$ *is a solution of the perturbated equation* (3).

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