Hyperbolicity and genuine nonlinearity conditions for certain p-systems of conservation laws, week solutions and the entropy condition

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Abstract. We consider a p-system of conservation laws that emerges in one dimensional elasticity theory. Such system is determined by a function W. We consider four forms of W. These are St.Venant-Kirchhoff, Ogden, Kirchhoff modified, Blatz-Ko-Ogden forms. In each of those cases we determine the conditions for the parameters μ, λ and f, under which the corresponding system is hyperbolic and genuinely nonlinear. We also establish what it means a weak solution of an initial and boundary value problem. Finally we ask if such solutions satisfy the entropy condition. For a standard entropy function we provide a complete answer, except of the Blatz-Ko-Ogden case. For a general strictly convex entropy function the result is that for the initial value of velocity function near zero these solutions satisfy the entropy condition, under the assumption of hyperbolicity and genuine nonlinearity.

Keywords: Weak solution, entropy condition, conservation laws, genuinely nonlinear, p-system.

MSC2010: 35L04, 35L60, 35L65, 35L67.

Condiciones de hiperbolicidad y no linealidad genuina para ciertos p-sistemas de leyes de conservación, soluciones débiles y condición de entropía

Resumen. Consideramos un p-sistema de leyes de conservación que surge de la teoría de elasticidad unidimensional. Tal sistema es determinado por una función W. Consideramos cuatro formas de W. Estas son los modelos de St.

Venant-Kirchhoff, Ogden, Kirchhoff modificado y Blatz-Ko-Ogden. En cada uno de estos casos determinamos las condiciones de los parámetros μ, λ y f bajo las cuales el correspondiente sistema es hiperbólico y genuinamente no lineal. Establecemos qué significa una solución débil de un problema de valor inicial y de frontera. Finalmente nos preguntamos si tales soluciones satisfacen la condición de entropía. Para una función de entropía estándar damos una respuesta completa, excepto del caso de Blatz-Ko-Ogden. Para una función de entropía general estrictamente convexa, el resultado es que para el valor inicial de la función velocidad cerca de cero estas soluciones satisfacen la condición de entropía, bajo la restricción de hiperbolicidad y no linealidad genuina.

Palabras clave: Solución débil, condición de entropía, leyes de conservación, genuinamente no lineal, p-sistema.

1. Introduction

The mathematical theory of hyperbolic systems of conservation laws was started by Eberhardt Hopf in 1950, followed in a series of studies by Olga Oleinik, Peter D. Lax and James Glimm [8] . The class of conservation laws is a very important class of partial differential equations because as their name indicates, they include those equations that model conservation laws of physics (mass, momentum, energy, etc).

As important examples of hyperbolic systems of balance laws arising in continuum physics we have: Euler's equations for compressible gas flow, the one dimensional shallow water equations [5], Maxwell's equations in nonlinear dielectrics, Lundquist's equations of magnetohydrodynamics and Boltzmann equation in thermodynamics [3] and equations of elasticity [9].

One of the main motivations of the theory of hyperbolic systems is that they describe for the most part real physical problems, because they are consistent with the fact that the physical signals have a finite propagation speed [9]. Such systems, even with smooth initial conditions, may fail to have a solution for all time; in such cases we have to extend the concept of classical solutions to the concept of a weak solution or generalized solution [5].

In the case of hyperbolic systems, the notion of weak solution based on distributions does not guarantee uniqueness, and it is necessary to devise admissibility criteria that will hopefully single out a unique weak solution. Several such criteria have indeed been proposed, motivated by physical and/or mathematical considerations. It seems that a consensus has been reached on this issue for such solutions, they are called entropy conditions [3]. Nevertheless, to the question about existence and uniqueness of generalized solutions subject to the entropy conditions, the answer is, in general, open. For the scalar conservation law, the questions existence and uniqueness are basically settled [5]. For genuinely nonlinear systems, existence (but not uniqueness) is known for initial data of small total variation [12]. Some of the main contributors to the field are: Lax, Glimm, Dafermos, Bressan, Marsden and Ogden [1], [2], [3], [4], [7], [9], and [10].

All of this motivates us to study systems of conservation laws that emerge in the theory of elasticity. These systems are determined by constitutive relations between the stress

and strain. For hyperelastic materials, the constitutive relations can be written in a simpler form. Now the stress is determined by a scalar function of the strain called the strain-energy function W. A further simplification of a stress-strain relation is obtained for isotropic materials.

In applications some specific strain-energy functions are used; in our work we consider four different forms of W. In all our studies we restrict ourselves to the case of one dimensional elasticity.

The first important question that arises is the following: given the function W, is the corresponding system of PDE's hyperbolic? By answering it, we can assess how good the model corresponding to that particular W is.

There exists also another important condition called genuine nonlinearity condition, which is related to the entropy condition [12]. According to our previous remarks the entropy condition can be considered a physical one. This implies an importance of genuine nonlinearity condition as well. For that reason our second question is about the validity of that particular condition for the models under study.

Our third important question is how manageable is the entropy condition, that is: given a weak solution of the elasticity system, can we conclude if it is or not an entropy solution? In general, except of the linear case, it is not easy to answer that question, because in the entropy condition there appear two functions: entropy and entropy-flux, which satisfy a given nonlinear system of PDE's, the first of them is convex and otherwise they are arbitrary.

For this reason we restrict ourselves to study the entropy condition for a relatively simple weak solutions, which correspond to a well understood physical situation of what can be called a compression shock. Such solutions are obtained easily in linear case by means of D'Alembert's formula and by analogy in nonlinear case, employing the Rankine-Hugoniot conditions. If for a given model (W function) such solution does not satisfy the entropy condition, we can consider the model as inadequate to describe the compression shock.

In this work we give answers to all mentioned above questions. The obtained results do not appear in the reviewed literature.

It has to be added also that the concept of a weak solution is well known in the literature. For example in [5] one can find a definition of a weak solution of an initial value problem for a system of conservation laws in two variables. Using a general idea of that concept we define what it means to be a weak solution of an initial and boundary value problem for p-systems. This definition does not appear explicitly in the reviewed literature.

The paper is organized as follows: In Section 2 the main notation and concepts are introduced: conservation laws, hyperbolic system, weak solution, Rankine-Hugoniot condition, genuine nonlinearity, entropy/entropy-flux pair. Next, we give a brief presentation of basic concepts of the theory of elasticity, such as, deformation gradient, deformation tensor, second Piola-Kirchhoff stress tensor and first Piola-tensor. We also present four forms of W (strain-energy function) appearing in the theory of elasticity, to model a behavior of certain materials. We refer to them as: St.Venant-Kirchhoff, Kirchhoff modified, Ogden and Blatz-Ko-Ogden functions.

In Section 3 we consider one dimensional reduction of the system of partial differential equations for elasticity, which depends on the strain-energy function W and results in a

p-system. Also, we introduce the notions of hyperbolicity, no interpenetration of matter and genuine nonlinearity.

In Section 4 we provide the concept of weak solutions for various versions of an IBVP (initial and boundary value problem) for a p-system, including a particular case of IBVP, $IBVP_{V_0}$, and we find its solutions employing the Rankine-Hugoniot conditions; we denote such solution by $S(V_0)$.

In Section 5 we discuss the notions of an entropy/entropy-flux pair for a p-system, entropy condition, entropy condition for a solution of $IBVP_{V_0}$ and standard entropy function. We also establish the importance of the requirements of hyperbolicity (strict) and genuine nonlinearity, as being essential in proving if a weak solution is an entropy solution.

In Section 6 we show the results concerning to hyperbolicity and genuine nonlinearity for the models under consideration and the entropy condition corresponding to a standard entropy function for a solution of $IBVP_{V_0}$.

Finally, in Section 7 we present a summary of the main conclusions of our research.

We should remark that the results of this paper are substantially contained in [11], the MSc Thesis of E. Pérez.

2. Preliminaries

2.1. Conservation laws and related concepts

We begin this section with some essential definitions, that we will use in the course of this work.

A conservation law asserts that the change in the total amount of a physical entity contained in any bounded region $G \subset \mathbb{R}^n$ of space is due to the flux of that entity across the boundary of G. In particular, the rate of change is

$$\frac{d}{dt} \int_G \mathbf{u} dX = - \int_{\partial G} \mathbf{F}(\mathbf{u})\mathbf{n} dS, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(X,t) = (u^1(X,t),\dots,u^m(X,t))$ $(X \in \mathbb{R}^n, t \geq 0)$ measures the density of the physical entity under discussion, the vector $\mathbf{F} : \mathbb{R}^m \to \mathbb{M}^{m \times n}$ describes its flux and $\mathbf{n}$ is the outward normal to the boundary ∂G of G. Here $\mathbf{u}$ and $\mathbf{F}$ are C^1 functions. Rewriting (1), we deduce

$$\int_G \mathbf{u}_t dX = - \int_{\partial G} \mathbf{F}(\mathbf{u})\mathbf{n} dS = - \int_G \operatorname{div} \mathbf{F}(\mathbf{u}) dX. \tag{2}$$

As the region $G \subset \mathbb{R}^n$ was arbitrary, we derive from (2) this initial-value problem for a general system of conservation laws:

$$\begin{cases} \mathbf{u}_t + \operatorname{div} \mathbf{F}(\mathbf{u}) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ \mathbf{u} = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{3}$$

where $g = (g^1, \ldots, g^m)$ is a given function describing the initial distribution of $\mathbf{u} = (u^1, \ldots, u^m)$. In particular, the initial-value problem for a system of conservation laws in one-dimensional space, takes the form

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_X = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{4}$$

with initial condition given by

$$\mathbf{u}(X, t) = g \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{5}$$

where $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R} \to \mathbb{R}^m$ are given and $\mathbf{u} : \mathbb{R} \times [0, \infty) \to \mathbb{R}^m$ is the unknown, $\mathbf{u} = \mathbf{u}(X, t)$ [5].

For C^1 functions the conservation law (4) is equivalent to

$$\mathbf{u}_t + \mathbf{B}(\mathbf{u})\mathbf{u}_X = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{6}$$

where $\mathbf{B} : \mathbb{R}^m \to \mathbb{M}^{m \times m}$ is given by $B(z) = DF(z)$, for $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, where

$$D\mathbf{F}(z) = \begin{pmatrix} F_{z_1}^1 & \cdots & F_{z_m}^1 \\ \vdots & \ddots & \vdots \\ F_{z_1}^m & \cdots & F_{z_m}^m \end{pmatrix}. \tag{7}$$

If for each $z \in \mathbb{R}^m$ the eigenvalues of $\mathbf{B}(z)$ are real and distinct, we call the system (6) **strictly hyperbolic** [5].

A system of conservation laws (6) is said to be **genuinely nonlinear** in a region $\Omega \subseteq \mathbb{R}^n$ if

$$\nabla \lambda_k \cdot \mathbf{r}_k \neq 0,$$

for $k = 1, 2, \ldots, n$ at all points in Ω, where $\lambda_k(z)$ are the eigenvalues of $\mathbf{B}(z)$, with corresponding eigenvectors $\mathbf{r}_k(z)$ [12].

Definition 2.1. The p-system is a conservation law, being this a collection of two equations:

$$\begin{cases} u_t^2 - p(u^1)_X = 0 \text{ (Newton's law)}, \\ \quad u_t^1 - u_X^2 = 0 \text{ (compatibility condition)}, \end{cases} \tag{8}$$

in $\mathbb{R} \times (0, \infty)$, where $p : \mathbb{R} \to \mathbb{R}$ is given. Here $F(z) = (-p(z_1), -z_2)$ for $z = (z_1, z_2)$ [5].

Definition 2.2. A weak solution of (4) is a function $\mathbf{u} \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R}^m)$ such that

$$\int_0^\infty \int_{-\infty}^\infty (\mathbf{u} \cdot \phi_t + \mathbf{F}(\mathbf{u}) \cdot \phi_X) dX dt + \int_{-\infty}^\infty (\mathbf{g} \cdot \phi)|_{t=0} \; dX = 0$$

for every smooth $\phi : \mathbb{R} \times [0, \infty) \to \mathbb{R}^m$, with compact support [5].

2.2.   Basic notions of Elasticity Theory

We consider a continuous body which occupies a connected open subset of a three-dimensional Euclidean point space, and we refer to such a subset as a configuration of the body. We identify an arbitrary configuration as a reference configuration and denote this by $\mathfrak{B}_o$. Let points in $\mathfrak{B}_o$ be labelled by their position vectors $\mathbf{X} = (X^1, X^2, X^3)$, where X^1, X^2 and X^3 are coordinates relative to an arbitrary chosen Cartesian orthogonal coordinate system. Now, suppose that the body is deformed from $\mathfrak{B}_o$ so that it occupies a new configuration, which is denoted by $\mathfrak{B}_t$. We refer to $\mathfrak{B}_t$ as the deformed configuration of the body. The deformation is represented by the mapping $\phi_t : \mathfrak{B}_o \to \mathfrak{B}_t$ which takes points $\mathbf{X}$ in $\mathfrak{B}_o$ to points $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathfrak{B}_t$, where x_1, x_2 and x_3 are coordinates relative to the same Cartesian orthogonal coordinate system as X^1, X^2 and X^3. Thus, the position vector of the point $\mathbf{X}$ in $\mathfrak{B}_t$, which is denoted by $\mathbf{x}$, is

$$\mathbf{x} = \phi(\mathbf{X}, \mathbf{t}) \equiv \phi_{\mathbf{t}}(\mathbf{X}).$$

The mapping ϕ is called the deformation from $\mathfrak{B}_o$ to $\mathfrak{B}_t$. We require ϕ_t to be sufficiently smooth, orientation preserving and invertible. The last two requirements mean physically, that no interpenetration of matter occurs.

Deformation gradient, deformation tensor, strain-energy function and time evolution of an elastic body

Now, we introduce some basic definitions of Elasticity theory, namely: deformation gradient, deformation tensor, second Piola-Kirchhoff stress tensor, first Piola-tensor [9]. We restrict our discussion to hyperelastic, homogeneous and isotropic materials.

- $F_A^a(X, t) = \frac{\partial \phi^a}{\partial X^A}$ (deformation gradient), $a, A \in \{1, 2, 3\}$.

- $C = F^T F$ or componentwise by $C_{AB} = \delta_{ij} F_A^i F_B^j$, $A, B \in \{1, 2, 3\}$ (deformation tensor).

- Principal invariants of C:
  $I_1 = \mathrm{tr}(C), I_2 = (\det(C))\mathrm{tr}(C^{-1}), I_3(C) = \det(C)$.

- Second Piola-Kirchhoff stress tensor:

$$S^{AB} = 2\left\{ \frac{\partial W}{\partial I_1} G^{AB} + \left(\frac{\partial W}{\partial I_2} I_2 + \frac{\partial W}{\partial I_3} I_3 \right) C^{-1} - \frac{\partial W}{\partial I_2} I_3 C^{-2} \right\},$$

  where G^{AB} is Kronecker's delta and W is the strain-energy function.

- The first Piola-tensor:

$$P^{iA} = F_B^i S^{BA} = F_1^i S^{1A} + F_2^i S^{2A} + F_3^i S^{3A}, \text{ where } i, A, B \in \{1, 2, 3\}.$$

We consider the following four forms of W, [9]:

1. St.Venant-Kirchhoff:

$$W = \frac{\lambda}{8}(I_1 - 3)^2 + \frac{\mu}{4}(I_1^2 - 2I_2 - 2I_1 + 3). \tag{9}$$

2. Kirchhoff modified:

$$W = \frac{\lambda}{8}(\ln I_3)^2 + \frac{\mu}{4}(I_1^2 - 2I_2 - 2I_1 + 3). \tag{10}$$

3. Ogden:

$$W = \frac{\mu}{2}\left(I_1 - 3 - 2\ln(\sqrt{I_3})\right) + \frac{\lambda}{2}(\sqrt{I_3} - 1)^2. \tag{11}$$

4. Blatz-Ko-Ogden:

$$W = f \cdot \frac{\mu}{2}[(I_1 - 3) + \frac{1}{\beta}(I_3^{-\beta} - 1)] + (1 - f)\frac{\mu}{2}\left[\frac{I_2}{I_3} - 3 + \frac{1}{\beta}(I_3^{\beta} - 1)\right]. \tag{12}$$

We can see that the functions (9)-(11) depend on two parameters: Lamé moduli λ and μ, where $\lambda, \mu > 0$. In (12) $\beta = \frac{\lambda}{2\mu}$ and this W depends also on a parameter f restricted by $0 < f < 1$.

Finally, the components of the mapping

$$\phi(\mathbf{X}, t) = (\phi^1(\mathbf{X}, t), \phi^2(\mathbf{X}, t), \phi^3(\mathbf{X}, t))$$

are subject to the following system of PDE's, describing the evolution of an elastic body:

$$\rho_0 \frac{\partial^2 \phi^i}{\partial t^2} = \frac{\partial P^{iA}}{\partial X^A}. \tag{13}$$

Here $\rho_0 = \rho_0(\mathbf{X})$ is the mass density in reference configuration assumed further to be constant.

3. One-dimensional reduction for certain models of elastic materials

In this section we present the reduction to the one-dimensional case, which we will maintain in all the paper. Also, we rewrite the requirements of hyperbolicity, no inter-penetration of matter and genuine nonlinearity, to the one-dimensional case.

We assume that there is a motion of particles only in the direction of X^1-axis, that is:

$$\begin{cases} \phi^1(\mathbf{X}, t) = X^1 + U(X^1, t), \\ \phi^2(\mathbf{X}, t) = X^2, \\ \phi^3(\mathbf{X}, t) = X^3. \end{cases} \tag{14}$$

Then $F_A^i, C_{AB}, C_{AB}^{-1}, I_1, I_2$ and I_3 become

$$F_A^i = \begin{pmatrix} \frac{\partial \phi^1}{\partial X^1} & \frac{\partial \phi^1}{\partial X^2} & \frac{\partial \phi^1}{\partial X^3} \\[6pt] \frac{\partial \phi^2}{\partial X^1} & \frac{\partial \phi^2}{\partial X^2} & \frac{\partial \phi^2}{\partial X^3} \\[6pt] \frac{\partial \phi^3}{\partial X^1} & \frac{\partial \phi^3}{\partial X^2} & \frac{\partial \phi^3}{\partial X^3} \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi^1}{\partial X^1} & 0 & 0 \\[6pt] 0 & 1 & 0 \\[6pt] 0 & 0 & 1 \end{pmatrix}, \qquad (15)$$

$$C_{AB} = \begin{pmatrix} (F_1^1)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ C_{AB}^{-1} = \begin{pmatrix} 1/(F_1^1)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (16)$$

$$I_1 = 2 + (F_1^1)^2, I_2 = 2(F_1^1)^2 + 1, I_3 = (F_1^1)^2.$$

Therefore the system (13) becomes

$$\rho_0 \frac{\partial^2 \phi^i}{\partial t^2} = \frac{\partial (P^{iA})}{\partial X^A}.$$

More specifically,

$$\begin{aligned} \rho_0 \frac{\partial^2 \phi^1}{\partial t} &= P,_1^{11} + P,_2^{12} + P,_3^{13}, \\ \rho_0 \frac{\partial^2 \phi^2}{\partial t} &= P,_1^{21} + P,_2^{22} + P,_3^{23}, \\ \rho_0 \frac{\partial^2 \phi^3}{\partial t} &= P,_1^{31} + P,_2^{32} + P,_3^{33}. \end{aligned} \qquad (17)$$

Notice that $P^{11} = F_1^1 S^{11} + F_2^1 S^{21} + F_3^1 S^{31} = F_1^1 S^{11}$, $P^{12} = P^{21} = P^{13} = P^{31} = P^{23} = P^{32} = 0, P^{22} = S^{22}, P^{33} = S^{33}$, and $\frac{\partial \phi^1}{\partial t} = \frac{\partial U}{\partial t}$, $\frac{\partial \phi^2}{\partial t} = 0 = \frac{\partial \phi^3}{\partial t}$. Consequently, (17) is reduced to one equation, which after denoting X^1 by X and putting $P = \frac{P^{11}}{\rho_0}$, reads

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial P}{\partial X}. \qquad (18)$$

Setting $V = \frac{\partial U}{\partial t}$ and $\Gamma = \frac{\partial U}{\partial X}$, one obtains a p-system of first order PDE's:

$$\begin{cases} V_t - (P(\Gamma))_X &= 0, \\ \Gamma_t - V_X &= 0. \end{cases} \qquad (19)$$

Remark 3.1. Under the assumption (14) the requirement of no interpenetration of matter means that $\phi_X > 0$, i.e., $1 + \mathbf{U}_X > 0$.

Notice that, the p-system (19) can be rewritten as

$$\mathbf{u}_t + \mathbf{B}(\mathbf{u})\mathbf{u}_X = 0, \qquad (20)$$

where $\mathbf{u} = (V, \Gamma)$ and $\mathbf{B} = \begin{pmatrix} 0 & -P'(\Gamma) \\ -1 & 0 \end{pmatrix}$. The eigenvalues of $\mathbf{B}$ are $\lambda_1 = -\sqrt{P'(\Gamma)}$ and $\lambda_2 = \sqrt{P'(\Gamma)}$ with corresponding eigenvectors $\mathbf{r}_1 = (\sqrt{P'(\Gamma)}, 1)$ and $\mathbf{r}_2 = (-\sqrt{P'(\Gamma)}, 1)$.

Remark 3.2. Note that for our case of a p-system, no interpenetration of matter condition, $\phi_X > 0$, is equivalent to $\Gamma > -1$, since $\phi_X = 1 + U_X$.

Remark 3.3. ■ The p-system (20) is **strictly hyperbolic** if $P' > 0$, everywhere in the domain of $P(\Gamma)$.

■ The p-system (20) is **genuinely nonlinear** in a region Ω of the domain of $P(\Gamma)$ if $P'' \neq 0$ everywhere in Ω.

Indeed, it is so since $-\nabla\lambda_1 \cdot \mathbf{r}_1 = \nabla\lambda_2 \cdot \mathbf{r}_2 = \frac{P''(\Gamma)}{2\sqrt{P'(\Gamma)}}$.

By continuity of $P''(\Gamma)$, genuine nonlinearity means that $P''(\Gamma)$ is of constant sign in Ω. However, we will call a p-system (20) genuinely nonlinear if $P'' < 0$, since this requirement plays an important role in studying entropy inequality.

We remark also that hyperbolicity condition is an essential physical requirement, since it guarantees that particles have a finite propagation speed. Now, we obtain explicit forms of the function P for the models under consideration. Indeed,

St.Venant-Kirchhoff: $P(\Gamma) = \left(\frac{\lambda+2\mu}{2\rho_0}\right)(1+\Gamma)(2+\Gamma)\Gamma$.

Modified Kirchhoff: $P(\Gamma) = \frac{1}{\rho_0}\left(\mu(1+\Gamma)^3 - \mu(1+\Gamma) + \lambda\frac{\ln(1+\Gamma)}{(1+\Gamma)}\right)$.

Ogden: $P(\Gamma) = \frac{1}{\rho_0}\left(\lambda\Gamma + \mu\frac{(2+\Gamma)\Gamma}{\Gamma+1}\right)$.

Blatz-Ko and Ogden:

$$P(\Gamma) = \frac{\mu(1+\Gamma)}{\rho_0}\left\{f\left[1 - (1+\Gamma)^{-2\beta-2}\right] + \frac{(1-f)}{(1+\Gamma)^4}\left[(1+\Gamma)^{2\beta+2} - 1\right]\right\}. \tag{21}$$

Definition 3.4. If $P(\Gamma) = \frac{(\lambda+2\mu)}{\rho_0}\Gamma$, the model is called linear model.

4. Weak solution of an IBVP for a p-system

In this section we give the concept of weak solutions for various versions of an IBVP (initial and boundary value problem), for a p-system, including a particular case of IBVP, $IBVP_{V_0}$. We also provide notions of an entropy/entropy-flux pair and entropy condition for a solution of $IBVP_{V_0}$.

Our aim is to give an answer to the question about a weak solution for an IVBP for (19) with these initial and boundary conditions:

$$\begin{cases} V(X, 0) = f(X), \\ \Gamma(X, 0) = g(X), \\ P(\Gamma(0, t)) + a(t)V(0, t) = c(t), \end{cases} \tag{22}$$

or

$$\begin{cases} V(X,0) = f(X), \\ \Gamma(X,0) = g(X), \\ V(0,t) + b(t)P(\Gamma(0,t)) = c(t). \end{cases} \qquad (23)$$

To define a weak solution of such IBVP in the first quadrant of the Xt-plane, we use arbitrary C^1 functions φ, ψ and χ,

$$\varphi, \psi, \chi : [0, \infty) \times [0, \infty) \to \mathbb{R}$$

of compact supports. We refer to those functions as test functions.

Proposition 4.1. *Let f, g, a and c be C^1 functions on $[0, \infty)$, and let $V(X,t)$, $\Gamma(X,t)$ be C^1 functions on $[0, \infty)^2$, such that $P(\Gamma(X,t))$ is C^1 on $([0, \infty)^2)$. Then the pair (V, Γ) is a classical solution of IBVP (19), (22), if and only if for all φ and ψ, with ψ satisfying the condition $\psi(0,t) = 0$, it holds*

$$-\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma\psi_t dtdX + \int_0^\infty \int_0^\infty V\psi_X dXdt = 0 \qquad (24)$$

and

$$-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty \int_0^\infty V\varphi_t dtdX + \int_0^\infty c(t)\varphi(0,t)dt$$
$$+ \int_0^\infty \int_0^\infty P(\Gamma)\varphi_X dXdt - \int_0^\infty g(X)a(0)\varphi(X,0)dX$$
$$- \int_0^\infty \int_0^\infty (a(t)\varphi(X,t))_t \Gamma dtdX + \int_0^\infty \int_0^\infty a(t)\varphi_X(X,t)V dXdt = 0. \qquad (25)$$

Proof. Indeed, assuming that (V, Γ) is a classical solution of IBVP (19),(22), we multiply the first equation in (19) by φ; integrating by parts and using the initial and boundary conditions (22) we obtain

$$-\int_0^\infty f(X)\varphi(X,0)dX - \int_0^\infty \int_0^\infty V(X,t)\varphi_t(X,t)dtdX + \int_0^\infty c(t)\varphi(0,t)dt$$
$$-\int_0^\infty a(t)V(0,t)\varphi(0,t)dt + \int_0^\infty \int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt = 0. \qquad (26)$$

Similarly, multiplying the second equation in (19) by a test function χ and integrating by parts results in

$$-\int_0^\infty g(X)\chi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)\chi_t(X,t)dtdX + \int_0^\infty V(0,t)\chi(0,t)dt$$
$$+ \int_0^\infty \int_0^\infty V(X,t)\chi_X(X,t)dXdt = 0. \qquad (27)$$

For $\chi = \psi$ the equation (27) is equivalent to

$$-\int_0^\infty g(X)\psi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)\psi_t(X,t)dtdX$$
$$+ \int_0^\infty \int_0^\infty V(X,t)\psi_X(X,t)dXdt = 0,$$

which is (24). Next, assuming that $\chi(X,t) = a(t)\varphi(X,t)$, the equation (27) becomes

$$-\int_0^\infty g(X)a(0)\varphi(X,0)dX - \int_0^\infty \int_0^\infty \Gamma(X,t)(a(t)\varphi(X,t))_t dtdX +$$
$$\int_0^\infty a(t)V(0,t)\varphi(0,t)dt + \int_0^\infty \int_0^\infty V(X,t)(a(t)\varphi(X,t))_X dXdt = 0. \quad (28)$$

Now, adding (26) to (28), we get

$$-\int_0^\infty f(X)\varphi(X,0)dx - \int_0^\infty \int_0^\infty V(X,t)\varphi_t(X,t)dtdX + \int_0^\infty c(t)\varphi(0,t)dt$$
$$+ \int_0^\infty \int_0^\infty P(\Gamma)\varphi_X(X,t)dXdt - \int_0^\infty g(X)a(0)\varphi(X,0)dX$$
$$- \int_0^\infty \int_0^\infty \Gamma(X,t)(a(t)\varphi(X,t))_t dtdX + \int_0^\infty \int_0^\infty V(X,t)(a(t)\varphi(X,t))_X dXdt = 0,$$

which is (25). Next, it remains to verify that if (V,Γ) satisfies (24) and (25) for all φ and ψ, then (V,Γ) is a classical solution of the IBVP (19), (22). Indeed, integrating by parts the equation (24) we obtain

$$\int_0^\infty \psi(X,0)(\Gamma(X,0) - g(X))dX + \int_0^\infty \int_0^\infty \psi(X,t)(\Gamma_t(X,t) - V_X(X,t))dXdt = 0. \quad (29)$$

If in addition ψ has compact support in $(0,\infty) \times (0,\infty)$, we obtain

$$\int_0^\infty \int_0^\infty \psi(X,t)(\Gamma_t(X,t) - V_X(X,t))dXdt = 0,$$

for all such test functions ψ. Therefore we conclude

$$\Gamma_t(X,t) - V_X(X,t) = 0. \quad (30)$$

Now, since $\Gamma_t(X,t) - V_X(X,t) = 0$, then (29) reduces to

$$\int_0^\infty \psi(X,0)(\Gamma(X,0) - g(X))dX = 0,$$

and this holds for all function ψ, with compact support in $[0,\infty) \times [0,\infty)$, containing points on the X-axis, and subject to $\psi(0,t) = 0$. So, we get

$$\Gamma(X,0) = g(X). \quad (31)$$

Similarly, integrating by parts the equation (25), we obtain

$$\int_0^\infty \varphi(X,0)(V(X,0) - f(X))dX + \int_0^\infty \varphi(0,t)(c(t) - P(\Gamma(0,t)) - V(0,t)a(t))dt$$

$$+ \int_0^\infty \int_0^\infty \varphi(X,t)(V_t(X,t) - (P(\Gamma))_X)dXdt + \int_0^\infty a(0)\varphi(X,0)(\Gamma(X,0) - g(X))dX$$

$$+ \int_0^\infty \int_0^\infty a(t)\varphi(X,t)(\Gamma_t(X,t) - V_X(X,t))dXdt = 0, \quad (32)$$

which because of (30) and (31) becomes

$$\int_0^\infty \varphi(X,0)(V(X,0) - f(X))dX + \int_0^\infty \varphi(0,t)(c(t) - P(\Gamma(0,t)) - V(0,t)a(t))dt$$

$$+ \int_0^\infty \int_0^\infty \varphi(X,t)(V_t(X,t) - (P(\Gamma))_X)dXdt = 0. \quad (33)$$

If in addition φ has compact support in $(0,\infty) \times (0,\infty)$, we obtain

$$\int_0^\infty \int_0^\infty \varphi(X,t)(V_t(X,t) - (P(\Gamma))_X)dXdt = 0$$

for all such test functions φ. Therefore we conclude that $V_t(X,t) - (P(\Gamma))_X = 0$. Now, if $V_t(X,t) - (P(\Gamma))_X = 0$, then the equation (33) becomes

$$\int_0^\infty \varphi(X,0)(V(X,0) - f(X))dX + \int_0^\infty \varphi(0,t)(c(t) - P(\Gamma(0,t)) - V(0,t)a(t))dt = 0. \quad (34)$$

Assuming that φ has compact support in $[0,\infty) \times [0,\infty)$ containing points on the X-axis, but not on the t-axis, we get $\int_0^\infty \varphi(X,0)(V(X,0) - f(X))dX = 0$, thus $V(X,0) - f(X) = 0$; then (34) reduces to

$$\int_0^\infty \varphi(0,t)(c(t) - P(\Gamma(0,t)) - V(0,t)a(t))dt = 0.$$

Since this holds for all function φ with compact support in $[0,\infty) \times [0,\infty)$, and containing points on the t-axis, then it follows that

$$P(\Gamma(0,t)) + a(t)V(0,t) = c(t).$$

Therefore we conclude that (V,Γ) is a classical solution for the IVBP (19), (22).   ☑

Proposition 4.1 suggests the following definition:

Definition 4.2. Let f, g and $c \in L^\infty([0,\infty))$, and $a \in C^1([0,\infty))$. We say that the pair $(V, \Gamma) \in L^\infty([0,\infty)^2)$, such that $P(\Gamma(X,t)) \in L^\infty([0,\infty)^2)$, is a weak solution of IBVP (19), (22), provided (24) and (25) hold for all test functions φ and ψ with ψ restricted by $\psi(0,t) = 0$.

Next, similarly we have an proposition analogous to the Proposition 4.1, for the IBVP (19), (23).

Proposition 4.3. *Let f, g, b and c be C^1 functions on $[0, \infty)$, and let $V(X, t)$, $\Gamma(X, t)$ be C^1 functions on $[0, \infty)^2$, such that $P(\Gamma(X, t))$ is C^1 on $[0, \infty)^2$. Then the pair (V, Γ) is a classical solution of IBVP (19),(23), if and only if, for all φ and ψ, with φ satisfying the condition $\varphi(0, t) = 0$, it holds*

$$-\int_0^\infty f(X)\varphi(X, 0)dX - \int_0^\infty \int_0^\infty V(X, t)\varphi_t(X, t)dt dX$$
$$+\int_0^\infty \int_0^\infty P(\Gamma)\varphi_X(X, t)dX dt = 0 \quad (35)$$

and

$$-\int_0^\infty g(X)\psi(X, 0)dX - \int_0^\infty \int_0^\infty \Gamma(X, t)\psi_t(X, t)dt dX + \int_0^\infty c(t)\psi(0, t)dt$$
$$+\int_0^\infty \int_0^\infty V(X, t)\psi_X(X, t)dX dt - \int_0^\infty f(X)b(0)\psi(X, 0)dX$$
$$-\int_0^\infty \int_0^\infty (b(t)\psi(X, t))_t V dt dX + \int_0^\infty \int_0^\infty b(t)\psi_X P(\Gamma)dX dt = 0. \quad (36)$$

Proposition 4.3, suggests the following definition:

Definition 4.4. Let $f, g, c \in L^\infty([0, \infty))$, and $b \in C^1([0, \infty))$. We say that the pair $(V, \Gamma) \in L^\infty([0, \infty)^2)$, such that $P(\Gamma(X, t)) \in L^\infty([0, \infty)^2)$, is a weak solution of the IBVP (19), (23), provided (35) and (36) hold for all test functions φ and ψ with φ restricted by $\varphi(0, t) = 0$.

Remark 4.5. Note that setting $b(t) = 0$ and $c(t) = h(t)$ in (36) we obtain (39). This is consistent with the fact that the initial and boundary conditions (23) are equivalent to (37) for this choice of b and c.

Now, we consider a p-system (19), with the following initial and boundary conditions:

$$\begin{cases} V(X, 0) = f(X), \\ \Gamma(X, 0) = g(X), \\ V(0, t) = h(t). \end{cases} \quad (37)$$

Notice that (37) is a particular case of the condition (23).

Proposition 4.6. *Let f, g and h be C^1 functions on $[0, \infty)$, and let $V(X, t)$, $\Gamma(X, t)$ be C^1 functions on $[0, \infty)^2$, such that $P(\Gamma(X, t))$ is C^1 on $[0, \infty)^2$. Then the pair (V, Γ) is a classical solution of IBVP (19), (37), if and only if for all φ and ψ, with φ satisfying the condition $\varphi(0, t) = 0$, it holds*

$$\int_0^\infty \int_0^\infty (V\varphi_t - P(\Gamma)\varphi_X)dtdX + \int_0^\infty f(X)\varphi(X,0)dX = 0 \qquad (38)$$

and

$$\int_0^\infty \int_0^\infty (\Gamma\psi_t - V\psi_X)dtdX - \int_0^\infty h(t)\psi(0,t)dt + \int_0^\infty g(X)\psi(X,0)dt = 0. \qquad (39)$$

Proposition 4.6, suggests the following definition:

Definition 4.7. Let $f, g, h \in L^\infty([0,\infty))$. We say that the pair $(V,\Gamma) \in L^\infty([0,\infty)^2)$, such that $P(\Gamma(X,t)) \in L^\infty([0,\infty)^2)$, is a weak solution of IBVP (19), (37), provided (38) and (39) hold for all test functions φ and ψ, with φ restricted by $\varphi(0,t) = 0$.

Remark 4.8. An interesting question arises in what sense (V,Γ) satisfies (37). To answer that question we need to prove that the traces, [5], [12], of (V,Γ) on the positive part of the X axes and of V on the positive part of the t axes exist and are equal to $f(X)$, $g(X)$ and $h(t)$, respectively. That problem seems to be non trivial. Its solution does not appear in the revised literature.

Remark 4.9. A similar argument shows that by replacing $\varphi(0,t) = 0$ by $\psi(0,t) = 0$, in the Definition 4.7, we can arrive to an analogous definition of a weak solution of the system (19), with the following initial and boundary conditions

$$\begin{cases} V(X,0) = f(X), \\ \Gamma(X,0) = g(X), \\ P(\Gamma(0,t)) = h(t). \end{cases} \qquad (40)$$

4.1. An initial and boundary value problem, $IBVP_{V_0}$, for a p-system

We consider a particular case of an IBVP, (19), (37), denoted further by $IBVP_{V_0}$:

$$\begin{cases} V(X,0) = -V_0, \\ \Gamma(X,0) = 0, \\ V(0,t) = 0. \end{cases} \qquad (41)$$

Theorem 4.10. *The pair (V,Γ), given by the equations (42) and (43), is a weak solution of $IBVP_{V_0}$ (19), (41),*

$$V(X,t) = \begin{cases} -V_0, & \text{if } X > \sigma t, \\ 0, & \text{if } X < \sigma t, \end{cases} \qquad (42)$$

$$\Gamma(X,t) = \begin{cases} 0, & \text{if } X > \sigma t, \\ \Gamma_l, & \text{if } X < \sigma t, \end{cases} \qquad (43)$$

where Γ_l and σ are determined by the Rankine-Hugoniot conditions, that is:

$$\Gamma_l P(\Gamma_l) = V_0^2,$$
$$\sigma = -\frac{V_0}{\Gamma_l}. \qquad (44)$$

We observe that the system (44) has an unique solution (σ, Γ_l) provided the first equation has an unique solution for Γ_l. We denote such solution by $S(V_0)$, and because of the relation between V_0 and Γ_l, by $S(\Gamma_l)$ as well. Concerning solvability of the first equation, we notice the following fact.

Lemma 4.11. *Let* $P(0) = 0$, $\lim\limits_{\Gamma \to -1^+} P(\Gamma) = -\infty$ *and for all* $\Gamma \in (-1, 0)$, $P'(\Gamma) > 0$. *Then, for each* $V_0 > 0$ *there exists an unique* $\Gamma_l \in (-1, 0)$ *such that* $\Gamma_l P(\Gamma_l) = V_0^2$.

Proof. Let $H(\Gamma) := \Gamma P(\Gamma)$. Then $H'(\Gamma) = P(\Gamma) + \Gamma P'(\Gamma) < 0$ for all $\Gamma \in (-1, 0)$, therefore $H(\Gamma)$ is decreasing. Since $H(0) = 0$ and $\lim\limits_{\Gamma \to -1^+} P(\Gamma) = -\infty$, we conclude that for each $V_0 > 0$, there exists an unique $\Gamma_l \in (-1, 0)$ such that $\Gamma_l P(\Gamma_l) = V_0^2$. ☑

Proof of Theorem 4.10

We verify that $S(\Gamma_l)$ is indeed a weak solution of $IBVP_{V_0}$ (19), (41), i.e., we verify that $S(\Gamma_l)$ satisfies the equations (38) and (39) for all test functions φ and ψ, with φ restricted by the condition $\varphi(0, t) = 0$. We also assume that $P(0) = 0$. Here $f(X) = -V_0$, $g(X) = 0$, and $h(t) = 0$ in (37). We verify (38) first. Its left-hand side is:

$$\int_0^\infty \int_0^\infty (V\varphi_t - P(\Gamma)\varphi_X)dtdX + \int_0^\infty f(X)\varphi(X, 0)dX$$

$$= -V_0 \int_0^\infty \int_0^{X/\sigma} \varphi_t dtdX + \int_0^\infty \int_{X/\sigma}^\infty 0 \cdot \varphi_t dtdX - P(\Gamma_l) \int_0^\infty \int_0^{\sigma t} \varphi_X dXdt$$

$$- P(0) \int_0^\infty \int_{\sigma t}^\infty \varphi_X dXdt - V_0 \int_0^\infty \varphi(X, 0)dX$$

$$= -V_0 \int_0^\infty \varphi(X, X/\sigma)dX + V_0 \int_0^\infty \varphi(X, 0)dX - P(\Gamma_l) \int_0^\infty \varphi(\sigma t, t)dt$$

$$+ P(\Gamma_l) \int_0^\infty \varphi(0, t)dt - V_0 \int_0^\infty \varphi(X, 0)dX$$

$$= -V_0 \int_0^\infty \varphi(X, X/\sigma)dX - \frac{P(\Gamma_l)}{\sigma} \int_0^\infty \varphi(X, X/\sigma)dX,$$

which is zero due to (44), so that (38) holds. Next, for the left-hand side of (39) we have:

$$\int_0^\infty \int_0^\infty (\Gamma\psi_t - V\psi_X)dtdX$$

$$= \int_0^\infty \int_0^{X/\sigma} 0 \cdot \psi_t dtdX + \Gamma_l \int_0^\infty \int_{X/\sigma}^\infty \psi_t dtdX - \int_0^\infty \int_0^{\sigma t} 0 \cdot \psi_X dXdt$$

$$+ V_0 \int_0^\infty \int_{\sigma t}^\infty \psi_X dXdt$$

$$= \Gamma_l \int_0^\infty \int_{X/\sigma}^\infty \psi_t dtdX + V_0 \int_0^\infty \int_{\sigma t}^\infty \psi_X dXdt$$

$$= -\Gamma_l \int_0^\infty \psi(X, X/\sigma)dX - \frac{V_0}{\sigma} \int_0^\infty \psi(X, X/\sigma)dX,$$

which is zero due to (44), so that (39) holds.

5.   Entropy condition and entropy solution for a p-system

The entropy/entropy-flux pair for a p-system is a pair of real valued $C^2(\mathbb{R}^2)$ functions $\Phi(V,\Gamma)$ and $\Psi(V,\Gamma)$, where Φ is convex, and such that

$$D\Phi(V,\Gamma)DF(V,\Gamma) = D\Psi(V,\Gamma), \tag{45}$$

where $F(V,\Gamma) = (-P(\Gamma),-V)$. Working out that condition one obtains

$$
\begin{aligned}
\Psi_V &= -\Phi_\Gamma, \\
\Psi_\Gamma &= -P'(\Gamma)\Phi_V.
\end{aligned}
\tag{46}
$$

Now, the integrability condition of the system (46) for Ψ is

$$\Phi_{\Gamma\Gamma} - P'(\Gamma)\Phi_{VV} = 0. \tag{47}$$

Given a convex function Φ that fulfills this equation, we can obtain Ψ by solving the system (46).

Definition 5.1. A weak solution $V(X,t)$, $\Gamma(X,t)$ of an IBVP is an entropy solution provided for each nonnegative $\varphi \in C_0^\infty((0,\infty) \times (0,\infty))$ and for each entropy/entropy-flux pair Φ, Ψ it holds

$$\int_0^\infty \int_0^\infty [\Phi(V,\Gamma)\varphi_t(X,t) + \Psi(V,\Gamma)\varphi_X(X,t)]dX dt \geq 0. \tag{48}$$

We refer to (48) as the entropy condition corresponding to (Φ,Ψ).

Remark 5.2. If a trivial solution $(V,\Gamma) = (0,0)$ is a solution of an IBVP, then it is an entropy solution.

The following proposition translates the entropy condition (48) into a jump condition for piecewise continuous weak solutions.

Proposition 5.3. *Suppose that $\mathbf{u} = (V,\Gamma)$ is a piecewise continuous weak solution of (4) that satisfies the entropy condition corresponding to (Φ,Ψ). Suppose $\mathbf{u}$ has a jump discontinuity along a shock curve with slope σ. Then,*

$$\sigma[\Phi(\mathbf{u})] - [\Psi(\mathbf{u})] \geq 0. \tag{49}$$

We call (49) the entropy jump condition corresponding to (Φ,Ψ).

Proof. We demonstrate (49), for our solution, (42) and (43).

Here $\mathbf{u} = (V,\Gamma)$ satisfies the inequality (48); therefore, we get

$$\left(\int_0^\infty \varphi(X,X/\sigma)dX\right)(\Phi(-V_0,0) - \Phi(0,\Gamma)) + \left(\int_0^\infty \varphi(t\sigma,t)dt\right)(\Psi(0,\Gamma) - \Psi(-V_0,0)) \geq 0.$$

And after the substitution $X = t\sigma$, in the integral with respect to t, we obtain:

$$\left(\int_0^\infty \varphi(X, X/\sigma) dX \right) \left[(\Phi(-V_0, 0) - \Phi(0, \Gamma)) + \frac{1}{\sigma}((\Psi(0, \Gamma) - \Psi(-V_0, 0)) \right] \geq 0.$$

Here $\varphi \geq 0$, therefore this last inequality is equivalent to

$$\Phi(-V_0, 0) - \Phi(0, \Gamma) \geq \frac{\Psi(-V_0, 0) - \Psi(0, \Gamma)}{\sigma}, \tag{50}$$

whose compact form is (49). ☑

The following theorem states that in the case of genuine nonlinear systems, the entropy condition is satisfied for Γ sufficiently close to zero.

Theorem 5.4. *If $P(0) = 0$, $P'(0) > 0$ and $P''(0) < 0$, then for each entropy/entropy-flux pair (Φ, Ψ), where Φ is strictly convex, $S(\Gamma)$ satisfies the entropy condition corresponding to (Φ, Ψ), for all Γ sufficiently close to zero and $\Gamma \leq 0$.*

Proof. The proof is based on Taylor's expansion formula. We notice that for $\Gamma = 0$, $S(0) = 0$. Therefore by Remark 5.2 this solution is an entropy solution. Now, we consider $\Gamma < 0$. Let $\epsilon = -V_0$; then $\epsilon = -\sqrt{\Gamma P(\Gamma)}$, and $\sigma = \frac{\epsilon}{\Gamma}$. Define

$$E(\Gamma) = \frac{\epsilon}{\Gamma}[\Phi(0, \Gamma) - \Phi(\epsilon, 0)] - [\Psi(0, \Gamma) - \Psi(\epsilon, 0)].$$

Consequently, the entropy jump condition, (49), holds if and only if $E(\Gamma) \leq 0$. We now let a "prime" indicate differentiation with respect to Γ. Notice that

$$\lim_{\Gamma \to 0^-} \left(\frac{\epsilon}{\Gamma} \right) = \sqrt{P'(0)} \text{ and } \lim_{\Gamma \to 0^-} \left(\frac{\epsilon}{\Gamma} \right)' = \frac{P''(0)}{\sqrt{P'(0)}}.$$

Thus, $\lim_{\Gamma \to 0^-} E(\Gamma) = 0$. Now, using (46) we obtain

$$E'(\Gamma) = \left(\frac{\epsilon}{\Gamma} \right)' [\Phi(0, \Gamma) - \Phi(\epsilon, 0)] + \frac{\epsilon}{\Gamma}[\Phi_\Gamma(0, \Gamma) - \Phi_\epsilon(\epsilon, 0)\epsilon'] + \Phi_\epsilon(0, \Gamma)P'(\Gamma) - \Phi_\Gamma(\epsilon, 0)\epsilon'.$$

Thus,

$$\lim_{\Gamma \to 0^-} E'(\Gamma) = \sqrt{P'(0)}\left(\Phi_\Gamma(0, 0) - \Phi_\epsilon(0, 0)\sqrt{P'(0)} \right)$$
$$+ \Phi_\epsilon(0, 0)P'(0) - \Phi_\Gamma(0, 0)\sqrt{P'(0)} = 0.$$

Furthermore,

$$E''(\Gamma) = \left(\frac{\epsilon}{\Gamma} \right)'' \left(\Phi(0, \Gamma) - \Phi(\epsilon, 0) \right) + 2\left(\frac{\epsilon}{\Gamma} \right)' \left(\Phi_\Gamma(0, \Gamma) - \Phi_\epsilon(\epsilon, 0)\epsilon' \right)$$
$$+ \left(\frac{\epsilon}{\Gamma} \right) \left(\Phi_{\Gamma\Gamma}(0, \Gamma) - \epsilon''\Phi_\epsilon(\epsilon, 0) - \Phi_{\epsilon\epsilon}(\epsilon, 0)(\epsilon')^2 \right) + P''(\Gamma)\Phi_\epsilon(0, \Gamma)$$
$$+ P'(\Gamma)\Phi_{\epsilon\Gamma}(0, \Gamma) - \epsilon''\Phi_\Gamma(\epsilon, 0) - (\epsilon')^2\Phi_{\Gamma\epsilon}(\epsilon, 0).$$

Hence,

$$
\begin{aligned}
\lim_{\Gamma \to 0^-} E''(\Gamma) =& 2\left(\frac{P''(0)}{4\sqrt{P'(0)}}\right)\left[\Phi_\Gamma(0,0) - \Phi_\epsilon(0,0)\sqrt{P'(0)}\right] \\
&+ \sqrt{P'(0)}\left[\Phi_{\Gamma\Gamma}(0,0) - \frac{P''(0)}{2\sqrt{P'(0)}}\Phi_\epsilon(0,0) - \Phi_{\epsilon\epsilon}(0,0)P'(0)\right] \\
&+ P''(0)\Phi_\epsilon(0,0) + P'(0)\Phi_{\epsilon\Gamma}(0,0) - \frac{P''(0)}{2\sqrt{P'(0)}}\Phi_\Gamma(0,0) - P'(0)\Phi_{\Gamma\epsilon}(0,0) = 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
E'''(\Gamma) =& \left(\frac{\epsilon}{\Gamma}\right)'''\left(\Phi(0,\Gamma) - \Phi(\epsilon,0)\right) + 3\left(\frac{\epsilon}{\Gamma}\right)''\left(\Phi_\Gamma(0,\Gamma) - \Phi_\epsilon(\epsilon,0)\epsilon'\right) + \\
& 3\left(\frac{\epsilon}{\Gamma}\right)'\left(\Phi_{\Gamma\Gamma}(0,\Gamma) - \epsilon''\Phi_\epsilon(\epsilon,0) - (\epsilon')^2\Phi_{\epsilon\epsilon}(\epsilon,0)\right) + \\
& \left(\frac{\epsilon}{\Gamma}\right)\left[\Phi_{\Gamma\Gamma\Gamma}(0,\Gamma) - \epsilon'''\Phi_\epsilon(\epsilon,0) - 3\epsilon'\epsilon''\Phi_{\epsilon\epsilon}(\epsilon,0) - (\epsilon')^3\Phi_{\epsilon\epsilon\epsilon}(\epsilon,0)\right] + \\
& P'''(\Gamma)\Phi_\epsilon(0,\Gamma) + 2P''(\Gamma)\Phi_{\epsilon\Gamma}(0,\Gamma) + P'(\Gamma)\Phi_{\epsilon\Gamma\Gamma}(0,\Gamma) - \epsilon'''\Phi_\Gamma(\epsilon,0) \\
& - 3\epsilon'\epsilon''\Phi_{\Gamma\epsilon}(\epsilon,0) - (\epsilon')^3\Phi_{\Gamma\epsilon\epsilon}(\epsilon,0).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\Gamma \to 0^-} E'''(\Gamma) =& -3\left[\frac{(1/4)(P''(0))^2 - (2/3)P'(0)P'''(0)}{4\left(\sqrt{P'(0)}\right)^3}\right]\left[\Phi_\Gamma(0,0) - \Phi_\epsilon(0,0)\sqrt{P'(0)}\right] \\
& + \frac{3P''(0)}{4\sqrt{P'(0)}}\left[\Phi_{\Gamma\Gamma}(0,0) - \frac{P''(0)}{2\sqrt{P'(0)}}\Phi_\epsilon(0,0) - \Phi_{\epsilon\epsilon}(0,0)P'(0)\right] \\
& + \sqrt{P'(0)}\left[\Phi_{\Gamma\Gamma\Gamma}(0,0) - \left(\frac{4P'(0)P'''(0) - (3/2)(P''(0))^2}{8(P'(0))^{3/2}}\right)\Phi_\epsilon(0,0)\right. \\
& \left. - \frac{3\sqrt{P'(0)}P''(0)}{2\sqrt{P'(0)}}\Phi_{\epsilon\epsilon}(0,0) - \left(\sqrt{P'(0)}\right)^3\Phi_{\epsilon\epsilon\epsilon}(0,0)\right] + P'''(0)\Phi_\epsilon(0,0) \\
& + 2P''(0)\Phi_{\epsilon\Gamma}(0,0) + P'(0)\Phi_{\epsilon\Gamma\Gamma}(0,0) - \left(\frac{4P'(0)P'''(0) - (3/2)(P''(0))^2}{8(P'(0))^{3/2}}\right)\Phi_\Gamma(0,0) \\
& - \frac{3\sqrt{P'(0)}P''(0)}{2\sqrt{P'(0)}}\Phi_{\Gamma\epsilon}(0,0) - \left(\sqrt{P'(0)}\right)^3\Phi_{\Gamma\epsilon\epsilon}(0,0).
\end{aligned}
$$

However, from (47) it follows that

$$
\Phi_{\Gamma\Gamma}(\epsilon,\Gamma) - \Phi_{\epsilon\epsilon}(\epsilon,\Gamma)P'(\Gamma) = 0,
$$
$$
\Phi_{\Gamma\Gamma\epsilon}(\epsilon,\Gamma) - \Phi_{\epsilon\epsilon\epsilon}(\epsilon,\Gamma)P'(\Gamma) = 0,
$$

and

$$
\Phi_{\Gamma\Gamma\Gamma}(\epsilon,\Gamma) - P''(\Gamma)\Phi_{\epsilon\epsilon}(\epsilon,\Gamma) - \Phi_{\epsilon\epsilon\Gamma}(\epsilon,\Gamma)P'(\Gamma) = 0.
$$

Consequently,

$$\lim_{\Gamma \to 0^-} E'''(\Gamma) = -\frac{1}{2}P''(0)[\sqrt{P'(0)}\Phi_{\epsilon\epsilon}(0,0) - \Phi_{\Gamma\epsilon}(0,0)]. \tag{51}$$

On the other hand, since Φ is strictly convex, we know that for all nonzero $(a,b) \in \mathbb{R}^2$

$$\Phi_{\epsilon\epsilon}(0,0)a^2 + 2\Phi_{\epsilon\Gamma}(0,0)ab + \Phi_{\Gamma\Gamma}(0,0)b^2 > 0. \tag{52}$$

Now, we demonstrate that the expression $\sqrt{P'(0)}\Phi_{\epsilon\epsilon}(0,0) - \Phi_{\Gamma\epsilon}(0,0)$ in (51) can be rewritten in a form of the left hand side of (52) with a and b appropriately chosen. To prove that, we modify this expression by an additive, equal to zero term $\alpha\Phi_{\Gamma\Gamma}(0,0) - \alpha\Phi_{\epsilon\epsilon}(0,0)P'(0)$, with α to be determined. Thus a and b have to be chosen so that

$$(\sqrt{P'(0)} - \alpha P'(0))\Phi_{\epsilon\epsilon} - \Phi_{\Gamma\epsilon} + \alpha\Phi_{\Gamma\Gamma} = \Phi_{\epsilon\epsilon}a^2 + 2\Phi_{\epsilon\Gamma}ab + \Phi_{\Gamma\Gamma}b^2 \tag{53}$$

holds, where the derivatives of Φ are at $(0,0)$. Consequently, we require that

$$\begin{aligned} a^2 &= \sqrt{P'(0)} - \alpha P'(0), \\ 2ab &= -1, \\ \alpha &= b^2. \end{aligned}$$

Solving the system for a, b and α, we get $a = \pm\frac{\sqrt[4]{P'(0)}}{\sqrt{2}}$, $b = \mp\frac{1}{\sqrt{2}\sqrt[4]{P'(0)}}$, and $\alpha = \frac{1}{2\sqrt{P'(0)}}$. In this way, we conclude from (51) that $\lim_{\Gamma \to 0^-} E'''(\Gamma) > 0$. Now, to conclude the proof, we use the following two lemmas, whose proofs are straightforward.

Lemma 5.5. *Let f and f' be continuous functions for $\Gamma < 0$. If $\lim_{\Gamma \to 0^-} f(\Gamma) = 0$ and $\lim_{\Gamma \to 0^-} f'(\Gamma) > 0$, then there exists $\Gamma_0 \in (-1,0)$ such that $f(\Gamma) < 0$ for $\Gamma_0 < \Gamma < 0$.*

Lemma 5.6. *Let f and f' be continuous functions for $\Gamma < 0$. If $\lim_{\Gamma \to 0^-} f(\Gamma) = 0$ and $\lim_{\Gamma \to 0^-} f'(\Gamma) < 0$, then, $f(\Gamma) > 0$ for $\Gamma_1 < \Gamma < 0$, where $\Gamma_1 \in (-1,0)$.*

Applying Lemma 5.5 with $f(\Gamma) = E''(\Gamma)$, we conclude that $E''(\Gamma) < 0$ for Γ close to zero and, using Lemma 5.6, with $f(\Gamma) = E'(\Gamma)$, we infer that $E'(\Gamma) > 0$ near zero. Thus, using Lemma 5.5 again, now, with $f(\Gamma) = E(\Gamma)$, we conclude that $E(\Gamma) < 0$ near zero. ☑

5.1. Entropy condition for a solution of $IBVP_{V_0}$

It is difficult to describe explicitly all entropy functions Φ. Nevertheless, employing separation of variables, we can figure out one of them, which we shall call a standard entropy function. For this purpose we set $\Phi(V,\Gamma) = a(V) + b(\Gamma)$, where a and b are functions to be determined. Substituting it in (47), we obtain

$a''(V) = \frac{b''(\Gamma)}{P'(\Gamma)}$. Since V and Γ are independent variables, therefore there exists a constant denoted by c such that $a''(V) = \frac{b''(\Gamma)}{P'(\Gamma)} = c$, which implies that:

$$a(V) = c\frac{V^2}{2} + c_1 V + c_2; \text{ with } c_1, c_2 \in \mathbb{R},$$

and

$$b(\Gamma) = c \int_0^{\Gamma} P(w)dw + c_3\Gamma + c_4; \ c_3, c_4 \in \mathbb{R}.$$

Thus, putting $c_1 = c_2 = c_3 = c_4 = 0$, we get $\Phi(V, \Gamma) = c\frac{V^2}{2} + c \int_0^{\Gamma} P(w)dw$. Substituting Φ into (46), it results in

$$\begin{aligned}
\Psi_V &= -cP(\Gamma), \\
\Psi_{\Gamma} &= -cVP'(\Gamma).
\end{aligned} \tag{54}$$

A solution of the system (54) is $\Psi(V, \Gamma) = -cVP(\Gamma)$. Notice that, strict convexity of Φ implies that $c > 0$ and $P'(\Gamma) > 0$. Thus, without loss of generality, we may put $c = 1$, thereby we obtain

$$\Phi(V, \Gamma) = \frac{V^2}{2} + \int_0^{\Gamma} P(w)dw \tag{55}$$

and

$$\Psi(V, \Gamma) = -P(\Gamma)V. \tag{56}$$

The function (55) is well known entropy function for a p-system [5], which we call a standard entropy function. For the solution $S(\Gamma_l)$, (44), the condition (48) can be simplified into (49). Here, $P(0) = 0$, $\Phi(-V_0, 0) = \frac{V_0^2}{2}$, $\Phi(\Gamma, 0) = \int_0^{\Gamma} P(w)dw$, $\Psi(-V_0, 0) = 0$, and $\Psi(0, \Gamma) = 0$, so that (49) becomes

$$2\int_0^{\Gamma_l} P(w)dw \leq \Gamma_l P(\Gamma_l). \tag{57}$$

This is the entropy condition for $S(\Gamma_l)$ corresponding to a standard entropy function, (55), and Ψ given by (56).

Remark 5.7. The assertion of Theorem 5.4 does not say how far from 0 the inequality still holds or it already does not hold. It is rather difficult, except for a linear case, to answer this question without having more particular information about the entropy functions. That is why we concentrate ourselves on studying the inequality (57) for previously listed models of elastic materials.

We notice the following facts, which clarify the importance of genuine nonlinearity condition in studying the entropy condition (57).

Lemma 5.8. If $P(0) = 0$ and $P''(\Gamma) < 0$ for all $\Gamma \in (-1, 0)$, then $S(\Gamma_l)$ satisfies (57) for all $\Gamma_l \in (-1, 0]$.

Proof. Consider the function $G(\Gamma_l) = 2\int_0^{\Gamma_l} P(w)dw - \Gamma_l P(\Gamma_l)$; so, $G'' = -\Gamma_l P''$. Therefore G' is decreasing for $-1 < \Gamma_l < 0$. Since $G'(0) = 0$, we obtain that $G'(\Gamma_l) > 0$ for $-1 < \Gamma_l < 0$. Consequently, $G(\Gamma_l) < 0$ for $\Gamma_l \in (-1, 0)$. ☑

Similarly we have the following proposition:

Lemma 5.9. If $P(0) = 0$ and $P''(\Gamma_l) > 0$ for $\overline{\Gamma} < \Gamma_l < 0$, where $\overline{\Gamma} \in (-1, 0)$, then $S(\Gamma_l)$ does not satisfy (57). Therefore $S(\Gamma_l)$ does not satisfy the entropy condition for $\overline{\Gamma} < \Gamma_l < 0$.

6. Results on hyperbolicity, genuine nonlinearity and entropy condition with a standard entropy function

In this section we present the results about the conditions of hyperbolicity $(P'(\Gamma) > 0)$, genuine nonlinearity $(P''(\Gamma) < 0)$ and the entropy condition (see equation (57)) for the models under consideration. To attain this goal, we use basic techniques of differential calculus and the Maple software to perform symbolic computation and to study the graphs of functions.

In some cases it is convenient to use instead of Γ a variable $s = \Gamma + 1$, restricted by $s > 0$, since Γ is subject to $\Gamma > -1$.

1. St.Venant-Kirchhoff:

   a) It is hyperbolic for all $\Gamma > -1 + 1/\sqrt{3}$.

   b) The condition of genuine nonlinearity is satisfied for all $\Gamma > -1$.

   c) For all $s \in (0, 1)$, $S(\Gamma_l)$ do not satisfy the entropy condition.

2. Kirchhoff modified:

   a) It is hyperbolic for all $\Gamma > -1$ provided a parameter $\frac{\lambda}{\mu}$ satisfies $\alpha_1 < \alpha < \alpha_2$, where α_1 and α_2 are two positive solutions of the equation

   $$6(5 + 4\log 6)\alpha = 1 + 12\alpha \log(3 + 3\sqrt{1 + 12\alpha}) + \sqrt{1 + 12\alpha}.$$

   An approximate inequality for α is

   $$0.0446567295 < \alpha < 1732.05696.$$

   b) $P''(\Gamma) < 0$ holds only for all $s \in (0, S_\alpha)$, where

   $$S_\alpha = \left[\frac{\alpha}{2} \operatorname{Lambert} W\left(\frac{12e^6}{\alpha}\right)\right]^{1/4},$$

   and where $\operatorname{Lambert} W$ is the inverse of the function we^w. Consequently, $P''(s) < 0$ holds for all $s \in (0, 1]$ iff $s_\alpha > 1$, what is equivalent to $\alpha > 2$.

   c) ▪ If $\alpha \geq 2$, then $S(\Gamma_l)$ satisfies the entropy condition for all $s \in (0, 1)$.

   ▪ If $0 < \alpha < 2$, then $S(\Gamma_l)$ satisfies the entropy condition for all $s \in (0, s_e]$ and does not for $s \in (s_e, 1)$, where s_e is a unique solution in $(0, 1)$ of the equation

   $$\frac{s(s+1)(1-s)^3}{2(s-1-s\ln s)\ln s} = \alpha.$$

3. Ogden:

   a) It is hyperbolic for all $\Gamma > -1$.

   b) Satisfies that $P''(s) < 0$ for all $s > 0$.

   c) $S(\Gamma_l)$ satisfies the entropy condition, for all $s \in (0, 1]$.

4. Blatz-Ko-Ogden:

$a)$ There are two parameters involved, $\beta = \frac{\lambda}{2\mu}$ and $f \in (0,1)$.

- If $\beta \geq 1/2$, it is hyperbolic for all $\Gamma > -1$.
- If $0 < \beta < 1/2$, the hyperbolicity condition requires a restriction for f of the form $f > f_\beta$, where f_β is a certain number in $(0,1)$ determined according to $f_\beta = \max_{s > s_\beta} Q(s, \beta)$, where

$$Q(s, \beta) = \frac{s^{2\beta}[(1 - 2\beta)s^{2\beta+2} - 3]}{s^2[(1 + 2\beta) + s^{2\beta+2}] + s^{2\beta}[(1 - 2\beta)s^{2\beta+2} - 3]}$$

and

$$s_\beta = \Big(\frac{3}{1 - 2\beta}\Big)^{\frac{1}{2\beta+2}};$$

here $f_\beta \leq \frac{1-2\beta}{1-2\beta+s_\beta^{2-2\beta}}$.

$b)$
- If $\beta \in [1/2, 1]$, then $P''(s) < 0$ for all $s > 0$.
- If $\beta \in (0, 1/2) \cup (1, \infty)$, then for all $s \leq s_0$, where

$$s_0 = \left[\frac{6}{(2\beta - 1)(\beta - 1)}\right]^{\frac{1}{2\beta+2}},$$

it holds $P''(s) < 0$.

- If $\beta \in (0, 1/2) \cup (1, \infty)$ and $s > s_0$, then there exists s_2 such that $P''(s) < 0$ up to s_2 and then it changes its sign.

$c)$
- If $0 < \beta \leq 5/2$, then $P''(s) < 0$ for all $s \in (0, 1]$.
- An experimentation with plots indicates that for a given value of $\beta > 5/2$ there exists $f_\beta \in (0, 1)$ such that $S(\Gamma_l)$ satisfies the entropy condition, for all $s \in (0, 1]$, provided $f \geq f_\beta$. If however $f < f_\beta$, then there exists $s_\beta \in (0, 1]$ such that the condition holds for all $s \in (0, s_\beta)$ and does not for $s \in (s_\beta, 1)$, while at $s = 1$ it holds again. We have been able to confirm theoretically such behavior of the condition only for $\beta = n/2$, where n is an integer and $n > 5$.

7.   Conclusions

1. A definition of a weak solution of an initial and boundary problem for a p-system, in the first quadrant of the Xt-plane, is provided. There are two unknown functions $V(X, t)$ and $\Gamma(X, t)$. Consequently, there are two initial conditions (at $t = 0$) and only one boundary condition (at $X = 0$). There are four types of boundary conditions considered: the first, (37), for $V(0, t)$; the second, (40), for $\Gamma(0, t)$ and the other two are mixed boundary conditions involving $V(0, t)$ and $\Gamma(0, t)$, (22) and (23), respectively. The first two types of boundary conditions are particular cases of the other two. All of that is consistent with what is known in case of classical solutions of linear systems [5].

2. A particular weak solution of a p-system, called a compression shock, is constructed. It satisfies the initial and boundary conditions given by (41), which is a particular case of (37). This solution, denoted by $S(\Gamma_l)$, can be interpreted as an impact velocity. $S(\Gamma_l)$ is constant by parts, having jump discontinuities of V and Γ along the line $X = \sigma t$; $(V, \Gamma) = (-V_0, 0)$, for $X > \sigma t$ and $(V, \Gamma) = (0, \Gamma_l)$, $X < \sigma t$, where the constants $\sigma > 0$ and $\Gamma_l < 0$ are solutions of the Rankine-Hugoniot conditions.

3. For the St.Venant-Kirchhoff model, $S(\Gamma_l)$ does not satisfy the entropy condition. Consequently, we can consider this model as inadequate to describe the compression shock. For the Kirchhoff modified, Ogden and Blatz-Ogden models we can verify that they satisfy, under certain restrictions on the parameters, the hypothesis of the Theorem 5.4. Therefore, for those models $S(\Gamma_l)$ satisfies the entropy condition, for Γ_l sufficiently close to zero.

4. The Theorem 5.4 does not provide an exact information about the interval for Γ_l in which the entropy condition holds. That is why we concentrate on the entropy condition with a well known in literature [5] entropy/entropy-flux pair (Φ, Ψ), which we call a standard entropy/entropy-flux pair. We provide the conditions for the parameters μ, λ, f and for Γ_l, under which $S(\Gamma_l)$ fulfills the entropy condition with this standard entropy function. This discussion is complete, except of the Blazt-Ko and Ogden model for $\beta > \frac{5}{2}$. In this case we clarify the validity of the entropy condition only for $\beta = \frac{n}{2}$, where n is an integer number greater than 5.

5. An open question remains about the entropy condition with a general entropy function.

Appendix A: Numerical comparison of the compression shocks for various models

In this section we obtain numerical values of Γ_l for the compression shock corresponding to given values of V_0; more specifically, we use $\widetilde{V_0} = \frac{\rho_0 V_0^2}{\mu}$. We do this for the following models: Modified Krchhoff, Ogden and Blatz-Ko-Ogden.

Here Γ_l is determined by the first equation in (44), which after substituting $\lambda = 2\mu\beta$ can be rewritten in the form $Q(\Gamma) = \widetilde{V_0}$, where

$$Q(\Gamma) = \Gamma\big(\mu(1+\Gamma)^3 - (1+\Gamma) + 2\beta\frac{\ln(1+\Gamma)}{(1+\Gamma)}\big) \quad \text{(Modified Kirchhoff)},$$

$$Q(\Gamma) = \Gamma\big(2\beta\Gamma + \frac{(2+\Gamma)\Gamma}{\Gamma+1}\big) \quad \text{(Ogden model)},$$

$$Q(\Gamma) = \Gamma(1+\Gamma)\big\{f\big[1-(1+\Gamma)^{-2\beta-2}\big] + \frac{(1-f)}{(1+\Gamma)^4}\big[(1+\Gamma)^{2\beta+2}-1\big]\big\} \quad \text{(Blatz-Ko-Ogden model)}.$$

$\widetilde{V_0}\Gamma$	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	-0.1912	-0.217420	-0.373581	-0.447296
0.25	-0.2929	-0.351888	-0.386761	-0.457802
0.5	-0.3978	-0.486632	-0.407276	-0.474264
2	-0.6667	-0.733399	-0.495098	-0.547908
4	-0.7938	-0.818724	-0.559164	-0.604841
10	-0.9063	-0.897073	-0.646584	-0.684415
40	-0.9753	-0.960995	-0.760038	-0.787456

Table 1. $\beta = 0.25$

$\widetilde{V_0}\Gamma$	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	-0.1764	-0.187793	-0.396762	-0.467705
0.25	-0.2722	-0.294512	-0.406156	-0.475495
0.5	-0.3729	-0.401976	-0.42134	-0.488051
2	-0.6446	-0.638674	-0.495152	-0.550033
4	-0.7808	-0.739561	-0.556282	-0.603611
10	-0.9027	-0.842347	-0.643929	-0.682669
40	-0.9678	-0.9357075	-0.759011	-0.786753

Table 2. $\beta = 0.5$

$\widetilde{V_0\Gamma}$	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	-0.1276	-0.123866	-0.516574	-0.571569
0.25	-0.2	-0.189781	-0.518832	-0.573691
0.5	-0.2798	-0.257764	-0.522632	-0.577243
2	-0.5298	-0.440641	-0.546019	-0.598641
4	-0.6951	-0.546173	-0.576544	-0.625725
10	-0.8757	-0.681674	-0.645033	-0.685736
40	-0.9731	-0.843239	-0.758247	-0.786419

Table 3. $\beta = 2$

$\widetilde{V_0\Gamma}$	Ogden	M.Kirchhoff	Blatz-Ko ($f = 0.25$)	Blat-Ko ($f = 0.5$)
0.1	-0.0909	-0.145165	-0.646008	-0.68264
0.25	-0.1433	-0.223122	-0.646471	-0.683109
0.5	-0.2020	-0.302771	-0.647248	-0.683894
2	-0.3975	-0.506407	-0.652028	-0.688701
4	-0.5501	-0.614488	-0.658692	-0.695328
10	-0.7941	-0.743021	-0.680005	-0.715904
40	-0.9684	-0.881932	-0.759992	-0.788159

Table 4. $\beta = 5$

References

[1] Bressan A. and LeFloch P., "Uniqueness of weak solutions to systems of conservations laws", *Arch. Ration. Mech. Anal.* 140 (1997), No. 4, 301–317.

[2] Bressan A. and Goatin P., "Oleinik type estimates and uniqueness for $n \times n$ conservation laws", *J. Differential Equations* 156 (1999), No. 1, 26–49.

[3] Dafermos C.M., *Hyperbolic conservation laws in continuum physics*, Springer-Verlag, Berlin, 2005.

[4] Dafermos C.M., "Genuinely nonlinear hyperbolic systems of two conservation laws", in *Contemp. Math.* 238, Amer. Math. Soc. (1999), 115–126.

[5] Evans L., *Partial Differential Equations*, American Mathematical Society, Providence, 2002.

[6] Glimm J., "Solutions in the large for nonlinear hyperbolic systems of equations", *Comm. Pure Appl. Math.* 18 (1965), No. 4, 697–715.

[7] Lax P., "Hyperbolic systems of conservation laws", *Comm. Pure Appl. Math.* 10 (1957), No. 4, 537–566.

[8] Lax P., "Hyperbolic systems of conservation laws and the mathematical theory of shock waves", CBMS Regional conference series in Mathematics, Philadelphia, USA, No. 11, 1973.

[9] Marsden J. and Hughes T., *Mathematical foundations of elasticity*, Dover Publications, New York, 1994.

[10] Ogden R.W., *Non-linear elastic deformations*, Dover Publications, New York, 1997.

[11] Pérez E., "Hyperbolicity and genuine nonlinearity conditions for certain p-systems of conservation laws, weak solutions and the entropy condition", Thesis (MSc), University of Puerto Rico at Mayaguez, Mayaguez, 2010, 89 p.

[12] Renardy M. and Rogers R., *An introduction to Partial Differential Equations*, Springer-Verlag, New York, 2004.