

A recursive condition for the symmetric nonnegative inverse eigenvalue problem

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Abstract. In this paper we present a sufficient condition and a necessary condition for *Symmetric Nonnegative Inverse Eigenvalue Problem*. This condition is independent of the existing realizability criteria. This criterion is recursive, that is, it determines whether a list $\Lambda = \{\lambda_1, \dots, \lambda_n, \lambda_{n+1}\}$ is realizable by a nonnegative symmetric matrix, if the list $\mu = \{\mu_1, \dots, \mu_n\}$ associated to Λ is realizable. This result is easy to program and improves some existing criteria.

Keywords: Inverse problems, eigenvalues, orthogonal matrices, symmetric matrix.

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Una condición recursiva para el problema inverso del autovalor para matrices simétricas no negativas

Abstract. En este artículo presentamos una condición suficiente y una condición necesaria para el *Problema Inverso de Autovalores para Matrices Simétricas no Negativas*. Esta condición es independiente de los criterios de realizabilidad existentes. Este criterio es recursivo, es decir determina si una lista $\Lambda = \{\lambda_1, \dots, \lambda_n, \lambda_{n+1}\}$ es realizable por una matriz simétrica no negativa, si la lista $\mu = \{\mu_1, \dots, \mu_n\}$ asociada a Λ es realizable. Este resultado es fácil de programar y mejora algunos criterios existentes.

Palabras clave: Problemas inversos, autovalores, matrices ortogonales, matrices simétricas.

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1. Introduction

The *nonnegative inverse eigenvalue problem (NIEP)* is the problem of finding necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of complex numbers to be the spectrum of a $n \times n$ nonnegative matrix. The problem of finding necessary and sufficient conditions for a list of real numbers to be spectrum of a nonnegative matrix is called the *real nonnegative inverse eigenvalue problem (RNIEP)*. Particularly, the problem of finding necessary and sufficient conditions for a list of real numbers Λ , to be the spectrum of a nonnegative symmetric matrix is called *symmetric nonnegative inverse eigenvalue problem (SNIEP)*. These problems remain unsolved.

The *NIEP*, *RNIEP* and *SNIEP* are completely solved for $n \leq 4$. The *NIEP* has been solved for $n = 3$ in 1978 by Loewy and London [7], for $n = 4$ was solved by Meehan [9] in 1998, and subsequently independently in a different formulation by Torre-Mayo and others [17] in 2007.

The *SNIEP* has been solved when $n = 3$ by Fiedler [2] in 1974, and for $n = 4$ has been solved by Guo [3] in 1996. The *RNIEP* and *SNIEP* are equivalent for $n \leq 4$ [3], but are different otherwise [5]. In 2011 Spector gives a complete solution to *SNIEP* when $n = 5$ and the trace of nonnegative matrix is zero [16]. Partial results for the *SNIEP* have been obtained in [1], [2], [6], [8], [10], [11], [12], [13], [15].

This paper is organized as follows: In Section 2 we establish the notation and basic results in relation to *SNIEP*. In Section 3 we present the main results, a sufficient condition and a necessary condition. In Section 4, we show a programming algorithm in relation to the main results. The realizability criteria shown in Section 3 are independent of the criteria presented in [1], [13], and they improve those criteria.

2. Notations and basic results

Throughout this paper we use the following notation: Let $\mathbb{R}^{m \times n}$ the matrix set of order $m \times n$ with entries real numbers, in particular $\mathbb{R}^{n \times n}$ square matrices of order n . We denote $\rho(A)$ be the spectral radius of $A \in \mathbb{R}^{n \times n}$. We say that $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is nonnegative if $a_{ij} \geq 0$ for all $i, j \in \{1, 2, \dots, n\}$. We shall say that $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is realizable if there exists a nonnegative matrix $A \in \mathbb{R}^{n \times n}$ with spectrum Λ and $\rho(A) = \lambda_1$. If Λ is realizable for A , then it is said that A realiza Λ .

If $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{n+1}\}$ and $\mu = \{\mu_1, \mu_2, \dots, \mu_n\}$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n \geq \mu_n \geq \lambda_{n+1},$$

we define the functions $f(t) = \prod_{k=1}^{n+1} (t - \lambda_k)$, $g(t) = \prod_{k=1}^n (t - \mu_k)$.

Show that the vector $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_n]^T$, with

$$y_i^2 = -\frac{f(\mu_i)}{g'(\mu_i)} = -\frac{\prod_{k=1}^{n+1}(\mu_i - \lambda_k)}{\prod_{k=1}^{i-1}(\mu_i - \mu_k) \cdot \prod_{k=i+1}^n(\mu_i - \mu_k)}, \quad (*)$$

is well defined (see [4]).

Finally, we present three results, the first two results are due to Horn and Johnson [4], and the third result was presented by Guo [3]. These will be later used for the development of new necessary and sufficient conditions for *SNIEP*.

Theorem 2.1. Let $\bar{A} \in \mathbb{R}^{n \times n}$ be a given symmetric matrix with eigenvalues μ_1, \dots, μ_n , let $\mathbf{z} \in \mathbb{R}^{n \times 1}$ be a given vector (column matrix), and let α be a given real number. Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be the symmetric matrix obtained by bordering \bar{A} with \mathbf{z} and α as follows: $A = \begin{bmatrix} \bar{A} & \mathbf{z} \\ \mathbf{z}^T & \alpha \end{bmatrix}$, with eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n+1}$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. Then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}.$$

Theorem 2.2. Let n be a given positive integer, and let $\mu = \{\mu_1, \dots, \mu_n\}$ and $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{n+1}\}$ be two lists of real numbers arranged in descending order such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}.$$

Let $D = \text{diag}(\mu_1, \dots, \mu_n)$. Then there exists a real number α and real vector $\mathbf{y} \in \mathbb{R}^{n \times 1}$ such that $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{n+1}\}$ is the set of eigenvalues of the real symmetric matrix

$$A = \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & \alpha \end{bmatrix}.$$

Theorem 2.3. The list $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of real numbers is a realizable symmetric matrix if, and only if $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 0$ and $\lambda_1 \geq |\lambda_i|$ for $i = 2, 3, 4$.

Observation: The matrix A of Theorem 2.2 is similar to the matrix $\tilde{A} = \begin{bmatrix} a_{11} & z^T \\ z & D \end{bmatrix}$, which has a spectrum Λ and a main submatrix with spectrum μ . In consequence, the results of the following section can be reformulated in such a way that the matrix that realize Λ has the form of \tilde{A} .

3. Sufficient condition and necessary condition

We consider a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of real numbers, such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The next result is a sufficient condition for the *SNIEP* [14].

Theorem 3.1. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{n+1}\}$, $\mu = \{\mu_1, \mu_2, \dots, \mu_n\}$ be lists of real numbers, such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_n \geq \lambda_{n+1}$, with $\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k \geq 0$. Moreover let

P be an orthogonal matrix, $D = \text{diag}\{\mu_1, \dots, \mu_n\}$ such that $PDP^T \geq 0$, and $\mathbf{y} \in \mathbb{R}^{n+1}$ given as (*) such that $P\mathbf{y} \geq 0$. Then there exists a nonnegative symmetric matrix A with spectrum Λ .

Proof. We define $\mathbf{y} = [y_1 \ \dots \ y_n]$ where each component y_i is given as (*), that is:

$$y_i^2 = -\frac{\prod_{k=1}^{n+1} (\mu_i - \lambda_k)}{\prod_{k=1}^{i-1} (\mu_i - \mu_k) \cdot \prod_{k=i+1}^n (\mu_i - \mu_k)}.$$

By Theorem 2.1 the matrix $\bar{A} = \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & \sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k \end{bmatrix}$ has spectrum Λ .

We define the orthogonal matrix $\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, with $\mathbf{0} \in \mathbb{R}^{n \times 1}$. Since $PDP^T \geq 0$, $P\mathbf{y} \geq 0$, then the symmetric matrix

$$\begin{aligned} A &= \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & \sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k \end{bmatrix} \begin{bmatrix} P^T & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} PDP^T & P\mathbf{y} \\ (P\mathbf{y})^T & \sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k \end{bmatrix} \end{aligned}$$

is nonnegative and with spectrum Λ .

□

The following example shows that Theorem 3.1 is independent of the realizability criteria established in [13, Lemma 4].

Example 3.2. Let $\Lambda = \{6, 1, 1, -4, -4\}$. For this list consider $\mu = \{4, 1, -1, -4\}$, $a = 2$. The matrix

$$B = P \text{diag}\{4, 1, -1, 4\} P^T \geq 0,$$

where

$$P = \begin{bmatrix} \frac{2}{15}\sqrt{15} & -\frac{1}{30}\sqrt{210} & -\frac{1}{30}\sqrt{210} & \frac{2}{15}\sqrt{15} \\ \frac{8}{45}\sqrt{15} & -\frac{1}{90}\sqrt{210} & \frac{1}{90}\sqrt{210} & -\frac{8}{45}\sqrt{15} \\ \frac{1}{90}\sqrt{210} & \frac{8}{45}\sqrt{15} & -\frac{8}{45}\sqrt{15} & -\frac{1}{90}\sqrt{210} \\ \frac{1}{30}\sqrt{210} & \frac{2}{15}\sqrt{15} & \frac{2}{15}\sqrt{15} & \frac{1}{30}\sqrt{210} \end{bmatrix},$$

have spectrum μ .

We define $y = (\sqrt{\frac{48}{5}}, 0, -\sqrt{\frac{42}{5}}, 0)^T$; then, because of Theorem 3.1 the matrix

$$\begin{aligned}
 A &= \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{diag}(4, 1, -1, 4) & y \\ y^T & 0 \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 3 & 0 & 0 & 3 \\ 3 & 0 & 0 & \frac{2}{3}\sqrt{14} & \frac{5}{3} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{2}{3}\sqrt{14} \\ 0 & \frac{2}{3}\sqrt{14} & \frac{4}{3} & 0 & 0 \\ 3 & \frac{5}{3} & \frac{2}{3}\sqrt{14} & 0 & 0 \end{bmatrix}
 \end{aligned}$$

is nonnegative with spectrum Λ .

The next example show that the Theorem 3.1 is independent of the realizability criteria shown in the [1, Theorem 3.4] and [13, Theorem 6].

Example 3.3. We consider the list $\Lambda = \{3 + \sqrt{10}, 1, 1, 3 - \sqrt{10}, -4, -4\}$. For this list there exist a list $\mu = \{6, 1, 1, -4, -4\}$ and the orthogonal matrix

$$P = \begin{bmatrix} \frac{\sqrt{7}}{5} & -\frac{\sqrt{10}}{10} & -\frac{\sqrt{14}}{10} & -\frac{\sqrt{10}}{5} & \frac{\sqrt{2}}{5} \\ \frac{\sqrt{2}}{5} & \frac{2\sqrt{35}}{15} & \frac{2}{15} & -\frac{\sqrt{35}}{15} & -\frac{2\sqrt{7}}{15} \\ \frac{\sqrt{7}}{5} & -\frac{2\sqrt{10}}{15} & \frac{\sqrt{14}}{15} & \frac{\sqrt{10}}{15} & -\frac{7\sqrt{2}}{15} \\ \frac{\sqrt{2}}{5} & 0 & \frac{4}{5} & 0 & \frac{\sqrt{7}}{5} \\ \frac{\sqrt{7}}{5} & \frac{\sqrt{10}}{5} & -\frac{\sqrt{14}}{10} & \frac{\sqrt{10}}{5} & \frac{\sqrt{2}}{5} \end{bmatrix},$$

such that for $y = [1 \ 0 \ 0 \ 0 \ 0]$, defined as (*) we have

$$Py = \left[\frac{1}{5}\sqrt{7} \quad \frac{1}{5}\sqrt{2} \quad \frac{1}{5}\sqrt{7} \quad \frac{1}{5}\sqrt{2} \quad \frac{1}{5}\sqrt{7} \right]^T \geq 0.$$

Thus, by Theorem 3.1, with $D = \text{diag}\{6, 1, 1, -4, -4\}$, the matrix

$$\begin{aligned}
A &= \begin{bmatrix} PDP^T & P\mathbf{y} \\ (P\mathbf{y})^T & \sum_{i=1}^6 \lambda_i - \sum_{i=1}^5 \mu_i \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 3 & 0 & 3 & \frac{1}{5}\sqrt{7} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{2}{3}\sqrt{14} & \frac{1}{5}\sqrt{2} \\ 3 & 0 & 0 & \frac{2}{3}\sqrt{14} & \frac{5}{3} & \frac{1}{5}\sqrt{7} \\ 0 & \frac{4}{3} & \frac{2}{3}\sqrt{14} & 0 & 0 & \frac{1}{5}\sqrt{2} \\ 3 & \frac{2}{3}\sqrt{14} & \frac{5}{3} & 0 & 0 & \frac{1}{5}\sqrt{7} \\ \frac{1}{5}\sqrt{7} & \frac{1}{5}\sqrt{2} & \frac{1}{5}\sqrt{7} & \frac{1}{5}\sqrt{2} & \frac{1}{5}\sqrt{7} & 0 \end{bmatrix} \geq 0,
\end{aligned}$$

has desired eigenvalues.

These examples show that Theorem 3.1 improves the realizability criteria presented in [1, Theorem 3.4] and [13, Lemma 4 and Theorem 6], in the sense that the lists $\{6, 1, 1, -4, -4\}$, $\{3 + \sqrt{10}, 1, 1, 3 - \sqrt{10}, -4, -4\}$ are not realizable by those criteria.

The next result shows a necessary condition for the *SNIEP*.

Theorem 3.4. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{n+1}\}$ realizable for a nonnegative symmetric matrix A . Then there exists:*

1. *A list of real number $\mu = \{\mu_1, \dots, \mu_n\}$, such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_n \geq \lambda_{n+1}$.*
2. *A orthogonal matrix Q and $\mathbf{b} \in \mathbb{R}^{n \times 1}$ such that $Q\mathbf{b} \geq 0$, and $QDQ^T \geq 0$, where $D = \text{diag}\{\mu_1, \dots, \mu_n\}$.*

Also it holds $\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k \geq 0$.

Proof. Let A be a nonnegative symmetric matrix with eigenvalues Λ . Without loss of generality we can assume that $A = \begin{bmatrix} B & \mathbf{z} \\ \mathbf{z}^T & \alpha \end{bmatrix}$, with $B \in \mathbb{R}^{n \times n}$, $\mathbf{z} \in \mathbb{R}^{n \times 1}$. Let $\mu = \sigma(B) = \{\mu_1, \dots, \mu_n\}$. By Theorem 2.1 we get that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_n \geq \lambda_{n+1}.$$

Since B is a nonnegative symmetric matrix, there exists a orthogonal matrix Q such that $B = QDQ^T \geq 0$. Let $\mathbf{b} = Q^T \mathbf{z}$. Since $\mathbf{z} \geq 0$ then we have that $Q\mathbf{b} = Q(Q^T \mathbf{z}) = \mathbf{z} \geq 0$.

The matrix

$$\hat{A} = \begin{bmatrix} D & Q^T \mathbf{z} \\ (Q^T \mathbf{z})^T & \alpha \end{bmatrix}$$

is similar to A , then $\sigma(\hat{A}) = \{\lambda_1, \dots, \lambda_{n+1}\}$. Since \hat{A} is a symmetric matrix, then eigenvalues and the diagonal entries satisfy the following relation:

$$\sum_{k=1}^{n+1} \lambda_k = \sum_{k=1}^n \mu_k + \alpha;$$

so,

$$\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k = \alpha \geq 0,$$

therefore $\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^n \mu_k \geq 0$. □

Note that if $P\mathbf{y} = \mathbf{z}$, with P given in Theorem 3.1, \mathbf{y} defined as (*) and \mathbf{z} given in Theorem 3.4, then Theorem 3.1 and Theorem 3.4 establish necessary and sufficient conditions for *SNIEP*.

Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ a list of real numbers. If $n = 2, 3$, it is easy to see that $P\mathbf{y} = \mathbf{z}$. For $n = 4$, with some ideas presented in [3], we consider the following cases:

1. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$, then the symmetric matrix

$$A_1 = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$$

has eigenvalues Λ . In this case we have $\mu = \{\lambda_1, \lambda_2, \lambda_3\}$, $P\mathbf{y} = \mathbf{z}$ with $P = I \in \mathbb{R}^{3 \times 3}$, and $\mathbf{y} = \mathbf{z} = [0 \ 0 \ 0]^T$.

2. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 > \lambda_4$, then the symmetric matrix

$$A_2 = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix},$$

with

$$B_1 = \begin{bmatrix} \frac{\lambda_1 + \lambda_4}{2} & \frac{\lambda_1 - \lambda_4}{2} \\ \frac{\lambda_1 - \lambda_4}{2} & \frac{\lambda_1 + \lambda_4}{2} \end{bmatrix}, B_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

has spectrum Λ . In this case we have $\mu = \{\lambda_1, \lambda_2, \lambda_4\}$, $P\mathbf{y} = \mathbf{z}$, with

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{y} = \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. If $\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3 \geq \lambda_4$ and $\lambda_2 \geq |\lambda_3|$, the matrix

$$A = \begin{bmatrix} \frac{\lambda_2 + \lambda_3}{2} & \frac{\lambda_2 - \lambda_3}{2} & 0 & 0 \\ \frac{\lambda_2 - \lambda_3}{2} & \frac{\lambda_2 + \lambda_3}{2} & 0 & 0 \\ 0 & 0 & \frac{\lambda_1 + \lambda_4}{2} & \frac{\lambda_1 - \lambda_4}{2} \\ 0 & 0 & \frac{\lambda_1 - \lambda_4}{2} & \frac{\lambda_1 + \lambda_4}{2} \end{bmatrix}$$

has spectrum Λ . In this case we have

$$\mu = \left\{ \lambda_2, \lambda_3, \frac{\lambda_1 + \lambda_4}{2} \right\}, \quad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} 0 \\ \frac{\lambda_1 - \lambda_4}{2} \\ 0 \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ \frac{\lambda_1 - \lambda_4}{2} \end{bmatrix},$$

and so, $P\mathbf{y} = \mathbf{z}$.

The case $\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3 \geq \lambda_4$ and $\lambda_2 < |\lambda_3|$ is analogous to the previous case.

4. If $\lambda_1 \geq 0 > \lambda_2 \geq \lambda_3 \geq \lambda_4$, the matrix $\begin{bmatrix} A_1 & \mathbf{z} \\ \mathbf{z}^T & 0 \end{bmatrix}$ has eigenvalues Λ , with

$$\mathbf{z} = \left[\sqrt{\frac{-\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}{2(\lambda_1 + \lambda_2 - \lambda_3)}} \quad \sqrt{\frac{-\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}{2(\lambda_1 + \lambda_2 - \lambda_3)}} \quad \sqrt{\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 + \lambda_2 - \lambda_3}} \right]^T,$$

$$A_1 = \begin{bmatrix} \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{2} & \frac{\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4}{2} & \sqrt{\frac{-\lambda_3 (\lambda_1 + \lambda_2)}{2}} \\ \frac{\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4}{2} & \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{2} & \sqrt{\frac{-\lambda_3 (\lambda_1 + \lambda_2)}{2}} \\ \sqrt{\frac{-\lambda_3 (\lambda_1 + \lambda_2)}{2}} & \sqrt{\frac{-\lambda_3 (\lambda_1 + \lambda_2)}{2}} & 0 \end{bmatrix}.$$

In this case we have $\mu = \{\lambda_1 + \lambda_2, \lambda_3, \lambda_4\}$,

$$P = \begin{bmatrix} \sqrt{\frac{\lambda_1 + \lambda_2}{2(\lambda_1 + \lambda_2 - \lambda_3)}} & -\sqrt{\frac{-\lambda_3}{2(\lambda_1 + \lambda_2 - \lambda_3)}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{\lambda_1 + \lambda_2}{2(\lambda_1 + \lambda_2 - \lambda_3)}} & -\sqrt{\frac{-\lambda_3}{2(\lambda_1 + \lambda_2 - \lambda_3)}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{-\lambda_3}{\lambda_1 + \lambda_2 - \lambda_3}} & \sqrt{\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3}} & 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \sqrt{-\lambda_1 \lambda_2} \\ 0 \\ 0 \end{bmatrix},$$

and it holds that $P\mathbf{y} = \mathbf{z}$.

We conjecture that the Theorem 3.4 and Theorem 3.1 establish a necessary and sufficient condition for *SNIEP* when $n \geq 5$, that is, it holds $P\mathbf{y} = \mathbf{z}$, with P given as in Theorem 3.1, \mathbf{y} given as (*) and \mathbf{z} given as in the Theorem 3.4.

The *SNIEP* for a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is equivalent to find an orthogonal n -dimensional matrix P , such that $PD_\Lambda P^T \geq 0$, where $D_\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

We consider the following orthogonal matrices:

1.

$$P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

2. For the list of real numbers $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that $\mu_1 \geq |\lambda_i|$, for $i = 2, 3, 4$, $\mu_3 < 0$ and $\mu_2 < |\mu_3|$, we define

$$P_3 = \begin{bmatrix} \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & -\frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & -\frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

3. For the list of real numbers $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that $\mu_1 \geq |\lambda_i|$, for $i = 2, 3, 4$ and $\mu_2 < 0$, we define

$$P_4 = \begin{bmatrix} \frac{\sqrt{\mu_1}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_2}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_3}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{\mu_1}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_2}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_3}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{\mu_1}\sqrt{-\mu_3}}{\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_2}\sqrt{-\mu_3}}{\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1+\mu_2}}{\sqrt{\mu_1+\mu_2-\mu_3}} & 0 \\ \frac{\sqrt{-\mu_2}}{\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1}}{\sqrt{\mu_1-\mu_2}} & 0 & 0 \end{bmatrix}.$$

The following result gives a sufficient condition for realizability of list Λ when $n = 5$, by means of orthogonal matrices.

Corollary 3.5. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ lists of real numbers such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4 \geq \mu_4 \geq \lambda_5$; $\sum_{i=1}^5 \lambda_i - \sum_{i=1}^4 \mu_i \geq 0$; $\mu_1 \geq |\lambda_i|$ for $i = 2, 3, 4$ and $\mathbf{y} \in \mathbb{R}^4$ defined as (*). If one of the following conditions is true:

1. $P_1\mathbf{y} \geq 0 \wedge \mu_3 \geq 0 > \mu_4$,
2. $P_2\mathbf{y} \geq 0 \wedge \mu_2 \geq 0 > \mu_3 \wedge \mu_2 \geq |\mu_3|$,
3. $P_3\mathbf{y} \geq 0 \wedge \mu_2 \geq 0 > \mu_3 \wedge \mu_2 < |\mu_3|$,

$$4. P_4 \mathbf{y} \geq 0 \wedge \mu_1 \geq 0 > \mu_2,$$

then Λ is the spectrum of a nonnegative symmetric matrix.

Proof. We recall that $\mathbf{y} \in \mathbb{R}^{4 \times 1}$ is defined by

$$y_i = -\frac{\prod_{k=1}^5 (\mu_i - \lambda_k)}{\prod_{k=1}^{i-1} (\mu_i - \mu_k) \cdot \prod_{k=i+1}^4 (\mu_i - \mu_k)}.$$

We study the case (1): If $P_1 \mathbf{y} \geq 0$, $\mu_3 \geq 0 > \mu_4$.

We define the symmetric matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & 0 & 0 & y_1 \\ 0 & \mu_2 & 0 & 0 & y_2 \\ 0 & 0 & \mu_3 & 0 & y_3 \\ 0 & 0 & 0 & \mu_4 & y_4 \\ y_1 & y_2 & y_3 & y_4 & a \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

that is

$$A = \begin{bmatrix} \frac{1}{2}\mu_1 + \frac{1}{2}\mu_4 & \frac{1}{2}\mu_1 - \frac{1}{2}\mu_4 & 0 & 0 & \frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_4 \\ \frac{1}{2}\mu_1 - \frac{1}{2}\mu_4 & \frac{1}{2}\mu_1 + \frac{1}{2}\mu_4 & 0 & 0 & \frac{1}{2}\sqrt{2}y_1 + \frac{1}{2}\sqrt{2}y_4 \\ 0 & 0 & \mu_2 & 0 & y_2 \\ 0 & 0 & 0 & \mu_3 & y_3 \\ \frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_4 & \frac{1}{2}\sqrt{2}y_1 + \frac{1}{2}\sqrt{2}y_4 & y_2 & y_3 & a \end{bmatrix},$$

$$\text{whit } a = \sum_{i=1}^5 \lambda_i - \sum_{i=1}^4 \mu_i \geq 0.$$

Since, by hypothesis $\mu_1 \geq |\mu_i|$, $i = 2, 3, 4$, we have $\frac{1}{2}\mu_1 \pm \frac{1}{2}\mu_4 \geq 0$. Moreover, since $\mu_3 \geq 0$, then $\mu_2 \geq 0$. Therefore the matrix A is nonnegative.

Also

$$P_1 \mathbf{y} = \left[\frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_4 \quad \frac{1}{2}\sqrt{2}y_1 + \frac{1}{2}\sqrt{2}y_4 \quad y_3 \quad y_4 \right]^T;$$

by Theorem 3.1 the nonnegative symmetric matrix A has spectrum Λ .

Otherwise it is similarly derived. \square

Example 3.6. Consider the list $\Lambda = \{6, 3, 3, -5, -5\}$. If the list Λ is realizable, then there exists a realizable list $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$. By theorems 2.1, 2.2, 2.3, we have $6 \geq \mu_1 \geq 5$, $\mu_2 = 3$, $-1 \geq \mu_3 \geq -4$, $\mu_4 = -5$ and $4 \geq \mu_1 + \mu_3 \geq 2$.

On the other hand, for Λ and μ define $\mathbf{y} = [y_1 \ y_2 \ y_3 \ y_4]^T$, with y_i , $i = 1, 2, 3, 4$, as (*). There is not P_i of the Corollary 3.5 such that $P_i \mathbf{y} \geq 0$, thus by Theorems 3.1, 3.4, the list $\Lambda = \{6, 3, 3, -5, -5\}$ is not realizable.

Observation: In [6] it is shown that this list can not be performed by any criterion.

4. Algorithm

In this section, we will implement an algorithm to decide whether a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is the spectrum of a nonnegative symmetric matrix, by using the results presented in the previous section. We introduce the following notation:

$$\begin{aligned} \mathbb{O}_n &= \{P \in \mathbb{R}^{n \times n} : PP^T = P^T P = I\}, \\ \mathbb{S}_n &= \{\Lambda = \{\lambda_1, \dots, \lambda_n\} : \exists A = A^T, \sigma(A) = \Lambda\}, \\ D_\Lambda &= \text{diag}\{\lambda_1, \dots, \lambda_n\}, \\ \mathbb{S}_n(\Lambda) &= \{A \geq 0 : A = A^T \geq 0, \sigma(A) = \Lambda\}, \\ \mathbb{O}_n(\Lambda) &= \{P \in \mathbb{O}_n : PD_\Lambda P^T \geq 0\}. \end{aligned}$$

Note that $\mathbb{S}_n(\Lambda) \neq \emptyset$ if only if $\mathbb{O}_n(\Lambda) \neq \emptyset$, and if $\lambda_n \geq 0$ then $\mathbb{O}_n(\Lambda) \neq \emptyset$. Also $\Lambda \in \mathbb{S}_n$ if and only if $\mathbb{S}_n(\Lambda) \neq \emptyset$ or $\mathbb{O}_n(\Lambda) \neq \emptyset$.

Algorithm

1. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{n+1}\}$
2. Let $\mu = \{\mu_1, \dots, \mu_n\}$ be such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}$, $\sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^n \mu_i \geq 0$ and $\sum_{i=1}^n \mu_i \geq 0$.
3. If $\mu \in \mathbb{S}_n$, define $\mathbf{y} = [y_1 \ \dots \ y_n]^T$ as (*). If else return step 2.
4. Let $P \in \mathbb{O}_n(\mu)$.
5. If $P\mathbf{y} \geq 0$, define

$$A = \begin{bmatrix} PD_\mu P^T & P\mathbf{y} \\ (P\mathbf{y})^T & \sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^n \mu_i \end{bmatrix} \in \mathbb{S}_n(\Lambda);$$

if else, return to step 4.

Naturally there exists several ways to select the μ_i , as well as several ways to determine if $\mu \in \mathbb{S}_n$. For the case $n = 5$, the selection of μ is limited, since for $n = 4$ there are necessary and sufficient conditions to determine if $\mu \in \mathbb{S}_n$.

The following example shows that the algorithm presented is recursive.

Example 4.1. We consider the list $\Lambda = \{9, 1, -1, -2, -6\}$.

We select $\mu = \{6, -1, -2, -3\}$. To show that μ is realizable, select the list $\nu = \{3, -1, -2\}$.

For the list ν , we define $\mathbf{x}^T = [3\sqrt{2} \ 0 \ 0]$, and the orthogonal matrix

$$P_1 = \begin{bmatrix} \frac{1}{10}\sqrt{30} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{5}}{5} \\ \frac{1}{10}\sqrt{30} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{5}}{5} \\ \frac{1}{5}\sqrt{10} & 0 & \frac{1}{5}\sqrt{15} \end{bmatrix}.$$

Thus the matrix

$$\begin{aligned} A_1 &= \begin{bmatrix} P_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{diag}\{3, -1, -2\} & \mathbf{x} \\ & \mathbf{x}^T & 0 \end{bmatrix} \begin{bmatrix} P_1^T & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & \frac{1}{2}\sqrt{2}\sqrt{6} & \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} \\ 1 & 0 & \frac{1}{2}\sqrt{2}\sqrt{6} & \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} \\ \frac{1}{2}\sqrt{2}\sqrt{6} & \frac{1}{2}\sqrt{2}\sqrt{6} & 0 & \frac{6}{5}\sqrt{5} \\ \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} & \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} & \frac{6}{5}\sqrt{5} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1.0 & 1.7321 & 2.3238 \\ 1.0 & 0 & 1.7321 & 2.3238 \\ 1.7321 & 1.7321 & 0 & 2.6833 \\ 2.3238 & 2.3238 & 2.6833 & 0 \end{bmatrix} \end{aligned}$$

it is nonnegative symmetrical with spectrum μ .

For the list μ , we define $\mathbf{y} = [2\sqrt{5} \ 0 \ 0 \ 4]^T$ and the orthogonal matrix

$$P = \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{1}{\sqrt{2}} & -\frac{1}{5}\sqrt{5} & -\frac{1}{30}\sqrt{90} \\ \frac{1}{5}\sqrt{5} & \frac{1}{\sqrt{2}} & -\frac{1}{5}\sqrt{5} & -\frac{1}{30}\sqrt{90} \\ \frac{2}{15}\sqrt{15} & 0 & \frac{1}{5}\sqrt{15} & -\frac{1}{15}\sqrt{30} \\ \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{6} \end{bmatrix}.$$

Thus the matrix

$$\begin{aligned}
A &= \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{diag}\{6, -1, -2, -3\} & \mathbf{y} \\ \mathbf{y}^T & 0 \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & \frac{1}{2}\sqrt{12} & \frac{3}{10}\sqrt{60} & 2 - \frac{2\sqrt{90}}{15} \\ 1 & 0 & \frac{1}{2}\sqrt{12} & \frac{3}{10}\sqrt{60} & 2 - \frac{2\sqrt{90}}{15} \\ \frac{1}{2}\sqrt{12} & \frac{1}{2}\sqrt{12} & 0 & \frac{6}{5}\sqrt{5} & \frac{4}{3}\sqrt{3} - \frac{4\sqrt{30}}{15} \\ \frac{3}{10}\sqrt{60} & \frac{3}{10}\sqrt{60} & \frac{6}{5}\sqrt{5} & 0 & \frac{4}{3}\sqrt{6} + \frac{2\sqrt{15}}{3} \\ 2 - \frac{2}{15}\sqrt{90} & 2 - \frac{2}{15}\sqrt{90} & \frac{4}{3}\sqrt{3} - \frac{4}{15}\sqrt{30} & \frac{4}{3}\sqrt{6} + \frac{2\sqrt{15}}{3} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1.0 & 1.7321 & 2.3238 & 0.73509 \\ 1.0 & 0 & 1.7321 & 2.3238 & 0.73509 \\ 1.7321 & 1.7321 & 0 & 2.6833 & 0.84881 \\ 2.3238 & 2.3238 & 2.6833 & 0 & 5.8480 \\ 0.73509 & 0.73509 & 0.84881 & 5.8480 & 1.0 \end{bmatrix}
\end{aligned}$$

it is nonnegative symmetrical with spectrum $\Lambda = \{9, 1, -1, -2, -6\}$.

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