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Rotating Relativistic Thin Disks as Sources of the Taub-NUT Solution

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Abstract. Rotating disks with nonzero radial pressure and finite radius are studied. The models are based in the Taub-NUT metric and constructed using the well-known "displace, cut and reflect" method. We find that the disks are made of perfect fluids with constant energy density and pressure. The energy density is negative, but the effective Newtonian density is possitive as the strong energy condition requires. We also find that the disks are not stable under radial perturbations and that there are regions of the disks where the particles move with superluminal velocities.

1. Introduction

An important problem in general relativity is the obtention of exact solutions of Einstein equations corresponding to physically acceptable configurations of matter. Although exact solutions has been obtained only in simple, highly symmetric, cases, in the last twenty years several generation techniques to obtain solutions of Einstein equations from a given solution have been developed with success [1]. However, one of the shortcomings of these methods is that they do not give significant information about the physical or geometrical meaning of the generated solutions.

As recently has been shown [2, 3], many vacuum Weyl solutions can be interpreted as the metrics of static thin disks, constructed using the well-known "displace, cut and reflect" method. The idea of the method is simple. Given a solution of the vacuum Einstein equations, a cut is maked above all singularities or sources. The identification of this solution with its mirror image yields relativistic models of disks as a consequence of the jump in the normal derivative of the metric tensor. The disk model can also be used for the interpretation of vacuum stationary solutions.

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In the last years, disks models with radial pressure or tension [4], electric fields [5], magnetic fields [6] and magnetic and electric fields [7] have been studied. (See also the references [8, 9, 10].) In this work we apply the above method to study rotating disks with nonzero radial pressure and of finite radius. The models are based in the Taub-NUT (Newman-Unti-Tamburino) metric, which is one of the simplest axially symmetric stationary solutions of vacuum Einstein equations [1].

2. The Taub-NUT Solution

We can write the metric as the Weyl-Lewis-Papapetrou line element [1]:

$$ds^2 = e^{-2\Phi}[\mathcal{R}^2 d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)] - e^{2\Phi}(dt + \mathcal{W}d\varphi)^2, \tag{1}$$

where Φ, Λ, $\mathcal{W}$ and $\mathcal{R}$ are functions of r and z only. In order to obtain finite thin disks with nonzero radial pressure, we take a solution of vacuum Einstein equations defined by the following relations [4].

Let be $w = r + iz$ and $\mathcal{F}(w) = w + \alpha\sqrt{w^2 - 1}$, with $\alpha \geq 1$. Then

$$\mathcal{R}(r,z) = \text{Re } \mathcal{F}(w) , \tag{2}$$

$$\mathcal{Z}(r,z) = \text{Im } \mathcal{F}(w) , \tag{3}$$

$$\Phi(r,z) = \Psi(\mathcal{R},\mathcal{Z}) , \tag{4}$$

$$\Lambda(r,z) = \Pi(\mathcal{R},\mathcal{Z}) + \ln|\mathcal{F}'(w)| , \tag{5}$$

$$\mathcal{W}(r,z) = \mathcal{M}(\mathcal{R},\mathcal{Z}) . \tag{6}$$

Now we take Ψ, Π and $\mathcal{M}$ as given by the Taub-NUT solution, that can be written in prolate spheroidal coordinates as in [11]:

$$\Psi = \frac{1}{2}\ln\left[\frac{x^2 - 1}{x^2 + 2ux + 1}\right], \tag{7}$$

$$\Pi = \frac{1}{2}\ln\left[\frac{x^2 - 1}{x^2 - y^2}\right], \tag{8}$$

$$\mathcal{M} = 2kvy. \tag{9}$$

The possitive constants u and v can be written as $u = m/k$ and $v = l/k$, with $k^2 = m^2 + l^2$, where m is the mass and l is the NUT parameter [1], and thus $u^2 + v^2 = 1$. The relation between (x, y) and $(\mathcal{R}, \mathcal{Z})$ is given by

$$\mathcal{R}^2 = k^2(x^2 - 1)(1 - y^2), \qquad \mathcal{Z} = kxy, \tag{10}$$

where $1 \leq x \leq \infty$, $0 \leq y \leq 1$ and $k = \sqrt{\alpha^2 - 1}$.

3. The Energy-Momentum Tensor

The energy-momentum tensor of the disks can be computed by using the distributional approach (see [4, 10]), and can be written as

$$S^0_0 = \frac{e^{\Phi-\Lambda}}{\mathcal{R}^2}\{2\mathcal{R}^2(\Lambda_{,z} - 2\Phi_{,z}) + 2\mathcal{R}\mathcal{R}_{,z} - e^{4\Phi}\mathcal{W}\mathcal{W}_{,z}\}, \tag{11}$$

$$S^0_1 = \frac{e^{\Phi-\Lambda}}{\mathcal{R}^2}\{2\mathcal{R}\mathcal{W}(\mathcal{R}_{,z} - 2\mathcal{R}\Phi_{,z}) - (\mathcal{R}^2 + \mathcal{W}^2 e^{4\Phi})\mathcal{W}_{,z}\}, \tag{12}$$

$$S^1_0 = \frac{e^{\Phi-\Lambda}}{\mathcal{R}^2}\{e^{4\Phi}\mathcal{W}_{,z}\}, \tag{13}$$

$$S^1_1 = \frac{e^{\Phi-\Lambda}}{\mathcal{R}^2}\{2\mathcal{R}^2\Lambda_{,z} + e^{4\Phi}\mathcal{W}\mathcal{W}_{,z}\}, \tag{14}$$

$$S^2_2 = \frac{e^{\Phi-\Lambda}}{\mathcal{R}^2}\{2\mathcal{R}\mathcal{R}_{,z}\}, \tag{15}$$

where all the quantities are evaluated at $z = 0^+$, $0 \leq r \leq 1$. Is easy to see that, using (2) – (10), the energy-momentum tensor can be cast as

$$S_{ab} = (\sigma + p)V_a V_b + p\, h_{ab} , \tag{16}$$

where

$$\sigma = -\frac{2p}{\alpha}\left[\frac{\alpha + ku}{\alpha^2 + 2u\alpha k + k^2}\right] , \tag{17}$$

$$p = \frac{2\alpha}{\sqrt{\alpha^2 + 2u\alpha k + k^2}} . \tag{18}$$

V^a is the velocity vector of the disk, with components

$$V^a = e^{-\Phi}(1, 0, 0, 0) , \tag{19}$$

and h_{ab} is the metric of the $z = 0$ hypersurface. The disks so are made of perfect fluids with constant energy density and pressure. As we can see, $\sigma \leq 0$, and so the disks do not agree with the weak energy condition [12]. On the other hand, the effective Newtonian density, defined as $\varrho = \sigma + 2p$, is

$$\varrho = \frac{2kp}{\alpha}\left[\frac{u\alpha^2 + 2\alpha k + uk^2}{\alpha^2 + 2u\alpha k + k^2}\right] , \tag{20}$$

so that $\varrho \geq 0$, as the strong energy condition requires.

4. The Motion of the Disks

In order to analize the motion of the disks we compute its tangential velocity with respect to the locally nonrotating frames [13, 14],

$$\mathrm{V} = \frac{g_{11}(\Omega - \omega)}{\sqrt{g_{01}^2 - g_{00}g_{11}}} \;, \tag{21}$$

were $\omega = -g_{01}/g_{11}$ and $\Omega = V^1/V^0$. By using (2)–(10) we obtain

$$\mathrm{V} \;=\; -\left[\frac{kvp^2}{2\alpha^2}\right]\frac{\sqrt{1-r^2}}{r}. \tag{22}$$

As we can see from the above expression, the particles of the disks move with superluminal velocities for $r < r_0$, where

$$r_0 \;=\; \frac{kvp^2}{\sqrt{4\alpha^2 + k^2v^2p^4}} \;. \tag{23}$$

The specific angular momentum of a particle of the disk, with mass μ, is given by $h = p_\varphi/\mu = g_{\varphi a}V^a$. Thus we have

$$h^2 \;=\; \left[\frac{k^2p^2}{1+k^2}\right](1 - r^2), \tag{24}$$

and is easy to see that

$$\frac{d(h^2)}{dr} \; < 0 \;. \tag{25}$$

That is, the disks are not stable under radial perturbations, as can be concluded by an extension of Rayleigh criteria of stability of a fluid in rest in a gravitational field; see, for instance, [15].

5. Concluding Remarks

We do not know of any exact axially symmetric stationary solution of Einstein equations with the kind of physical properties of the above model. We find that the disks are made of perfect fluids with constant energy density and pressure. The energy density is negative, but the effective Newotnian density is possitive as the strong energy condition requires. We also find that the disks are not stable under radial perturbationsand that there are regions of the disks where the particles move with superluminal velocities.

We are now working in rotating disks models with nonzero radial pressure based in the Kerr metric. In this case the energy-momentum tensor of the disks may be not so simple and we can have zones with heat flow. The inclusion of electric or magnetic fields to these models is also under consideration in order to obtain "hot" rotaing disks, with or without radial pressure.

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