

## *Existence of periodic solutions for seasonal epidemic models with quarantine*

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**Abstract.** In this work, we establish the existence of periodic orbits for a seasonal saturated epidemiological model of a population consisting of susceptible, infectious and quarantined individuals (an SIQS model). To do so, we use Leray-Schauder degree theory. We also provide numerical examples of these solutions.

**Keywords:** Leray-Schauder degree, SIQS models, periodic orbits, reproductive number.

**MSC2010:** 37J45, 34C25, 92D30, 34D23.

## *Existencia de soluciones periódicas para modelos epidemiológicos estacionales con cuarentena*

**Resumen.** En este trabajo establecemos la existencia de órbitas periódicas para un modelo epidemiológico estacional con cuarentena y tasa de incidencia saturada. Para realizar lo anterior, usamos un esquema variacional basado en la teoría de grado de Leray-Schauder. También presentamos algunos ejemplos numéricos para ilustrar nuestros resultados analíticos.

**Palabras clave:** Grado de Leray-Schauder, modelos SIQS, órbitas periódicas, número reproductivo básico.

### **1. Introduction**

In epidemiological models it is of importance the consideration of possible ways of controlling infectious diseases, such as vaccination or quarantine. Hence the addition of a compartment consisting of quarantined individuals gives a way of exploring possible

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Received: 12 February 2018, Accepted: 21 June 2018.

To cite this article: S. Guerrero-Flores, O. Osuna, G. Villavicencio-Pulido, Existence of periodic solutions for seasonal epidemic models with quarantine, *Rev. Integr. temas mat.* 36 (2018), No. 1, 37-47. doi: 10.18273/revint.v36n1-2018003.

mitigation effects for controlling or eliminating the disease. We consider a disease model in a population in the following way. At any given time,  $t$ , the total population  $N$  is divided into three compartments with  $N(t) = S(t) + I(t) + Q(t)$  where  $S$  is the number of individuals in the susceptible class,  $I$  is the number of individuals who are infectious but not quarantined, and  $Q$  is the number of individuals who are quarantined. Based in the autonomous bilinear model with quarantine of [5], we formulate a quarantined models with seasonally-dependent saturated incidence rate given by

$$\begin{aligned} S' &= \Lambda - \beta(t)IS/(1 + k_1I^{n_1} + k_2I^{n_2} + k_3S^{m_1}I^{n_3}) - dS + \gamma I + \epsilon Q, \\ I' &= \beta(t)IS/(1 + k_1I^{n_1} + k_2I^{n_2} + k_3S^{m_1}I^{n_3}) - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \tag{1}$$

where parameters  $\Lambda, \delta, d$  and  $\alpha$  are positive constants;  $\gamma, k_i, \epsilon$  are non-negative constants, and  $n_i, m_1 \geq 1, i = 1, 2, 3$ . The constant  $\Lambda$  is the recruitment rate of susceptibles corresponding to births and immigration,  $d$  is the per capita natural mortality rate,  $\delta$  is the rate constant for individuals leaving the infective compartment  $I$  for the quarantine compartment  $Q$ ,  $\alpha$  is the disease-related death rate constant in compartments  $I$  and  $Q$ , and  $\gamma$  and  $\epsilon$  are the rates at which individuals recover and return to susceptible compartment  $S$  from compartments  $I$  and  $Q$ , respectively. In short, we call these models as *SIQS* models.

The case of a bilinear autonomous *SIQS* model with  $\beta$ -constant was considered in [5], where thresholds, equilibria, and their stability are studied. In [10] a bilinear non-autonomous epidemic model with quarantine was studied and some conditions for thresholds and eradication of the infectious disease were obtained. In this work, we assume that the interactions between susceptible and infected individuals are modeled by the incidence function  $\beta(t)IS/(1 + k_1I^{n_1} + k_2I^{n_2} + k_3S^{m_1}I^{n_3})$ , where  $\beta(t)$  is a non constant non-negative continuous  $T$ -periodic function. Since several time-dependent factors are very important for the spread of diseases, with the use of a periodic incidence function we can take into account the variability of diseases according to climate seasons, school calendars, etc. Besides with the saturated term  $1 + k_1I^{n_1} + k_2I^{n_2} + k_3S^{m_1}I^{n_3}$ , we measure the inhibition effect from the behavioral change of the susceptible or infected individuals when the number of infected increases. In the case autonomous (with  $\beta$ -constant), several different saturated incidence rates have been proposed by authors, for example: Capasso and Serio [2] introduced a saturated incidence function  $\frac{\beta SI}{(1+kI)}$  in an epidemic model when they studied the cholera epidemic in 1973. Xiao and Ruan [12] proposed an *SIRS* epidemic model with non-monotonic incidence function  $\frac{\beta SI}{(1+kI^2)}$ . A general saturation incidence function  $\frac{\beta I^l S}{(1+kI^n)}$  was proposed by Liu and coworkers [8], [9] and used by a number of authors. A incidence function with form  $\frac{\beta IS}{(1+k_1I+k_2I^2)}$  was studied in [6]. In [1] was introduced a model with incidence function  $\frac{\beta IS}{(1+k_1I+k_2SI)}$ . So with our incidence rate can analyze similar situations, but in the non-autonomous case.

In this paper we are interested in the existence of periodic orbits for system (1). To do so, we shall use the Leray-Schauder degree theory, we reformulate the problem in a functional setting, in our development we generalize the Katriel's frame [7] to our context of quarantined system with saturated incidence rate making appropriate adjustments. In particular, we must adapt its scheme to three dimensional systems, use a Hurwitz

condition to control the roots of a characteristic polynomial, also we need use a second homotopy to deal with the term of saturation. In the study of epidemiological models the analysis of periodic solutions is seen as an important goal as this periodicity reveals the recurrence of an epidemic in a population. Hence, determining existence of such solutions under different parameter configurations and incidence functions is crucial.

## 2. Existence of periodic orbits

The total population size  $N$  is variable with  $N' = \Lambda - dN - \alpha(I + Q)$ . Following [5] we can see that, in the absence of disease, the population size  $N$  approaches the carrying capacity  $\Lambda/d$ . The differential equation for  $N$  implies that solutions of (1) starting in the positive orthant  $R_+^3$  either approach, enter, or remain in the subset of  $R^3$  defined by

$$\Sigma := \{(S, I, Q) \mid S \geq 0, I \geq 0, Q \geq 0, S + I + Q \leq \Lambda/d\}.$$

Thus it suffices to consider solutions in the region  $\Sigma$ .

The system (1) always has the disease-free equilibrium  $(S_0, I_0, Q_0) = (\Lambda/d, 0, 0)$ . The basic reproductive number  $\mathcal{R}_0$  has been defined as the average number of secondary infections that occur when one infective is introduced into a completely susceptible host population. When  $\beta$  is constant, it is easily computed by the van den Driessche and Watmough approach [11]: for system (1) we have that

$$\mathcal{R}_0 := \frac{1}{\gamma + \delta + d + \alpha} \frac{\partial f}{\partial I}(S_0, I_0, Q_0);$$

a direct calculation yields

$$\mathcal{R}_0 = \frac{\beta(\Lambda/d)}{\gamma + \delta + d + \alpha}.$$

Motivated by this, we consider  $\mathcal{R}_0$  for system (1) as follows:

$$\mathcal{R}_0 := \frac{(\Lambda/d)\bar{\beta}}{\gamma + \delta + d + \alpha}, \text{ where } \bar{\beta} := \frac{1}{T} \int_0^T \beta(t) dt.$$

We write

$$\beta(t) = \bar{\beta} + \beta_0(t), \text{ where } \int_0^T \beta_0(t) dt = 0.$$

The proof of the existence of periodic orbits for systems of type (1) will be done in two steps. First, we consider the bilinear case  $k_i = 0$ ,

$$\begin{aligned} S' &= \Lambda - \beta(t)IS - dS + \gamma I + \epsilon Q, \\ I' &= \beta(t)IS - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \tag{2}$$

and prove the existence of solutions on this system. Then, we construct an homotopy between (2) and (1).

For  $\lambda \in [0, 1]$  we define the homotopy

$$\begin{aligned} S' &= \Lambda - \beta_\lambda IS - dS + \gamma I + \epsilon Q, \\ I' &= \beta_\lambda IS - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \tag{3}$$

where  $\beta_\lambda := \bar{\beta} + \lambda\beta_0(t)$ .

To show the existence of a positive periodic solution, we shall use the Leray-Schauder degree theory. To do so, we need to reformulate our system as a functional problem defined on an adequate Banach space where periodic solutions correspond to the zeroes of convenient family of operators. Then, we need to find an open bounded subset on the Banach space such that the family of operators does not support zeroes on the boundary of such open set. After that, we can proceed to determine the Leray-Schauder degree and, if applicable, establish the existence of periodic solutions. In our methods, we are following the approach used in [7], but we establish suitable modifications for the non-linear incidence case, in particular, we need to use a double homotopy, in the first we work with the linear incidence case, so, we must adapt its scheme to three dimensional systems unlike [7] where it was possible to reduce the problem to a two-dimensional system; after that, for to estimate the Leray-Schauder degree we use a Hurwitz condition to control the roots of a characteristic polynomial; later in our context, we shall use a second homotopy to deal with the saturation term.

We start the proof by introducing for  $l = 0, 1$  the Banach spaces

$$\mathcal{C}_T^l := \{(S, I, Q) : S, I, Q \in C^l(\mathbb{R}, \mathbb{R}), S(t+T) = S(t), I(t+T) = I(t), Q(t+T) = Q(t)\}.$$

Let  $L : \mathcal{C}_T^1 \rightarrow \mathcal{C}_T^0$  and  $N_\lambda : \mathcal{C}_T^0 \rightarrow \mathcal{C}_T^0$  be the operators given by

$$L(S, I, Q) := (S' + dS, I' + (\gamma + \delta + d + \alpha)I, Q' + (\epsilon + d + \alpha)Q), \tag{4}$$

and

$$N_\lambda(S, I, Q) := (\Lambda + \gamma I + \epsilon Q - \beta_\lambda IS, \beta_\lambda IS, \delta I).$$

Since  $L$  is invertible we define

$$F_\lambda(S, I, Q) := (S, I, Q) - L^{-1} \circ N_\lambda(S, I, Q). \tag{5}$$

Since  $\mathcal{C}_T^1$  is compactly embedded in  $\mathcal{C}_T^0$ , we can think of  $L^{-1}$  as going from  $\mathcal{C}_T^0$  to  $\mathcal{C}_T^0$ ; therefore,  $L^{-1} \circ N_\lambda : \mathcal{C}_T^0 \rightarrow \mathcal{C}_T^0$  is a compact operator. In a similar fashion, we can consider  $F_\lambda : \mathcal{C}_T^0 \rightarrow \mathcal{C}_T^0$ . Thus, (5) is a functional reformulation of problem (3); in particular, periodic solutions of (3) correspond to zeroes of  $F_\lambda$ .

We consider the open sets

$$D := \{(S, I, Q) \in \mathcal{C}_T^0 : S > 0, I > 0, Q > 0, S + I + Q < \Lambda/d\}$$

and

$$G := \{(S, I, Q) \in D : \min_{[0, T]} S(t) < r(\Lambda/d)\},$$

for a fixed  $r$ .

Recall that the existence of a solution for  $F_1$  in  $G$  via Leray-Schauder degree is guaranteed if  $\deg(F_0, G) \neq 0$  and  $F_\lambda$  is an admissible homotopy, i.e.,  $0 \notin F_\lambda(\partial G), \forall \lambda \in [0, 1]$ . The next result says that  $F_\lambda$  is admissible.

**Lemma 2.1.** *If  $\mathcal{R}_0 > 1$  and  $r$  is such that  $\frac{1}{\mathcal{R}_0} < r < 1$ , then for any  $\lambda \in [0, 1]$  there are no solutions  $(S, I, Q)$  of (3) on  $\partial G$ .*

*Proof.* First note that  $(S_0, I_0, Q_0)$  is the only solution of (3) on  $\partial D$  for any  $\lambda \in [0, 1]$ . If  $(S, I, Q) \in \partial G$ , then  $(S, I, Q) \notin \partial D$ , so

$$(S, I, Q) \in D \text{ and } S(t) \geq r(\Lambda/d), \forall t. \quad (6)$$

By integrating the second equation in (3) on the interval  $[0, T]$ , we have that

$$\int_0^T \frac{I'}{I} dt + (\gamma + \delta + d + \alpha)T = \int_0^T \beta_\lambda S dt,$$

but  $\int_0^T \frac{I'}{I} dt = 0$  because  $I$  is  $T$ -periodic; using the inequality (6) one gets

$$\gamma + \delta + d + \alpha = \frac{1}{T} \int_0^T \beta_\lambda S dt \geq r(\Lambda/d)\bar{\beta}.$$

Now, from our hypothesis,

$$\gamma + \delta + d + \alpha \geq r(\Lambda/d)\bar{\beta} > \bar{\beta}(\Lambda/d)\frac{1}{\mathcal{R}_0} = \gamma + \delta + d + \alpha, \quad (7)$$

which is a contradiction.  $\square$

By Theorem 2 in [5], when  $\lambda = 0$ , the system (3) has exactly two periodic orbits in  $\Sigma$  being these:  $S_0 = \Lambda/d, I_0 = 0, Q_0 = 0$  and

$$S_1 = \frac{\Lambda/d}{\mathcal{R}_0}, \quad I_1 = \frac{\Lambda(1 - \frac{1}{\mathcal{R}_0})}{(d + \alpha)[1 + \frac{\delta}{\epsilon + d + \alpha}]}, \quad Q_1 = \frac{\delta I_1}{\epsilon + d + \alpha},$$

which in fact are critical points.

We recall that a polynomial is Hurwitz if its eigenvalues have negative real parts. It is well known that if the characteristic polynomial of a linear differential equation is Hurwitz, then any solution converges to the origin, therefore the unique possible periodic solution that a linear system with Hurwitz characteristic polynomial can have is the origin. The following lemma, which is a direct consequence of the Routh-Hurwitz stability criterion, gives us a characterization so that a cubic polynomial is Hurwitz.

**Lemma 2.2.** *The polynomial  $p(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$  is Hurwitz if and only if  $a_i$  have the same sign and  $a_1 a_2 - a_3 a_0 > 0$ .*

The following result determine the degree of  $F_0$  on  $G$ .

**Proposition 2.3.** *For the open set  $G$ , we have that  $\deg(F_0, G) \neq 0$ .*

*Proof.* Note that if  $\mathcal{R}_0 > 1$ , then  $(S_1, I_1, Q_1)$  is the unique periodic solution of  $F_0(S, I, Q) = 0$  in  $G$ . So, to establish the degree  $\deg(F_0, G) \neq 0$  we need only to prove that  $DF_0(S_1, I_1, Q_1)$  is invertible. We have that  $F_0$  is a compact perturbation of the identity, so by the Fredholm alternative it is enough to prove that

$$\text{Ker}(DF_0(S_1, I_1, Q_1)) = \{0\}. \quad (8)$$

Consider  $(U, V, W) \in \mathcal{C}_T^0$  so that  $(U, V, W) \in \text{Ker}(DF_0(S_1, I_1, Q_1))$ ; by the definition of  $F_0$ , we get that  $L(U, V, W) = DN_0(S_1, I_1, Q_1)(U, V, W)$ , since

$$N_0(S, I, Q) = (\Lambda + \gamma I + \epsilon Q - \bar{\beta}IS, \bar{\beta}IS, \delta I).$$

Then, we obtain

$$DN_0(S_1, I_1, Q_1)(U, V, W) = (\gamma V + \epsilon W - \bar{\beta}(S_1 V + I_1 U), \bar{\beta}(S_1 V + I_1 U), \delta V).$$

Using the definition of

$$L(U, V, W) = (U' + dU, V' + (\gamma + \delta + d + \alpha)V, W' + (\epsilon + d + \alpha)W),$$

we get

$$\begin{aligned} (U', V', W') &= (-dU + \gamma V + \epsilon W - \bar{\beta}(S_1 V + I_1 U), -(\gamma + \delta + d + \alpha)V + \bar{\beta}(S_1 V + I_1 U), \\ &\quad -(\epsilon + d + \alpha)W + \delta V). \end{aligned}$$

Rewriting and substituting  $S_1$ , we have

$$\begin{pmatrix} U' \\ V' \\ W' \end{pmatrix} = \begin{pmatrix} -(d + \bar{\beta}I_1) & -(\delta + d + \alpha) & \epsilon \\ \bar{\beta}I_1 & 0 & 0 \\ 0 & \delta & -(\epsilon + d + \alpha) \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}. \quad (9)$$

Thus any possible periodic element  $(U, V, W) \in \text{Ker}(DF_0(S_1, I_1, Q_1))$  is in fact a solution of the linear system (9). Therefore it would be enough for us to see that the characteristic polynomial of (9) is Hurwitz.

Denoting by  $A = (a_{ij})$  the above matrix, we have that its characteristic polynomial is given by

$$p(\lambda) = -\lambda^3 + \text{tr}(A)\lambda^2 - m\lambda + \det(A), \quad (10)$$

where  $m := M_{11} + M_{22} + M_{33}$ , the sum of the minors for the elements of the main diagonal. We have

$$\text{tr}(A) = -(d + \bar{\beta}I_1 + \epsilon + d + \alpha), \quad (11)$$

$$\begin{aligned} \det(A) &= -\bar{\beta}I_1 \begin{vmatrix} -(\delta + d + \alpha) & \epsilon \\ \delta & -(\epsilon + d + \alpha) \end{vmatrix} = -\bar{\beta}I_1 [(\delta + d + \alpha)(\epsilon + d + \alpha) - \delta\epsilon] \\ &= -\bar{\beta}I_1 (d + \alpha)(\delta + \epsilon + d + \alpha) \end{aligned} \quad (12)$$

and

$$m = \begin{vmatrix} 0 & 0 \\ \delta & -(\epsilon + d + \alpha) \end{vmatrix} + \begin{vmatrix} -(d + \bar{\beta}I_1) & \epsilon \\ 0 & -(\epsilon + d + \alpha) \end{vmatrix} + \begin{vmatrix} -(d + \bar{\beta}I_1) & -(\delta + d + \alpha) \\ \bar{\beta}I_1 & 0 \end{vmatrix};$$

hence,

$$m = (d + \bar{\beta}I_1)(\epsilon + d + \alpha) + \bar{\beta}I_1(\delta + d + \alpha) = \bar{\beta}I_1(\delta + \epsilon + d + \alpha) + \bar{\beta}I_1(d + \alpha) + d(\epsilon + d + \alpha). \quad (13)$$

Note that  $\text{tr}(A)$ ,  $-m$  and  $\det(A)$  are negative, thus all coefficients of the characteristic polynomial are negative; therefore, to apply Lemma 2.2 we need to verify that

$$\text{tr}(A)(-m) - \det(A)(-1) > 0, \text{ i.e., } \text{tr}(A)(-m) > -\det(A). \quad (14)$$

So,

$$\begin{aligned} \text{tr}(A)(-m) &= [d + \bar{\beta}I_1 + \epsilon + (d + \alpha)][\bar{\beta}I_1(\delta + \epsilon + d + \alpha) + \bar{\beta}I_1(d + \alpha) + d(\epsilon + d + \alpha)] \quad (15) \\ &= \bar{\beta}I_1(d + \alpha)(\delta + \epsilon + d + \alpha) + (d + \bar{\beta}I_1 + \epsilon)[\bar{\beta}I_1(\delta + \epsilon + d + \alpha)] \\ &\quad + [d + \bar{\beta}I_1 + \epsilon + (d + \alpha)][\bar{\beta}I_1(d + \alpha) + d(\epsilon + d + \alpha)] \\ &= -\det(A) + (d + \bar{\beta}I_1 + \epsilon)[\bar{\beta}I_1(\delta + \epsilon + d + \alpha)] + [d + \bar{\beta}I_1 + \epsilon + (d + \alpha)][\bar{\beta}I_1(d + \alpha) + d(\epsilon + d + \alpha)]. \end{aligned}$$

Thus the inequality (14) is valid, and by Lemma 2.2 the characteristic polynomial of system (9) is Hurwitz. So the periodic element  $(U, V, W)$  is solution of (9) and as any solution of this system converges to the origin by having a characteristic Hurwitz polynomial, then the only option is that  $(U, V, W) = (0, 0, 0)$ , which proves the result.  $\square$

Thus we have established our first result

**Theorem 2.4.** *If  $\mathcal{R}_0 > 1$ , then the system (2) admits a non-trivial periodic solution.*

*Proof.* Using the invariance of the Leray-Schauder degree under homotopy by Lemma 2.1 and Proposition 2.3 we obtain that  $\text{deg}(F_1, G) \neq 0$ , then the system (2) admits a non-trivial periodic solution, which proves Theorem 2.4.  $\square$

### 2.1. Saturated case

We now establish the existence of periodic solutions in the case of saturated system (1), when  $k > 0$ . Assume that  $\mathcal{R}_0 > 1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3}$ ; then, by taking (and for the rest of the work) in Theorem 2.4

$$r := \frac{1}{1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3}},$$

we obtain that  $\text{deg}(F_1, G) \neq 0$ .

For  $\tau \in [0, 1]$  we define the homotopy

$$\begin{aligned} S' &= \Lambda - \beta(t)IS/(1 + \tau B) - dS + \gamma I + \epsilon Q, \\ I' &= \beta(t)IS/(1 + \tau B) - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \quad (16)$$

where  $B := k_1 I^{n_1} + k_2 I^{n_2} + k_3 S^{m_1} I^{n_3}$ . We consider the operator  $M_\tau : \mathcal{C}_T^0 \rightarrow \mathcal{C}_T^0$  given by

$$M_\tau(S, I, Q) := (\Lambda + \gamma I + \epsilon Q - \beta(t)IS/(1 + \tau B), \beta(t)IS/(1 + \tau B), \delta I).$$

Taking  $L$  as (4), we define

$$H_\tau(S, I, Q) := (S, I, Q) - L^{-1} \circ M_\tau(S, I, Q). \quad (17)$$

Thus, (17) is a functional reformulation of problem (1); in particular, periodic solutions of (1) correspond to zeroes of  $H_\tau$ . Note that  $H_0 = F_1$ , therefore  $\deg(H_0, G) \neq 0$ . Recall that the existence of a solution for  $H_1$  in  $G$  is guaranteed via Leray-Schauder degree if  $\deg(H_0, G) \neq 0$  and  $H_\tau$  is an admissible homotopy, i.e.,  $0 \notin H_\tau(\partial G), \forall \tau \in [0, 1]$ . So we need only establish that  $H_\tau$  is an admissible homotopy.

**Lemma 2.5.** *If  $\mathcal{R}_0 > 1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3}$ , then for any  $\tau \in [0, 1]$  there are no solutions  $(S, I, Q)$  of (16) on  $\partial G$ .*

*Proof.* Note that if  $(S, I, Q) \in \partial G$ , then  $(S, I, Q) \notin \partial D$ ; therefore,

$$(S, I, Q) \in D \text{ and } S(t) \geq r(\Lambda/d), \forall t; \quad (18)$$

recall that  $r = \frac{1}{1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3}}$ .

Multiplying by  $(1 + \tau B)/I$  and integrating the second equation in (16) on the interval  $[0, T]$ , we have that

$$(1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3})(\gamma + \delta + d + \alpha)T \geq \int_0^T \beta S dt.$$

Now, from our hypothesis,

$$\gamma + \delta + d + \alpha \geq \frac{(\Lambda/d)\bar{\beta}}{(1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3})} > \bar{\beta}(\Lambda/d) \frac{1}{\mathcal{R}_0} = \gamma + \delta + d + \alpha, \quad (19)$$

which is a contradiction. Hence  $H_\tau$  is an admissible homotopy.  $\square$

By combining our observations we get:

**Theorem 2.6.** *If  $\mathcal{R}_0 > 1 + k_1(\frac{\Lambda}{d})^{n_1} + k_2(\frac{\Lambda}{d})^{n_2} + k_3(\frac{\Lambda}{d})^{m_1+n_3}$ , then there is at least one  $T$ -periodic orbit of (1) whose components are positive.*

## 2.2. Examples

We now provide numerical evidence of the existence of periodic solutions.

### Example 2.7.

$$\begin{aligned} S' &= \Lambda - \beta(t)IS - dS + \gamma I + \epsilon Q, \\ I' &= \beta(t)IS - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \quad (20)$$

where parameters are determined by  $\Lambda = 22$ ,  $d = 0.0001$ ,  $\gamma = 15$ ,  $\delta = 0.095$ ,  $\epsilon = 0.28999$  and  $\alpha = 0.026$ . The incidence function is determined by  $\beta(t) = 0.15 (1 + 0.92 \cos(2\pi t))$ .

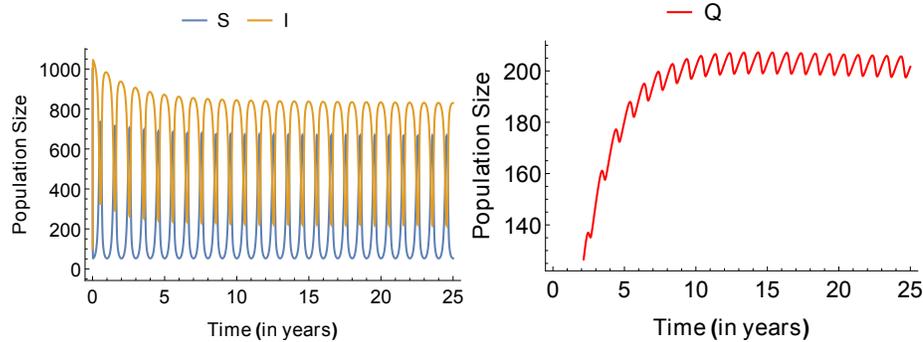


Figure 1. (A) SIQS model. The initial conditions are  $S_0 = 1000$ ,  $I_0 = 100$  and  $Q_0 = 10$ .

**Example 2.8.**

$$\begin{aligned} S' &= \Lambda - \beta(t)IS/(1 + kI) - dS + \gamma I + \epsilon Q, \\ I' &= \beta(t)IS/(1 + kI) - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \tag{21}$$

where parameters are determined by  $\Lambda = 22$ ,  $d = 0.001$ ,  $\gamma = 0.8$ ,  $\delta = 0.2$ ,  $\epsilon = 0.19$ ,  $k = 0.8$  and  $\alpha = 0.029$ . The incidence function is determined by  $\beta(t) = 0.8 (1 + 0.92 \cos(2\pi t))$ .

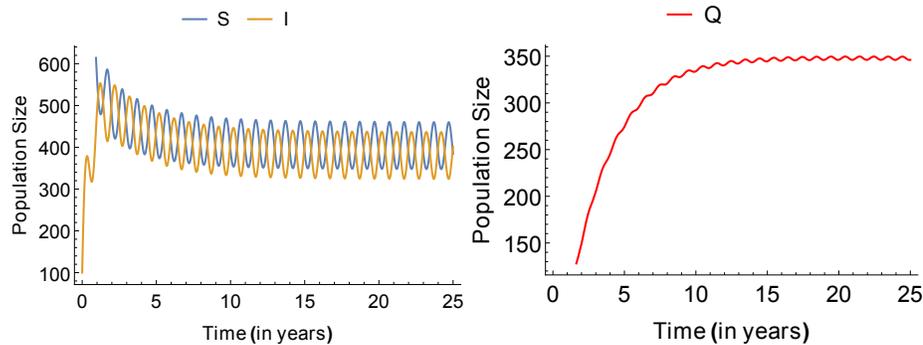


Figure 2. (B) The initial conditions are  $S_0 = 1000$ ,  $I_0 = 100$  and  $Q_0 = 20$ .

**Example 2.9.**

$$\begin{aligned} S' &= \Lambda - \beta(t)IS/(1 + kI^3) - dS + \gamma I + \epsilon Q, \\ I' &= \beta(t)IS/(1 + kI^3) - (\gamma + \delta + d + \alpha)I, \\ Q' &= \delta I - (\epsilon + d + \alpha)Q, \end{aligned} \tag{22}$$

where parameters are determined by  $\Lambda = 1.5$ ,  $d = 0.0001$ ,  $\gamma = 0.2$ ,  $\delta = 0.199$ ,  $\epsilon = 0.19$ ,  $k = 0.0009$  and  $\alpha = 0.005$ . The incidence function is determined by  $\beta(t) = 0.5(1 + 2\cos(2\pi t))$ .

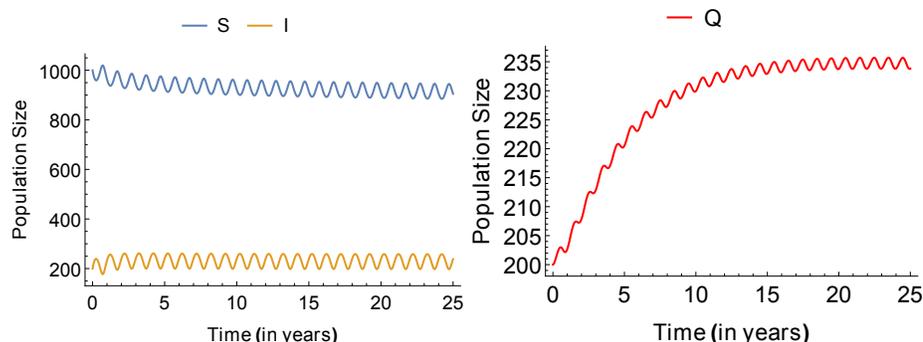


Figure 3. (B) The initial conditions are  $S_0 = 1000$ ,  $I_0 = 200$  and  $Q_0 = 200$ .

### 3. Conclusions

Public health policies have consequences in the behaviour of infectious diseases; in particular, the strategy of quarantine. Feng and Thieme pointed out in [3], [4], that the quarantine process could contribute to sustained oscillations in diseases by combining with other factors such as seasonal variation in the contact rates, stochastic effects, and density dependent demographics.

In our work, we establish analytically the occurrence of sustained oscillations in the some epidemiological model with the quarantine and periodic contact rate. We exhibit numerical evidence of the existence of such periodic solutions by means of numerical simulations of convenient SIQS epidemiological models.

### Acknowledgement

The authors would like to thank the anonymous referees for their helpful comments.

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