# A family of relativistic charged thin disks with an inner edge

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**Abstract.** A new family of exact solutions of the Einstein-Maxwell equations for static axially symmetric spacetimes is presented. The metric functions of the solutions are explicitly computed and are simply written in terms of the oblate spheroidal coordinates. The solutions, obtained by applying the Ernst method of complex potentials, describe an infinite family of static charged dust disks with an inner edge. The energy density, pressure and charge density of all the disks of the family are everywhere well behaved, in such a way that the energy-momentum tensor fully agrees with all the energy conditions.

## 1. Introduction

The study of axially symmetric solutions of the Einstein and Einstein-Maxwell field equations corresponding to disklike configurations of matter, apart from its purely mathematical interest, has a clear astrophysical relevance. Now, although normally it is considered that disks with electric fields do not have clear astrophysical importance, there exists the possibility that some galaxies be positively charged [1], so the study of charged disks may be of interest not only in the context of exact solutions. On the other hand,

the observational data supporting the existence of black holes at the nucleus of some galaxies is today so abundant, with the strongest dynamical evidence coming from the center of the Milky Way, that there is no doubt about the relevance of the study of binary systems composed by a thin disk surrounding a central black hole (see [2, 3] for recent reviews on the observational evidence).

Now, as was shown by Lemos and Letelier [4, 5, 6], it is clear that the obtainment of exact solutions that properly describe thin disklike sources with an inner edge has a clear relevance in the study of binary systems that involve a central black hole. Indeed, as was pointed in [7], the main properties of these annular disks strongly depend on their specific density profiles. Therefore, it is worth to have other solutions, in addition to those obtained previously, in such a way that a thoroughly study of such systems can be made. Also, the obtainment of static solutions can be considered as the first step in obtaining of the more realistic stationary solutions, which can be obtained from the previous ones by means of well known methods of solutions generation.

In agreement with the above considerations, in this paper we present an infinite family of new exact solutions of the Einstein-Maxwell equations for static axially symmetric spacetimes. These solutions describe a family of relativistic charged thin disks with an inner edge, the charged version of the family of relativistic thin dust disks with an inner edge recently presented in [8, 9]. The solutions were obtained by applying the Ernst method of complex potentials [10, 11], which generates a solution of the Einstein-Maxwell equations starting from a solution of the vacuum Einstein equations. The paper is organized as follows. First, in Section 2, we present the formalism to describe thin disks. Then, in Section 3, we present a family of charged thin disks with an inner edge, while in Section 4 we analyze the behavior of the solutions. Finally, in Section 5, we conclude by summarizing our main results.

## 2. Einstein-Maxwell equations and thin disks

The vacuum Einstein-Maxwell equations, in geometrized units such that $c = 8\pi G = \mu_0 = \epsilon_0 = 1$, can be written as

$$G_{ab} = T_{ab}, \tag{1}$$

$$F^{ab}{}_{;b} = 0, \tag{2}$$

with the electromagnetic energy-momentum tensor given by

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \tag{3}$$

where

$$F_{ab} = A_{b,a} - A_{a,b} \tag{4}$$

is the electromagnetic field tensor and $A_a$ is the electromagnetic four potential.

Now, for a static axially symmetric spacetime the line element can be written in cylindrical coordinates $x^a = (t, \varphi, r, z)$ as [12]

$$ds^2 = -e^{2\Phi}dt^2 + e^{-2\Phi}[r^2 d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)], \tag{5}$$

where the metric functions $\Phi$ and $\Lambda$ depend on $r$ and $z$ only. We take the electromagnetic potential as

$$A_a = (\psi, 0, 0, 0), \tag{6}$$

where it is assumed that the electric potential $\psi$ also depends on $r$ and $z$ only.

The solutions of the Einstein-Maxwell equations corresponding to a disklike source are even functions of the $z$ coordinate. Therefore, they are everywhere continuous functions but with their first $z$-derivatives discontinuous at the disk surface. Accordingly, in order to obtain the energy-momentum tensor and the current density of the source, we will express the jump across the disk of the first $z$-derivatives of the metric tensor as

$$b_{ab} = [g_{ab,z}] = 2g_{ab,z}|_{z=0^+}, \tag{7}$$

and the jump across the disk of the electromagnetic field tensor as

$$[F_{za}] = [A_{a,z}] = 2A_{a,z}|_{z=0^+}, \tag{8}$$

where the reflection symmetry of the functions with respect to $z = 0$ has been used.

Then, by using the distributional approach [13, 14, 15] or the junction conditions on the extrinsic curvature of thin shells [16, 17, 18], the Einstein-Maxwell equations yield an energy-momentum tensor as

$$T^{ab} = T_+^{ab}\theta(z) + T_-^{ab}[1 - \theta(z)] + Q^{ab}\delta(z), \tag{9}$$

and a current density as

$$J^a = I^a \delta(z), \tag{10}$$

where $\theta(z)$ and $\delta(z)$ are, respectively, the Heaviside and Dirac distributions with support on $z = 0$. Here $T_\pm^{ab}$ are the electromagnetic energy-momentum tensors as defined by (3) for the $z \geq 0$ and $z \leq 0$ regions, respectively, whereas

$$Q_b^a = \frac{1}{2}\{b^{az}\delta_b^z - b^{zz}\delta_b^a + g^{az}b_b^z - g^{zz}b_b^a + b_c^c(g^{zz}\delta_b^a - g^{az}\delta_b^z)\} \tag{11}$$

gives the part of the energy-momentum tensor corresponding to the disk source, and

$$I^a = [F^{az}] \tag{12}$$

is the contribution of the disk source to the current density. Now, the surface energy-momentum tensor of the disk, $S_{ab}$, and the surface current density, $j^a$, can be obtained through the relations

$$S_{ab} = \int Q_{ab}\,\delta(z)\,ds_n \;=\; e^{\Lambda-\Phi}\,Q_{ab}, \tag{13}$$

$$j_a \;=\; \int I_a\,\delta(z)\,ds_n \;=\; e^{\Lambda-\Phi}I_a, \tag{14}$$

where $ds_n = \sqrt{g_{zz}}\,dz$ is the physical measurement of length in the direction normal to the disk.

For the metric (5), the only non-zero components of $S_b^a$ and $j_0$ are

$$S_0^0 = 2e^{\Phi-\Lambda}\{\Lambda_{,z} - 2\Phi_{,z}\}, \tag{15}$$

$$S_1^1 = 2e^{\Phi-\Lambda}\Lambda_{,z}, \tag{16}$$

$$j_0 = -2e^{\Phi-\Lambda}\psi_{,z}, \tag{17}$$

where all the quantities are evaluated at $z = 0^+$. Therefore, the surface energy-momentum tensor of the disk and the surface current density of the disk can be written as

$$S^{ab} = \varepsilon V^a V^b, \tag{18}$$

$$j^a = \sigma V^a, \tag{19}$$

where

$$V^a = e^{-\Phi}(1, 0, 0, 0) \tag{20}$$

is the velocity vector of the matter distribution.

## 3.   Charged dust disks with a central inner edge

In this section we consider a family of static axially symmetric solutions of the vacuum Einstein-Maxwell equations given, for $n = 1, 2, \ldots$, by

$$e^{\Phi_n} = \frac{2\beta e^{\phi_n}}{(1 + \beta) - e^{2\phi_n}(1 - \beta)}, \tag{21}$$

$$\psi_n = \frac{\sqrt{1 - \beta^2}(e^{2\phi_n} - 1)}{e^{2\phi_n}(1 - \beta) - (1 + \beta)}, \tag{22}$$

$$\phi_n = \frac{\alpha y F_n(x, y)}{a^n(x^2 + y^2)^{2n-1}}, \tag{23}$$

$$\Lambda_n = \frac{\alpha^2(2n - 2)!(y^2 - 1)A_n(x, y)}{4^n a^{2n}(x^2 + y^2)^{4n}}, \tag{24}$$

where $\alpha$ is an arbitrary constant and $\beta = \sqrt{1 - q^2}$, where $q^2 \leq 1$. The functions are all expressed in terms of the oblate spheroidal coordinates, connected with the cylindrical coordinates via the transformation formulas

$$r^2 = a^2(1 + x^2)(1 - y^2), \quad z = axy, \tag{25}$$

where $x \in (-\infty, \infty)$ and $y \in [0, 1]$. This solution was obtained, by means of the Ernst method of complex potentials [10, 11], from the solutions describing a family of relativistic static thin dust disks with an inner edge recently presented in [8, 9].

In the above expressions, the functions $F_n(x, y)$ and $A_n(x, y)$ can be easily obtained by means of the procedure described in [8]. So, the functions $F_n(x, y)$ are polynomial functions with highest degree $4n - 4$, the first two of them being

$$F_1 = 1, \tag{26}$$

$$F_2 = x^4 + 3x^2(1 - y^2) - y^2, \tag{27}$$

while the $A_n(x, y)$ are polynomial functions off highest degree $8n - 2$, of which the first two are

$$\begin{aligned}
A_1 &= x^4(9y^2 - 1) + 2x^2y^2(y^2 + 3) + y^4(y^2 - 1), \\
A_2 &= 2x^{12}(9y^2 - 1) - 4x^{10}(51y^4 - 41y^2 + 2) \\
&\quad + x^8(735y^6 - 1241y^4 + 419y^2 - 9) - x^6y^2(132y^6 \\
&\quad - 1644y^4 + 1604y^2 - 252) + x^4y^4(84y^6 - 384y^4 \\
&\quad + 1266y^2 - 630) + 4x^2y^6(6y^6 + 6y^4 - 39y^2 + 63) \\
&\quad + 3y^8(y^6 + y^4 + y^2 - 3),
\end{aligned}$$

all of them easily obtained as described in [8].

Now, the expressions for the surface energy density $\varepsilon$ and the surface charge density $\sigma$ of the disks can be written as

$$\varepsilon_n = 4e^{\Phi_n - \Lambda_n} \left\{1 - r\Phi_{n,r}\right\} \Phi_{n,z}, \tag{28}$$

$$\sigma_n = -2e^{\Phi_n - \Lambda_n} \psi_{n,z}, \tag{29}$$

where all of the quantities are evaluated at $z = 0^+$. Accordingly, by using the expressions (21) - (24), we obtain for the energy density and the charge density the expressions

$$\varepsilon_n = \frac{\epsilon_n}{\beta}, \tag{30}$$

$$\sigma_n = \frac{q\epsilon_n}{2\beta}, \tag{31}$$

where

$$\epsilon_n(x) = \frac{4\alpha E_n(x)}{a^{n+1}x^{2n+1}} \exp\left\{-\frac{\alpha^2(2n-2)!B_n(x)}{2^{2n}a^{2n}x^{4n}}\right\} \tag{32}$$

is the surface energy density of the static thin dust disks, without charge, as presented in [8, 9].

In the above expression, the $E_n(x)$ are positive definite polynomials of degree $2k$, with $k = (n-1)/2$ for odd $n$ and $k = n/2$ for even $n$, of which we only will write below the first three,

$$E_1(x) = 1, \tag{33}$$

$$E_2(x) = x^2 + 3, \tag{34}$$

$$E_3(x) = 3(x^2 + 5), \tag{35}$$

and the $B_n(x)$ are positive definite polynomials of degree $4k$, with $k = (n-1)/2$ for odd $n$ and $k = n/2$ for even $n$, being

$$B_1(x) = 1, \tag{36}$$

$$B_2(x) = 2x^4 + 8x^2 + 9, \tag{37}$$

$$B_3(x) = 27x^4 + 72x^2 + 50, \tag{38}$$

the first three of them.

## 4. Behavior of the solutions

From the expressions in the previous section we can see that, by taking $\alpha > 0$, the energy density of the disks will be everywhere positive,

$$\varepsilon_n(x) \geq 0, \tag{39}$$

so that, as the azimuthal pressure is zero, we have an infinite family of dust disks that are in fully agreement with all the energy conditions. Also is easy to see that, for any value of $n$,

$$\varepsilon_n(0) = 0, \tag{40}$$

$$\lim_{x \to \infty} \varepsilon_n(x) = 0. \tag{41}$$

That is, the energy density of the disks is zero at their inner edge and vanishes at infinite. Furthermore, the surface mass density of the disks reduces to their energy density,

$$\mu_n = \varepsilon_n, \tag{42}$$

so that its behavior is the same as of the energy density. Finally, by using equation (30) and (31) we can write

$$\sigma_n = \frac{q}{2}\varepsilon_n, \tag{43}$$

so that the charge density of the disks is equal, except by a constant, to their energy density. Accordingly, the electric and gravitational forces are in exact balance, as in the configurations of 'electrically counterpoised dust'.

Now, in order to show the behavior of the energy densities, we plot the dimensionless surface energy densities $\tilde{\varepsilon}_n = a\varepsilon_n$ as functions of the dimensionless radial coordinate $\tilde{r} = r/a$. So, in Figure 1, we plot $\tilde{\epsilon}_n$ as a function of $\tilde{r}$ for the first three disks of the family, with $n = 1$, 2 and 3, for different values of the parameter $\tilde{\alpha}_n = \alpha/a^n$. Then, for each value of $n$, we take $\tilde{\alpha}_n = 0.5$, 1 and 1.5. The first curve on left corresponds to $\tilde{\alpha}_n = 0.5$, while the last curve on right corresponds to $\tilde{\alpha}_n = 1.5$. The curves with greatest maximum corresponds to $\beta = 0.2$, whereas the curves with least maximum corresponds to $\beta = 1$, the disks without charge. As we can see, in all the cases the surface energy density is positive everywhere, having a maximum near the inner edge of the disks, and then rapidly decreasing as $\tilde{r}$ increases. We can also see that, for a fixed value of $n$, as the value of $\tilde{\alpha}_n$ increases, the value of the maximum diminishes and moves towards increasing
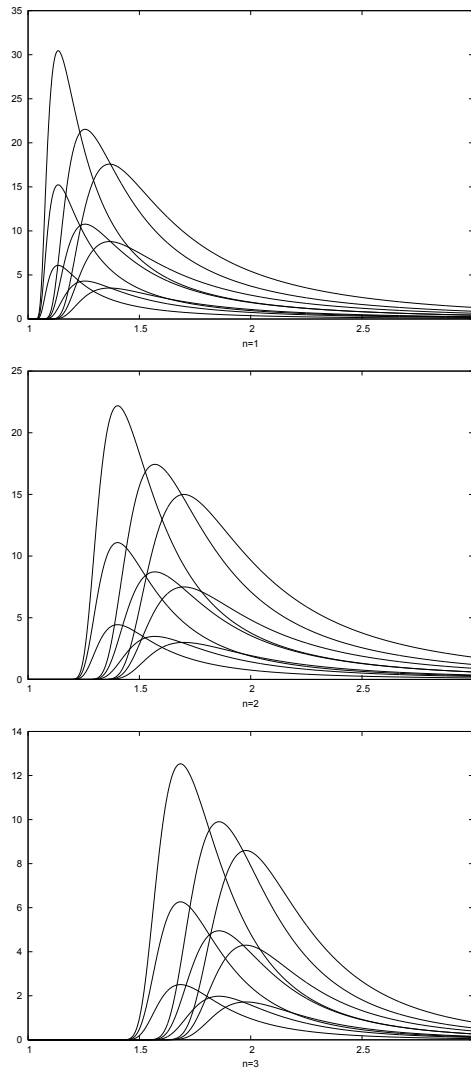
Figure 1. Energy density $\tilde{\varepsilon}_n$ as a function of $\tilde{r}$ for the first three disks of the family, with $\tilde{\alpha}_n = 0.5$, 1, 1.5 and $\beta = 1$, 0.4 and 0.2. For each value of $n$, the first curve on left corresponds to $\tilde{\alpha} = 0.5$, whereas the last curve on right corresponds to $\tilde{\alpha} = 4$. Also, for each value of $n$ the curves with greatest maximum corresponds to $\beta = 0.2$, whereas the curves with least maximum corresponds to $\beta = 1$, the disks without charge

values of $\tilde{r}$. Also, as there is a direct relationship between $\beta$ and the electric field, the maximum of the curves increases as the electric field increases. The same behavior is observed for a fixed value of $\tilde{\alpha}_n$ and increasing values of $n$.

## 5. Concluding remarks

We presented an infinite family of new exact solutions of the vacuum Einstein-Maxwell equations for static and axially symmetric spacetimes. The solutions describe an infinite family of charged thin dust disks with a central inner edge, the charged version of the family of relativistic thin dust disks with an inner edge recently presented in [8, 9]. Now, although the strange behavior of the Newtonian potentials may suggest that the disks do not correspond to reasonable astrophysical sources, their energy densities are everywhere positive and well behaved, in such a way that their energy-momentum tensor are in fully agreement with all the energy conditions.

On the other hand, since all the metric functions of the solutions were explicitly computed, these are the first fully integrated exact solutions for such kind of thin disk sources. Moreover, the method used here to obtain these explicit solutions may serve as a guideline to find more physical solutions in future works. Now, besides their importance as a new family of exact solutions of the Einstein-Maxwell vacuum equations, the main importance of this family of solutions is that they can be easily superposed with the Schwarzschild solution in order to describe binary systems composed by a thin disk surrounding a charged central black hole, as was be done in [19] with the first member of the family of relativistic thin dust disks presented in [8, 9].

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## References

[1]  J. Bally and E. R. Harrison. *Ap. J.*, **220**, 743 (1978).

[2]  A. Celotti, J. C. Miller and D. W. Sciama. *Class. Quantum Grav*, **16**, A3 (1999).

[3]  M. C. Begelman. *Science*, **300**, 1898 (2003).

[4]  J. P. S. Lemos and P. S. Letelier. *Class. Quantum Grav.*, **10**, L75 (1993).

[5] J. P. S. Lemos and P. S. Letelier. *Phys. Rev D*, **49**, 5135 (1994).

[6] J. P. S. Lemos and P. S. Letelier. *Int. J. Mod. Phys. D*, **5**, 53 (1995).

[7] O. Semerák. *Class. Quantum Grav.*, **21**, 2203 (2004).

[8] G. A. González, A. C. Gutiérrez-Piñeres and V. Viña-Cervantes. "Relativistic static thin dust disks with an inner edge: An infinite family of new exact solutions", *arXiv*: 0811.3869v1 (2008).

[9] G. A. González, A. C. Gutiérrez-Piñeres and V. Viña-Cervantes. *AIP Conf. Proc.*, **1122**, 284-287 (2009).

[10] F. J. Ernst. *Phys. Rev.*, **167**, 1175 (1968).

[11] F. J. Ernst. *Phys. Rev.*, **168**, 1415 (1968).

[12] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt. *Exact Solutions of Einstein's Field Equations.* Cambridge University Press, 2003.

[13] A. Papapetrou and A. Hamouni. *Ann. Inst. Henri Poincaré*, **9**, 179 (1968).

[14] A. Lichnerowicz. *C.R. Acad. Sci.*, **273**, 528 (1971).

[15] A. H. Taub. *J. Math. Phys.*, **21**, 1423 (1980).

[16] E. Israel. *Nuovo Cimento*, **44B**, 1 (1966).

[17] E. Israel. *Nuovo Cimento*, **48B**, 463 (1967).

[18] E. Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics.* Cambridge University Press, 2004.

[19] G. A. González and A. C. Gutiérrez-Piñeres. "Counterrotating Dust Disk Around a Schwarzschild Black Hole: New Fully Integrated Explicit Exact Solution", *arXiv*: 0811.3002v1 (2008).

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