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# A proof of Holsztyński theorem

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**Abstract.** For a compact Hausdorff space, we denote by $C(K)$ the Banach space of continuous functions defined in $K$ with values in $\mathbb{R}$ or $\mathbb{C}$. A well known result in Banach spaces of continuous functions is the Holsztyński theorem which establishes that if $C(K)$ is isometric to a subspace of $C(S)$, then $K$ is a continuous image of $S$. The aim of this paper is to give an alternative proof of this result for extremely regular subspaces of $C(K)$.
**Keywords**: $C(K)$ Banach spaces, Banach-Stone theorem.
**MSC2010**: 46B03, 46E15, 46E40, 46B25.

## Una prueba del teorema de Holsztyński

**Resumen.** Dado un espacio compacto Hausdorff, denotaremos por $C(K)$ el espacio de Banach de las funciones continuas definidas en $K$ con valores en $\mathbb{R}$ o $\mathbb{C}$. Un resultado clásico en la teoría de Espacios de Banach de funciones continuas es el teorema de Holsztyński el cual establece que si $C(K)$ es isométrico a un subespacio de $C(S)$, entonces $K$ es imagen continua de un subespacio de $S$. El objetivo de este artículo es dar una prueba alternativa de este resultado para subespacios extremadamente regulares de $C(K)$.
**Palabras clave**: Espacios de Banach $C(K)$, teorema de Banach-Stone.

## 1. Introduction and main theorems

We will use the standard terminology and notation of Banach space theory. For unexplained definitions and notation we refer to [1]-[10]. As usual $\mathbb{K}$ stands for the field $\mathbb{R}$ or $\mathbb{C}$. For a compact Hausdorff space $K$, we denote by $C(K)$ the Banach space of $\mathbb{K}$-valued continuous functions on $K$, provided with the supremum norm.

The classical Banach-Stone theorem states that the Banach space $C(K)$ determines the topology of $K$ [3], [4], [5], [11]. More precisely, if $T\colon C(K) \to C(S)$ is an onto isometry,

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then there are a homeomorphism $h\colon S \to K$ and a continuous function $\sigma\colon S \to \mathbb{K}$ with $|\sigma(s)| = 1$ for all $s \in S$ such that

$$Tf(s) = \sigma(s)f(h(s)) \quad \text{for all } f \in C(K) \text{ and } s \in S. \tag{1}$$

The conclusion of the Banach-Stone theorem is too far to be valid when we consider into isomorphisms between $C(K)$ spaces. Thus it seems natural to ask for topological properties which are preserved under into isomorphisms of $C(K)$ spaces. In this direction, Holsztyński [8] proved:

**Theorem 1.1.** *Let $K$ and $S$ be compact Hausdorff spaces. If $T\colon C(K) \to C(S)$ is an into isometry, then there are a closed subset $\Delta$ of $S$, a continuous surjection $\psi\colon \Delta \to K$ and a continuous function $\sigma\colon \Delta \to \mathbb{K}$ with $|\sigma(s)| = 1$ for all $s \in \Delta$ such that*

$$Tf(s) = \sigma(s)f(\psi(s)) \quad \textit{for all } f \in C(K) \textit{ and all } s \in \Delta.$$

In [2], it is established the following generalization of Theorem 1.1 for extremely regular spaces. According to [6], a closed subspace $A$ of $C(K)$ is called extremely regular if for each $k \in K$ and each neighborhood $U$ of $k$ and each $0 < \varepsilon < 1$, there exists $f \in A$ satisfying $\|f\| = f(k) = 1$ and $|f(w)| < \varepsilon$ for all $w \in K \setminus U$.

**Theorem 1.2.** *Let $K$ and $S$ be compact Hausdorff spaces. Let $A$ be an extremely regular subspace of $C(K)$ and $B$ a closed subspace of $C(S)$. Suppose that $T\colon A \to B$ is an into isometry. Then there exist a closed subset $\Delta$ of $S$, a continuous function $\psi$ from $\Delta$ onto $K$ and a continuous function $\sigma\colon \Delta \to \mathbb{K}$ with $|\sigma(s)| = 1$ for all $s \in \Delta$ such that*

$$Tf(s) = \sigma(s)f(\psi(s)) \quad \textit{for all } s \in \Delta \textit{ and } f \in A.$$

The aim of this note is to give an alternative proof of Theorem 1.2. The paper is divided as follows: in the second section we generalize a result which is proved by Plebanek in the setting of $C(K)$ spaces (see [9, Theorem 3.3]). In third section, we prove Theorem 1.2.

## 2. Preliminaries

Following [7, p. 222], we identify dual space $C(K)^*$ with the space of regular countably additive bounded measures, and we denote it by $M(K)$. We always consider $M(K)$ equipped with the *weak*$^*$ topology inherited from $C(K)^*$. The total variation of a measure $\mu \in M(K)$ on a Borel set $E$ is denoted by $|\mu|(E)$, and its norm by $\|\mu\| = |\mu|(K)$.

Let $K$ and $S$ be compact Hausdorff spaces. Throughout the paper $A$ denotes an extremely regular subspace of $C_0(K)$. Also $B$ will be a closed subspace of $C(S)$. If $s \in S$ is fixed and $T\colon A \to B$ is an embedding, $\nu_s$ will denote any norm-preserving extension to $C(K)$ of the functional $T^*\delta_s\colon A \to \mathbb{R}$ defined as $T^*\delta_s(f) = Tf(s)$ for $f \in A$. Also let us assume that $T$ satisfies $r\|f\| \le \|Tf\| \le \|f\|$ for all $f \in A$, where $r > 0$. Analogously if $E = TA \subset B$ and $k \in K$ is given, let $\mu_k$ be any norm-preserving extension to $C(S)$ of the functional $(T^{-1})^*\delta_k\colon E \to \mathbb{R}$.

Before stating our first result, we need to establish a notation.

Let $k \in K$ be given and $\mathcal{V}_k$ any fundamental system of open neighborhoods of $k$. Consider the set $\mathcal{C}_k = \mathcal{V}_k \times (0, \infty)$. In $\mathcal{C}_k$ we define a partial order as follows: $(U,t) \prec (V,s)$ iff $V \subset U$ and $s < t$. Note that $(\mathcal{C}_k, \prec)$ is a directed set. It is easy to see that there exists a net $(f_{(U,t)})_{(U,t)\in\mathcal{C}_k}$ in $A$ satisfying

1. $\|f_{(U,t)}\| = f_{(U,t)}(k) = 1$;
2. $|f_{(U,t)}(w)| < t$ for all $w \in K \setminus U$.

We will write $\{(U,t), f_{(U,t)}\}_{(U,t)\in\mathcal{C}_k} \leftrightarrow \{k\}$ to indicate that the above conditions are satisfied.

**Lemma 2.1.** *Let $A$ be an extremely regular subspace of $C(K)$ and $k \in K$ given. Suppose that $\{(U,t), f_{(U,t)}\}_{(U,t)\in\mathcal{C}_k} \leftrightarrow \{k\}$. If $\mu \in M(K)$, then*

$$\lim_{(U,t)\in\mathcal{C}_k} \int_K f_{(U,t)}\, d\mu = \mu(\{k\}).$$

*Proof.* The statement is obvious if $\|\mu\| = 0$, so we assume that $\|\mu\| \neq 0$. Let $\varepsilon > 0$ be given. Since $|\mu|$ is regular, there is $W \subset K$ open with $k \in W$ such that $|\mu|(W \setminus \{k\}) < \varepsilon/2$. Let $U_0 \in \mathcal{V}_k$ be such that $U_0 \subset W$. If $(U_0, \varepsilon/2\|\mu\|) \prec (V,t)$, we have

$$\begin{aligned} \left|\int_K f_{(V,t)}\, d\mu - \mu(\{k\})\right| &= \left|\int_{V\setminus\{k\}} f_{(V,t)}\, d\mu + \int_{K\setminus V} f_{(V,t)}\, d\mu\right| \\ &\leq \left|\int_{V\setminus\{k\}} f_{(V,t)}\, d\mu\right| + \left|\int_{K\setminus V} f_{(V,t)}\, d\mu\right| \\ &\leq |\mu|(V \setminus \{k\}) + t|\mu|(K \setminus V) < \varepsilon. \end{aligned}$$

☑

The next two results are proved in [9] for $C(K)$ spaces. However, we noted that they are also valid for extremely regular subspaces of $C(K)$. So, for sake of completeness we include a proof here.

**Lemma 2.2.** *Let $k \in K$ be fixed. If $\mu = \mu_k$, then $\|\nu_s\| \geq r$ $\mu$-almost everywhere.*

*Proof.* Let $N = \{s \in S : \|\delta_s|_E\| < 1\}$. We show that $\mu(N) = 0$. For $0 < h < 1$, define $N_h = \{s \in S : \|\delta_s|_E\| \leq h\}$; then $N_h$ is closed and $N = \bigcup_{h<1} N_h$. It suffices to prove that $|\mu|(N_h) = 0$ for all $h \in (0,1)$. If $\varepsilon > 0$ is given, then there is $f \in A$ with $\|Tf\| \leq 1$ such that $\|\mu\| - \varepsilon < |\mu(Tf)|$. Thus,

$$\begin{aligned} \|\mu\| - \varepsilon &< |\mu(Tf)| \\ &= \left|\int_S Tf\, d\mu\right| \\ &\leq \left|\int_{N_h} Tf\, d\mu\right| + \left|\int_{S\setminus N_h} Tf\, d\mu\right| \\ &\leq h|\mu|(N_h) + |\mu|(S \setminus N_h). \end{aligned}$$

Since $\|\mu\| = |\mu|(N_h) + |\mu|(S \setminus N_h)$, we infer that $|\mu|(N_h) \leq \varepsilon/1-h$. Thus, $|\mu|(N_h) = 0$, by the arbitrariness of $\varepsilon$.

Now let $s \in S \setminus N$; then $\|\delta_s|_E\| \geq 1$. For a positive number $\varepsilon$ there exists $f \in A$ with $\|Tf\| \leq 1$ such that $|Tf(s)| > 1-\varepsilon$. From the fact $\|f\| \leq 1/r$, we infer that $r(1-\varepsilon) < \|\nu_s\|$. So, the result follows when $\varepsilon \to 0$. ☑

If $h$ is a real valued function defined on a topological space $X$, the oscillation of $h$ at $x$ on a set $A$ is

$$\operatorname{osc}_x(h, A) = \inf_U \sup\{|h(x') - h(x'')| : x', x'' \in U \cap A\},$$

where the infimum is taken over all open neighborhoods $U$ of $x$.

**Lemma 2.3.** *Let $k \in K$ and $\varepsilon > 0$ be fixed. Consider the measure $\mu = \mu_k$. Suppose that there is a compact subset $F$ of $S$ such that*

1. $\|\nu_s\| \geq r$ *for all* $s \in F$*;*
2. $\operatorname{osc}_s(\|\nu_s\|, F) \leq \varepsilon$ *for all* $s \in F$*;*
3. $|\mu|(S \setminus F) < \varepsilon$.

*Then, there is $s \in F$ such that $|\nu_s(\{k\})| \geq r - 2\varepsilon$.*

*Proof.* Let $\delta > 0$ be given and let $U \subset K$ be open with $k \in K$. Since $A$ is extremely regular, there exists $f_U \in A$ such that $\|f_U\| = f_U(k) = 1$ and $|f_U(w)| < \delta$ for all $w \in K \setminus U$. We will show that there is $s_U \in F$ satisfying $|Tf_U(s_U)| > r - \varepsilon$. Indeed, if $|Tf_U(s)| < r - \varepsilon$ for all $s \in F$, then

$$\begin{aligned} 1 = f_U(k) &= \mu(Tf_U) \\ &= \int_S Tf_U \, d\mu = \int_F Tf_U \, d\mu + \int_{S\setminus F} Tf_U \, d\mu \\ &< (r-\varepsilon)|\mu|(F) + \varepsilon \\ &\leq \frac{r-\varepsilon}{r} + \varepsilon \leq 1, \end{aligned}$$

which is absurd. Now if $s_U \in F$ satisfies $|Tf_U(s_U)| > r - \varepsilon$, then

$$\begin{aligned} r - \varepsilon &< |Tf_U(s_U)| \\ &= \left| \int_K f_U \, d\nu_{s_U} \right| \\ &\leq \left| \int_U f_U \, d\nu_{s_U} \right| + \left| \int_{K\setminus U} f_U \, d\nu_{s_U} \right| \\ &\leq |\nu_{s_U}|(U) + \delta, \end{aligned}$$

since $\|\nu_{s_U}\| = \|T^*\delta_{s_U}\| \leq 1$. So if $\delta \to 0$, then $r - \varepsilon \leq |\nu_{s_U}|(U)$. Let $\mathcal{V}_k$ be a fundamental system of open neighborhoods of $k$ and consider the net $(s_U)_{U \in \mathcal{V}_k}$ in $F$. Since $F$ is compact, there is a subnet $(s_U)_{U\in\mathcal{W}}$ converging to $s \in F$. By (2), so we may assume that $\|\nu_{s_U}\| \leq \|\nu_s\| + \varepsilon$ for all $U \in \mathcal{W}$.

Now, if $U \subset K$ is open with $k \in U$, then we have $|\nu_s|(U) \geq r - 2\varepsilon$. Indeed, by Urysohn Lemma [7, Proposition 4.32] there exists $g\colon K \to [0,1]$ continuous such that $g = 1$ on an open set $V$ containing $k$ and $g = 0$ outside $U$. Thus, if $W \in \mathcal{W}$ satisfies $W \subset V$, then $|v_{s_W}|(g) \geq |v_{s_W}|(W) \geq r - \varepsilon$. Whence,

$$|\nu_{s_W}|(1-g) \leq |\nu_{s_W}|(K) - (r-\varepsilon) \leq |\nu_s|(K) - r + 2\varepsilon.$$

Since $\nu_{s_W} \to \nu_s$ in the weak* topology, by [9, Lemma 2.1] and the above inequality we have

$$|\nu_s|(1-g) \leq |\nu_s|(K) - r + 2\varepsilon.$$

Therefore, $|\nu_s|(U) \geq |\nu_s|(g) \geq r - 2\varepsilon$. Regularness of $\nu_s$ implies $|\nu_s(\{k\})| \geq r - 2\varepsilon$, and the proof is complete. ☑

The proof of the next result follows as in [9, Theorem 3.3] by using Lemmas 2.2 and 2.3.

**Theorem 2.4.** *Let $K$ and $S$ be compact Hausdorff spaces. Suppose that $T\colon A \to B$ is an embedding. For each $k \in K$ we have*

$$\sup\{|T^*\delta_s(\{k\})| \,:\, s \in S\} \geq \frac{1}{\|T\|\|T^{-1}\|}.$$

## 3. Proof of Theorem 1.2

Since $T$ is an isometry we have $\|T\| = \|T^{-1}\| = 1$. For $k \in K$ we set

$$\Delta_k = \{s \in S \,:\, |T^*\delta_s(\{k\})| = 1\}.$$

By Theorem 2.4 we have $\Delta_k \neq \emptyset$ for each $k \in K$.

**Claim 3.1.** If $k_1, k_2 \in K$ and $k_1 \neq k_2$, then $\Delta_{k_1} \cap \Delta_{k_2} = \emptyset$.

If not, let $s \in S$ be such that $s \in \Delta_{k_1} \cap \Delta_{k_2}$. Then

$$|T^*\delta_s(\{k_1\})| = 1 \quad \text{and} \quad |T^*\delta_s(\{k_2\})| = 1.$$

By taking $a, b \in \mathbb{K}$ with $aT^*\delta_s(\{k_1\}) = 1$ and $bT^*\delta_s(\{k_2\}) = 1$, we infer from definition of variation that

$$\begin{aligned} 1 &\geq \|T^*\delta_s\| \geq |T^*\delta_s|(\{k_1, k_2\}) \\ &\geq |aT^*\delta_s(\{k_1\}) + bT^*\delta_s(\{k_2\})| = 2, \end{aligned}$$

which is absurd. This proves the claim.

**Claim 3.2.** Let $k \in K$ be given. If $s \in \Delta_k$, then there is $a_s \in \mathbb{K}$ with $|a_s| = 1$ such that $Tf(s) = a_s f(k)$ for all $f \in A$.

Indeed, if $s \in \Delta_k$, then $a_s = T^*\delta_s(\{k\}) \in \mathbb{K}$ and $|a_s| = 1$. On the other hand, $T^*\delta_s = a_s\delta_k + \mu$, where $\mu \in M(K)$ satisfies $\mu(\{k\}) = 0$. So, it follows that

$$\begin{aligned} 1 &\geq \|T^*\delta_s\| = |a_s| + \|\mu\| \\ &= 1 + \|\mu\|. \end{aligned}$$

So, $\|\mu\| = 0$, which means that $\mu = 0$. Hence $T^*\delta_s = a_s\delta_k$, that is, $Tf(s) = a_s f(k)$ for all $f \in A$, as claimed.

Set $\Delta = \bigcup_{k\in K} \Delta_k$, and let $\psi\colon \Delta \to K$ and $\sigma\colon \Delta \to \mathbb{K}$ be defined as $\psi(s) = k$ and $\sigma(s) = a_s$, respectively, iff $s \in \Delta_k$, where $a_s$ is determined as in Claim 3.2. Note that $\psi$ is well-defined by Claim 3.1. The surjectivity of $\psi$ is consequence from the fact $\Delta_k \neq \emptyset$ for each $k \in K$. Clearly, $|\sigma(s)| = 1$ for all $s \in S$. Also, by Claim 3.2 we have

$$Tf(s) = \sigma(s)f(\psi(s)) \quad \text{for all } f \in A \text{ and } s \in \Delta. \tag{2}$$

**Claim 3.3.** $\psi\colon \Delta \to K$ and $\sigma\colon \Delta \to \mathbb{K}$ are continuous.

Let $s \in \Delta$ be given and $(s_\alpha)$ a net in $\Delta$ such that $s_\alpha \to s$. Suppose that $\psi(s_\alpha) = k_\alpha \not\to \psi(s) = k$. Thus, there is a compact neighborhood $V \subset K$ of $k$ such that for all $\alpha$, there is $\alpha' \geq \alpha$ with $k_{\alpha'} \notin V$. Since $A$ is extremely regular, there exists $f \in A$ such that $\|f\| = f(k) = 1$ and $|f(w)| < 1/2$ for all $w \in K \setminus V$. Note that $|Tf(s)| = |f(\psi(s))| = |f(k)| = 1$. By continuity of $Tf$, there is $\alpha_0$ such that $|Tf(s_\alpha)| > 1/2$ for all $\alpha \geq \alpha_0$. By taking $\alpha' \geq \alpha_0$ with $k_{\alpha'} \notin V$, we have $1/2 > |f(k_{\alpha'})| = |f(\psi(s_{\alpha'}))| = |Tf(s_{\alpha'})| > 1/2$, which is impossible.

Now we prove continuity of $\sigma$. Let $s \in \Delta$ be given and $\psi(s) = k$. Take $f \in A$ such that $\|f\| = f(k) = 1$. By Equation (2) we have $\sigma(s) = Tf(s)$, and continuity follows immediately.

**Claim 3.4.** $\Delta$ is closed.

Let $(s_\alpha)$ be a net in $\Delta$ and suppose that $s_\alpha \to s$ for some $s \in S$. Write $\psi(s_\alpha) = k_\alpha$ for all $\alpha$. By compactness of $K$, we may assume that $k_\alpha \to k$ for some $k \in K$. By Claim 3.2 we have $|Tf(s_\alpha)| = |f(\psi(s_\alpha))| = |f(k_\alpha)|$ for all $f \in A$. Thus, $|Tf(s)| = |f(k)|$ for all $f \in A$. Let $(f_{(U,t)})_{(U,t)\in\mathcal{C}_k}$ be a net in $A$ such that $\{(U,t), f_{(U,t)}\}_{(U,t)\in\mathcal{C}_k} \leftrightarrow \{k\}$. Then $|Tf_{(U,t)}(s)| = |f_{(U,t)}(k)| = 1$ for all $(U,t) \in \mathcal{C}_k$. Once again by Lemma 2.1, we have

$$\lim_{(U,t)\in\mathcal{C}_k} \int_K f_{(U,t)}\, dT^*\delta_s = T^*\delta_s(\{k\}).$$

So, $|T^*\delta_s(\{k\})| = 1$, that is, $s \in \Delta$.

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