Revista Integración, temas de matemáticas
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 37, N° 1, 2019, pág. 45–76

DOI: http://dx.doi.org/10.18273/revint.v37n1-2019004

An Introduction to the Theory of Local Zeta Functions from Scratch

Edwin León-Cardenal[a], W. A. Zúñiga-Galindo[b*]

[a] Centro de Investigación en Matemáticas, Unidad Zacatecas, Quantum, Ciudad del Conocimiento, Zacatecas, México.
[b] Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Unidad Querétaro, Departamento de Matemáticas, Libramiento Norponiente #2000, Fracc. Real de Juriquilla. Santiago de Querétaro, Qro. 76230, México.

Abstract. This survey article aims to provide an introduction to the theory of local zeta functions in the p-adic framework for beginners. We also give an extensive guide to the current literature on local zeta functions and its connections with other fields in mathematics and physics.
Keywords: Local zeta functions, p-adic analysis, local fields, stationary phase formula.
MSC2010: 11S40, 11S80, 11M41, 14G10.

Una Introducción a la Teoría de las Funciones Zeta Locales para principiantes

Resumen. En este artículo panorámico brindamos una introducción a la teoría de las funciones zeta locales p-ádicas para principiantes. También se presenta una revisión extensiva a la literatura especializada sobre funciones zeta locales y sus conexiones con otros campos de las matemáticas y la física.
Palabras clave: Funciones zeta locales, análisis p-ádico, cuerpos locales, fórmula de la fase estacionaria.

1. Introduction

In these notes we provide an introduction to the theory of local zeta functions from scratch. We assume essentially a basic knowledge of algebra, metric spaces and basic analysis, mainly measure theory. Let $\mathbb{K}$ be a local field, for instance $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, the field of p-adic numbers, or $\mathbb{F}_p((t))$ the field of formal Laurent series with coefficients in a finite

* E-mail: wazuniga@math.cinvestav.edu.mx
Received: 16 November 2018, Accepted: 11 December 2018.
To cite this article: E. León-Cardenal, W.A. Zúñiga-Galindo, An Introduction to the Theory of Local Zeta Functions from Scratch, *Rev. Integr. temas mat.* 37 (2019), No. 1, 45–76. doi: 10.18273/revint.v37n1-2019004.

field with p elements. Let $h(x) \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-constant polynomial and let φ be a test function. The local zeta function attached to the pair (h, φ) is defined as

$$Z_{\varphi}(s,h) = \int\limits_{\mathbb{K}^n \setminus h^{-1}(0)} \varphi(x)\ |h(x)|_{\mathbb{K}}^{s}\ d^n x, \quad \mathrm{Re}(s) > 0,$$

where $|\cdot|_{\mathbb{K}}$ denotes the absolute value of $\mathbb{K}$, $s \in \mathbb{C}$, and $d^n x$ denotes a normalized Haar measure of the topological group $(\mathbb{K}^n, +)$. These integrals give rise to holomorphic functions of s in the half-plane $\mathrm{Re}(s) > 0$. If $\mathbb{K}$ has characteristic zero, then $Z_{\varphi}(s,h)$ admits a meromorphic continuation to the whole complex plane. The p-adic local zeta functions (also called Igusa's local zeta functions) are connected with the number of solutions of polynomial congruences $\mod p^m$ and with exponential sums $\mod p^m$ (see e.g., [14], [28], [31]).

In the Archimedean case, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the study of local zeta functions was initiated by Gel'fand and Shilov [21]. The meromorphic continuation of the local zeta functions was established, independently, by Atiyah [4] and Bernstein [6] (see also [31, Theorem 5.5.1 and Corollary 5.5.1]). The main motivation was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. It is important to mention here, that in the p-adic framework, the existence of fundamental solutions for pseudodifferential operators is also a consequence of the fact that the Igusa local zeta functions admit a meromorphic continuation (see [33, Chapter 10] and [62, Chapter 5]).

On the other hand, in the middle 60s, Weil initiated the study of local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [59]. In the 70s, Igusa developed a uniform theory for local zeta functions over local fields of characteristic zero [28], [30]. More recently, Denef and Loeser introduced in [15] the topological zeta functions, and in [16] they introduce the motivic zeta functions, which constitute a vast generalization of the p-adic local zeta functions.

In the last thirty-five years there has been a strong interest on p-adic models of quantum field theory, which is motivated by the fact that these models are exactly solvable. There is a large list of p-adic type Feynman and string amplitudes that are related with local zeta functions of Igusa-type, and it is interesting to mention that it seems that the mathematical community working on local zeta functions is not aware of this fact (see e.g. [2], [5], [7], [10]-[13], [18]-[20], [22]-[24], [27], [38], [39], [42], [48]-[52], and the references therein).

The connections between Feynman amplitudes and local zeta functions are very old and deep. Let us mention that the works of Speer [50] and Bollini, Giambiagi and González Domínguez [11] on regularization of Feynman amplitudes in quantum field theory are based on the analytic continuation of distributions attached to complex powers of polynomial functions in the sense of Gel'fand and Shilov [21] (see also [5], [7], [10] and [42], among others). This analogy turns out to be very important in the rigorous construction of quantum scalar fields in the p-adic setting (see [43] and the references therein).

The local zeta functions are also deeply connected with p-adic string amplitudes. In [8], the authors proved that the p-adic Koba-Nielsen type string amplitudes are bona fide integrals. They attached to these amplitudes Igusa-type integrals depending on several complex parameters and show that these integrals admit meromorphic continuations as rational functions. Then they used these functions to regularize the Koba-Nielsen amplitudes. In [9], the authors discussed the limit p approaches to one of tree-level p-adic open string amplitudes and its connections with the topological zeta functions. There is empirical evidence that p-adic strings are related to the ordinary strings in the limit $p \to 1$. Denef and Loeser established that the limit $p \to 1$ of a Igusa's local zeta function gives rise to an object called topological zeta function. By using Denef-Loeser's theory of topological zeta functions, it is showed in [9] that limit $p \to 1$ of tree-level p-adic string amplitudes give rise to certain amplitudes, that we have named Denef-Loeser string amplitudes.

Finally, we want to mention about the remarkable connection between local zeta functions and algebraic statistics (see [40], [58]). In [58] is presented an interesting connection with machine learning.

This survey article is based on well-known references, mainly Igusa's book [31]. The work is organized as follows. In Section 2, we introduce the field of p-adic numbers, and we devote Section 3 to the integration theory over $\mathbb{Q}_p$. Section 4 is dedicated to the implicit function theorems on the p-adic field. In Section 5, we introduce the simplest type of local zeta function and show its connection with number of solutions of polynomial congruences $\mod p^m$. In Section 6, we introduce the stationary phase formula and use it to establish the rationality of local zeta functions for several type of polynomials. Finally, in Section 7, we state Hironaka's resolution of singularities theorem, and we use it to prove the rationality of the simplest type of local zeta functions in Section 8.

For an introduction to p-adic analysis the reader may consult [1], [25], [32], [35], [46], [47], [53] and [56]. For an in-depth discussion of the classical aspects of the local zeta functions, we recommend [3], [14], [21], [28], [30], [31], [41]. There are many excellent surveys about local zeta functions and their generalizations. For an introduction to Igusa's zeta function, topological zeta functions and motivic integration we refer the reader to [14], [16], [17], [44], [45] ,[55]. A good introduction to local zeta functions for pre-homogeneous vector spaces can be found in [30], [31] and [34]. Some general references for differential equations over non-Archimedean fields are [1], [33], [56], [62]. Finally, the reader interested in the relations between p-adic analysis and mathematical physics may enjoy [12], [13], [19], [20], [22]-[24], [27], [33],[37]-[39],[43], [48], [49], [52], [54], [56], [57] and [62].

2. p-adic Numbers- Essential Facts

2.1. Basic Facts

In this section we summarize the basic aspects of the field of p-adic numbers, for an in-depth discussion the reader may consult [1], [25], [32], [35], [46], [47], [53] and [56].

Definition 2.1. Let F be a field. An absolute value on F is a real-valued function, $|\cdot|$, satisfying

(i) $|x| = 0 \Leftrightarrow x = 0$;

(ii) $|xy| = |x||y|$;

(iii) $|x + y| \leq |x| + |y|$ (*triangle inequality*), for any $x, y \in F$.

Definition 2.2. An absolute value $|\cdot|$ is called non-Archimedean (or ultrametric), if it satisfies

$$|x + y| \leq \max\{|x|, |y|\}.$$

Example 2.3. *The trivial absolute value is defined as*

$$|x|_{trivial} = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

From now on we will work only with non trivial absolute values.

Definition 2.4. Given two absolute values $|\cdot|_1,\ |\cdot|_2$ defined on F, we say that they are equivalent, if there exists a positive constant c such that

$$|x|_1 = |x|_2^c$$

for any $x \in F$.

Definition 2.5. Let p be a fixed prime number, and let x be a nonzero rational number. Then, $x = p^k \frac{a}{b}$ for some $a, b, k \in \mathbb{Z}$, with $p \nmid ab$. The p-adic absolute value of x is defined as

$$|x|_p = \begin{cases} p^{-k}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Lemma 2.6. *The function $|\cdot|_p$ is a non-Archimedean absolute value on $\mathbb{Q}$.*

The proof is left to the reader. In fact, we kindly invite the reader to prove all the results labeled as Lemmas in these notes.

Theorem 2.7 (Ostrowski, [35])**.** *Any non trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ or to the standard absolute value $|\cdot|_\infty$.*

An absolute value $|\cdot|$ on F allow us to define a distance $d(x, y) := |x - y|,\ x, y \in F$. We now introduce a topology on F by giving a basis of open sets consisting of the *open balls* $B_r(a)$ with center a and radius $r > 0$:

$$B_r(a) = \{x \in F : |x - a| < r\}.$$

A sequence of points $\{x_i\}_{i\in\mathbb{N}} \subset F$ is called *Cauchy* if

$$|x_m - x_n| \to 0, \qquad m, n \to \infty.$$

A field F with a non trivial absolute value $|\cdot|$ is said to be *complete* if any Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ has a limit point $x^* \in F$, i.e. if $|x_n - x^*| \to 0,\ n \to \infty$. This is equivalent to the fact that (F, d), with $d(x, y) = |x - y|$, is *a complete metric space.*

Remark 2.8. Let (X, d), (Y, D) be two metric spaces. A bijection $\rho : X \to Y$ satisfying

$$D(\rho(x), \rho(x')) = d(x, x'),$$

is called an *isometry.*

The following fact is well-known (see e.g. [36]).

Theorem 2.9. *Let (M, d) be a metric space. There exists a complete metric space $(\widetilde{M}, \widetilde{d})$, such that M is isometric to a dense subset of $\widetilde{M}$.*

The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the distance induced by $|\cdot|_p$. Any p-adic number $x \neq 0$ has a unique representation of the form

$$x = p^{\gamma} \sum_{i=0}^{\infty} x_i p^i, \tag{1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_i \in \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The integer γ is called the *p-adic order of x*, and it will be denoted as $ord(x)$. By definition $ord(0) = +\infty$.

Lemma 2.10. *Let $(F, |\cdot|)$ be a valued field, where $|\cdot|$ is a non-Archimedean absolute value. Assume that F is complete with respect to $|\cdot|$. Then, the series $\sum_{k \geq 0} a_k$, $a_k \in F$ converges if, and only if, $\lim_{k\to\infty} |a_k| = 0$.*

Since $|x_i p^{i+r}|_p = p^{-i-r} \to 0$, $i \to \infty$, from Lemma 2.10 we conclude that series (1) converges in $|\cdot|_p$.

Example 2.11.

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$

Indeed, set

$$\begin{aligned} z^{(n)} &:= (p-1) + (p-1)p + \cdots (p-1)p^n \\ &= (p-1)\frac{p^{n+1}-1}{p-1} = p^{n+1} - 1. \end{aligned}$$

Then $\lim_{n\to\infty} z^{(n)} = \lim_{n\to\infty} p^{n+1} - 1 = 0 - 1 = -1$, *since* $|p^{n+1}|_p = p^{-n-1}$.

The unit ball

$$\begin{aligned} \mathbb{Z}_p &= \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \\ &= \{x \in \mathbb{Q}_p : x = \sum_{i=i_0}^{\infty} x_i p^i, i_0 \geq 0\}, \end{aligned}$$

is a ring, more precisely, it is a domain of principal ideals. Any ideal of $\mathbb{Z}_p$ has the form

$$p^m \mathbb{Z}_p = \{x \in \mathbb{Z}_p : x = \sum_{i \geq m} x_i p^i\},\ m \in \mathbb{N}.$$

Indeed, let $I \subseteq \mathbb{Z}_p$ be an ideal. Set $m_0 = \min_{x \in I} ord(x) \in \mathbb{N}$, and let $x_0 \in I$ such that $ord(x_0) = m_0$. Then $I = x_0\mathbb{Z}_p$.

From a geometric point of view, the ideals of the form $p^m\mathbb{Z}_p,\ m \in \mathbb{Z}$, constitute a fundamental system of neighborhoods around the origin in $\mathbb{Q}_p$. The *residue field* of $\mathbb{Q}_p$ is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ (the finite field with p elements).

The *group of units* of $\mathbb{Z}_p$ is

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : |x|_p = 1\}.$$

Lemma 2.12. *$x = x_0 + x_1 p + \cdots \in \mathbb{Z}_p$ is a unit if, and only if, $x_0 \neq 0$. Moreover if $x \in \mathbb{Q}_p \setminus \{0\}$, then $x = p^m u,\ m \in \mathbb{Z},\ u \in \mathbb{Z}_p^\times$.*

2.2. Topology of $\mathbb{Q}_p$

As we already mentioned, $\mathbb{Q}_p$ with $d(x, y) = |x - y|_p$ is a complete metric space. Define

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^r\},\ r \in \mathbb{Z},$$

as the *ball with center a and radius p^r*, and

$$S_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p = p^r\},\ r \in \mathbb{Z}$$

as the *sphere with center a and radius p^r*.

The topology of $\mathbb{Q}_p$ is quite different from the usual topology of $\mathbb{R}$. First of all, since $|\cdot|_p : \mathbb{Q}_p \to \{p^m, m \in \mathbb{Z}\} \cup \{0\}$, the radii are always integer powers of p; for the sake of brevity we just use the power in the notation $B_r(a)$ and $S_r(a)$. On the other hand, since the powers of p and zero form a discrete set in $\mathbb{R}$, in the definition of $B_r(a)$ and $S_r(a)$ we can always use '$\leq$'. Indeed,

$$\{x \in \mathbb{Q}_p : |x - a|_p < p^r\} = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{r-1}\} = B_{r-1}(a) \subset B_r(a).$$

Remark 2.13.

$$\begin{aligned} B_r(a) &= a + p^r\mathbb{Z}_p, \\ S_r(a) &= a + p^r\mathbb{Z}_p^\times. \end{aligned}$$

We declare $B_r(a),\ r \in \mathbb{Z},\ a \in \mathbb{Q}_p$, are open subsets; in addition, these sets form a basis for the topology of $\mathbb{Q}_p$.

Proposition 2.14. *$S_r(a),\ B_r(a)$ are open and closed sets in the topology of $\mathbb{Q}_p$.*

Proof. We first show that $S_r(a)$ is open. Note that $\mathbb{Z}_p^\times = \sqcup_{i \in \{1,\dots,p-1\}} i + p\mathbb{Z}_p$, then

$$S_r(a) = \bigsqcup_{i \in \{1,\dots,p-1\}} a + p^r i + p^{r+1}\mathbb{Z}_p = \bigsqcup_i B_{r+1}(a + p^r i)$$

is an open set.

In order to show that $S_r(a)$ is closed, we take a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of $S_r(a)$ converging to $\widetilde{x_0} \in \mathbb{Q}_p$. We must show that $\widetilde{x_0} \in S_r(a)$. Note that $x_n = a + p^r u_n,\ u_n \in \mathbb{Z}_p^\times$. Since $\{x_n\}$ is a Cauchy sequence, we have

$$|x_n - x_m|_p = p^{-r}|u_n - u_m|_p \to 0, \qquad n, m \to \infty,$$

thus $\{u_n\}_{n\in\mathbb{N}}$ is also Cauchy, and since $\mathbb{Q}_p$ is complete, $u_n \to \widetilde{u_0}$. Then $x_n \to a + p^r\widetilde{u_0}$, so in order to conclude our proof we must verify that $\widetilde{u_0} \in \mathbb{Z}_p^\times$. Because u_m is arbitrarily close to $\widetilde{u_0}$, their p-adic expansions must agree up to a big power of p, hence $\widetilde{u_0} \in \mathbb{Z}_p^\times$.

A similar argument shows that $B_r(a)$ is closed. ☑

Proposition 2.15. *If $b \in B_r(a)$ then $B_r(b) = B_r(a)$, i.e., any point of the ball $B_r(a)$ is its center.*

Proof. Let $x \in B_r(b)$; then,

$$|x - a|_p = |x - b + b - a|_p \leq \max\{|x - b|_p, |b - a|_p\} \leq p^r,$$

i.e., $B_r(b) \subseteq B_r(a)$. Since $a \in B_r(b)$ (i.e. $|b - a|_p = |a - b|_p \leq p^r$), we can repeat the previous argument to show that $B_r(a) \subseteq B_r(b)$. ☑

Lemma 2.16. *The following assertions hold:*

(i) any two balls in $\mathbb{Q}_p$ are either disjoint or one is contained in another;

(ii) the boundary of any ball is the empty set.

Theorem 2.17 ([1, Sec. 1.8])**.** *A set $K \subset \mathbb{Q}_p$ is compact if, and only if, it is closed and bounded in $\mathbb{Q}_p$.*

2.3. The $n-$dimensional p-adic space

We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1\leq i\leq d} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1\leq i\leq n}\{ord(x_i)\}$; then, $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a separable complete ultrametric space (here, separable means that $\mathbb{Q}_p^n$ contains a countable dense subset, which is $\mathbb{Q}^n$).

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n \ : \ ||x - a||_p \leq p^r\}$ *the ball of radius p^r with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x_i \in \mathbb{Q}_p \ : \ |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n \ : \ ||x - a||_p = p^r\}$ *the sphere of radius p^r with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^\times\right)^n \subsetneq S_0^n$, for $n \geq 2$.

As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e., the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. Two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other. As in the one dimensional case, a subset of $\mathbb{Q}_p^n$ is compact if, and only if, it is closed and bounded in $\mathbb{Q}_p^n$. Since the balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$, one has that $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

3. Integration on $\mathbb{Q}_p$

For this section we assume a basic knowledge of measure theory (see e.g. [26]).

Theorem 3.1 ([26, Thm B. Sec. 58]). *Let $(G,\cdot)$ be a locally compact topological group. There exists a Borel measure dx, unique up to multiplication by a positive constant, such that $\int_U dx > 0$ for every non empty Borel open set U, and $\int_{x\cdot E} dx = \int_E dx$, for every Borel set E.*

The measure dx is called a *Haar measure of G*. Since $(\mathbb{Q}_p,+)$ is a locally compact topological group, by Theorem 3.1 there exists a measure dx, which is invariant under translations, i.e., $d(x+a)=dx$. If we normalize this measure by the condition

$$\int_{\mathbb{Z}_p} dx = 1,$$

then dx is unique.

For the n-dimensional case we use that $(\mathbb{Q}_p^n,+)$ is a locally compact topological group. We denote by $d^n x$ the product measure $dx_1\cdots dx_n$ such that

$$\int_{\mathbb{Z}_p^n} d^n x = 1.$$

This measure also satisfies that $d^n(x+a)=d^n x$, for $a\in\mathbb{Q}_p^n$. The open compact balls of $\mathbb{Q}_p^n$, e.g. $a+p^m\mathbb{Z}_p^n$, generate the Borel σ-algebra of $\mathbb{Q}_p^n$. The measure $d^n x$ assigns to each open compact subset U a nonnegative real number $\int_U d^n x$, which satisfies

$$\int_{\cup_{k=1}^{\infty} U_k} d^n x = \sum_{k=1}^{\infty} \int_{U_k} d^n x \tag{2}$$

for all compact open subsets U_k in $\mathbb{Q}_p^n$, which are pairwise disjoint, and verify $\cup_{k=1}^{\infty} U_k$ is still compact. In addition,

$$\int_{a+U} d^n x = \int_U d^n x.$$

3.1. Integration of locally constant functions

A function $\varphi:\mathbb{Q}_p^n\to\mathbb{C}$ is said to be *locally constant* if for every $x\in\mathbb{Q}_p^n$ there exists an open compact subset U, containing x, and such that $f(x)=f(u)$ for all $u\in U$.

Lemma 3.2. *Every locally constant function is continuous.*

Remark 3.3. Set $I=\mathbb{Q}_p/\mathbb{Z}_p$; then I is countable. We fix a set of representatives for the elements of I of the form

$$x_{-m}p^{-m}+\cdots+x_{-1}p^{-1}.$$

If V is an open subset of $\mathbb{Q}_p$, then for any $x\in V$ there exists a ball contained in V of the form

$$p^m(j+\mathbb{Z}_p^n),$$

for some $j\in I^n$ and $m\in\mathbb{Z}$, containing x. Consequently, $\mathbb{Q}_p^n$ is a second-countable space.

Any locally constant function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form

$$\varphi(x) = \sum_{n=1}^{+\infty} c_k 1_{U_k}(x), \tag{3}$$

where $c_k \in \mathbb{C}$,

$$1_{U_k}(x) = \begin{cases} 1, & \text{if} \quad x \in U_k, \\ \\ 0, & \text{if} \quad x \notin U_k, \end{cases}$$

and $U_k \subseteq \mathbb{Q}_p^n$ is an open compact for every k. In the proof of this fact one may use Remark 3.3.

Let $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ be a locally constant function as in (3). Assume that $A = \bigsqcup_{i=1}^k U_i$, with U_i open compact. Then we define

$$\int_A \varphi(x)\, d^n x = c_1 \int_{U_1} d^n x + \cdots + c_n \int_{U_k} d^n x. \tag{4}$$

We recall that, given a function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$, the *support of* φ is the set

$$Supp(\varphi) = \overline{\{x \in \mathbb{Q}_p^n \ : \ \varphi(x) \neq 0\}}.$$

A locally constant function with compact support is called a *p-adic test function* or a *Bruhat-Schwartz function*. These functions form a $\mathbb{C}$-vector space denoted as $\mathcal{D}$. From (2) and (4) one has that the mapping

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathbb{C} \\ \varphi & \longmapsto & \int_{\mathbb{Q}_p^n} \varphi \ d^n x, \end{array} \tag{5}$$

is a well-defined linear functional.

3.2. Integration of continuous functions with compact support

We now extend the integration to a larger class of functions. Let U be a open compact subset of $\mathbb{Q}_p^n$. We denote by $C(U, \mathbb{C})$ the space of all the complex-valued continuous functions supported on U, endowed with the supremum norm. We denote by $C_0(\mathbb{Q}_p^n, \mathbb{C})$ the space of all the complex-valued continuous functions vanishing at infinity, endowed also with the supremum norm. The function φ vanishes at infinity, if given $\varepsilon > 0$, there exists a compact subset K such that $|\varphi(x)| < \varepsilon$, if $x \notin K$.

It is known that $\mathcal{D}$ is dense in $C_0(\mathbb{Q}_p^n, \mathbb{C})$ (see, e.g., [53, Prop. 1.3]). We identify $C(U, \mathbb{C})$ with a subspace of $C_0(\mathbb{Q}_p^n, \mathbb{C})$, therefore $\mathcal{D}$ is dense in $C(U, \mathbb{C})$.

We fix an open compact subset U and consider the functional (5), since

$$\left| \int_{\mathbb{Q}_p^n} \varphi \ d^n x \right| \leq \sup_{x \in U} |\varphi(x)| \int_U d^n x,$$

then functional (5) has a unique extension to $C(U, \mathbb{C})$.

This means that if $f \in C(U,\mathbb{C})$ and $\{f_m\}_{m\in\mathbb{N}}$ is any sequence in $\mathcal{D}$ approaching f in the supremum norm, then
$$\int_{\mathbb{Q}_p^n} f\ d^n x = \lim_{m\to\infty} \int_{\mathbb{Q}_p^n} f_m\ d^n x.$$

3.3. Improper Integrals

Our next task is the integration of functions that do not have compact support. A function $f : \mathbb{Q}_p \to \mathbb{C}$ is said to be *locally integrable*, $f \in L^1_{loc}$, if
$$\int\limits_K f(x)\,dx$$
exists for every compact K.

Example 3.4. *The function $|x|_p$ is locally integrable but not integrable.*

Definition 3.5 (Improper Integral)**.** A function $f \in L^1_{loc}$ is said to be integrable in $\mathbb{Q}_p^n$ if
$$\lim_{l\to+\infty} \int_{B_l(0)} f(x)\,d^n x = \lim_{l\to+\infty} \sum_{j=-\infty}^{l} \int_{S_j(0)} f(x)\,d^n x$$
exists. If the limit exists, it is denoted as $\int_{\mathbb{Q}_p^n} f(x)\,d^n x$, and we say that the *improper integral exists.*

Note that
$$\int_{\mathbb{Q}_p^n} f(x)\,d^n x = \sum_{j=-\infty}^{+\infty} \int_{S_j(0)} f(x)\,d^n x.$$

3.4. The change of variables formula in dimension one

Let us start with the formula
$$d(ax) = |a|_p\,dx,\ a \in \mathbb{Q}_p^\times, \tag{6}$$
which means the following:
$$\int\limits_{aU} dx = |a|_p \int\limits_U dx,$$
for every Borel set $U \subseteq \mathbb{Q}_p$, for instance an open compact subset. Consider
$$\begin{array}{cccc} T_a: & \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p \\ & x & \longmapsto & ax, \end{array}$$
with $a \in \mathbb{Q}_p^\times$. T_a is a topological and algebraic isomorphism. Then $aU \mapsto \int_{aU} dx$ is a Haar measure for $(\mathbb{Q}_p,+)$, and by the uniqueness of such measure, there exists a positive

constant $C(a)$ such that $\int_{aU} dx = C(a) \int_U dx$. To compute $C(a)$ we can pick any open compact set, for instance $U = \mathbb{Z}_p$, and then we must show

$$\int_{a\mathbb{Z}_p} dx = C(a) = |a|_p \,.$$

Let us consider first the case $a \in \mathbb{Z}_p$, i.e., $a = p^l u$, $l \in \mathbb{N}$, $u \in \mathbb{Z}_p^\times$. Fix a system of representatives $\{b\}$ of $\mathbb{Z}_p/p^l\mathbb{Z}_p$ in $\mathbb{Z}_p$; then,

$$\mathbb{Z}_p = \bigsqcup_{b \in \mathbb{Z}_p/p^l\mathbb{Z}_p} b + p^l\mathbb{Z}_p,$$

and

$$\begin{aligned} 1 = \int_{\mathbb{Z}_p} dx &= \sum_{b \in \mathbb{Z}_p/p^l\mathbb{Z}_p} \int_{b+p^l\mathbb{Z}_p} dx = \sum_{b \in \mathbb{Z}_p/p^l\mathbb{Z}_p} \int_{p^l\mathbb{Z}_p} dx \\ &= \#\left(\mathbb{Z}_p/p^l\mathbb{Z}_p\right) \int_{p^l\mathbb{Z}_p} dx, \end{aligned}$$

i.e.,

$$p^{-l} = |a|_p = \int_{p^l\mathbb{Z}_p} dx = \int_{a\mathbb{Z}_p} dx.$$

The case $a \notin \mathbb{Z}_p$ is treated in a similar way.

Now, if we take $f : U \to \mathbb{C}$, where U is a Borel set, then

$$\int_U f(x)\, dx = |a|_p \int_{a^{-1}U+b} f(ay+b)\ dy, \text{ for any } a \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p.$$

The formula follows by changing variables as $x = ay + b$. Then we get $dx = d(ay+b) = d(ay) = |a|_p\, dy$, because the Haar measure is invariant under translations and formula (6).

Example 3.6. *Take $U = \mathbb{Z}_p \smallsetminus \{0\}$. We show that*

$$\int_U dx = \int_{\mathbb{Z}_p} dx = 1.$$

Notice that U is not compact, since the sequence $\{p^n\}_{n\in\mathbb{N}} \subseteq U$ converges to $0 \notin U$. Now, by using

$$\mathbb{Z}_p \smallsetminus \{0\} = \bigsqcup_{j=0}^{\infty} \left\{x \in \mathbb{Z}_p : |x|_p = p^{-j}\right\},$$

we have

$$\int_{\mathbb{Z}_p\smallsetminus\{0\}} dx=\sum_{j=0}^{\infty}\int_{p^j\mathbb{Z}_p^{\times}} dx=\sum_{j=0}^{\infty}p^{-j}\int_{\mathbb{Z}_p^{\times}} dy,\quad (x=p^jy)$$

$$=\left(\frac{1}{1-p^{-1}}\right)\int_{\mathbb{Z}_p^{\times}} dy=\left(\frac{1}{1-p^{-1}}\right)\left\{\int_{\mathbb{Z}_p} dy-\int_{p\mathbb{Z}_p} dy\right\}=\frac{1-p^{-1}}{1-p^{-1}}=1.$$

This calculation shows that $\mathbb{Z}_p\smallsetminus\{0\}$ *has Haar measure* 1 *and that* $\{0\}$ *has Haar measure* 0.

Example 3.7. *For any* $r\in\mathbb{Z}$,

$$\int_{B_r(0)} dx=\int_{p^{-r}\mathbb{Z}_p} dx=p^r\int_{\mathbb{Z}_p} dy=p^r.$$

Example 3.8. *For any* $r\in\mathbb{Z}$,

$$\int_{S_r(0)} dx=\int_{B_r(0)} dx-\int_{B_{r-1}(0)} dx=p^r-p^{r-1}=p^r\left(1-p^{-1}\right).$$

Example 3.9. *Set*

$$Z(s):=\int_{\mathbb{Z}_p}|x|_p^s\ dx,\ s\in\mathbb{C}\ \text{with}\ \operatorname{Re}(s)>-1.$$

We prove that $Z(s)$ *has a meromorphic continuation to the whole complex plane as a rational function of* p^{-s}.

Indeed,

$$Z(s)=\int_{\mathbb{Z}_p\smallsetminus\{0\}}|x|_p^s\ dx=\sum_{j=0}^{\infty}\int_{|x|_p=p^{-j}}|x|_p^s\ dx=\sum_{j=0}^{\infty}p^{-js}\int_{|x|_p=p^{-j}} dx$$

$$=\left(1-p^{-1}\right)\sum_{j=0}^{\infty}p^{-j(s+1)}\ \textit{(here we need the hypothesis } \operatorname{Re}(s)>-1\textit{)}$$

$$=\frac{\left(1-p^{-1}\right)}{1-p^{-1-s}},\quad \textit{for}\ \operatorname{Re}(s)>-1.$$

We now note that the right hand-side is defined for any complex number $s\neq-1$, *therefore, it gives a meromorphic continuation of* $Z(s)$ *to the half-plane* $\operatorname{Re}(s)<-1$. *Thus we have shown that* $Z(s)$ *has a meromorphic continuation to the whole* $\mathbb{C}$ *with a simple pole at* $\operatorname{Re}(s)=-1$.

Example 3.10. *Let* $f:\mathbb{Q}_p\to\mathbb{C}$ *be a radial function, i.e.,* $f(x)=f(|x|_p)$. *If* $\sum_{j=-\infty}^{+\infty}f\left(p^j\right)p^j<+\infty$, *then*

$$\int_{\mathbb{Q}_p}f\left(|x|_p\right)dx=\sum_{j=-\infty}^{+\infty}\int_{|x|_p=p^j}f\left(|x|_p\right)\ dx=\left(1-p^{-1}\right)\sum_{j=-\infty}^{+\infty}f\left(p^j\right)p^j.$$

Example 3.11. *By using* $\sum_{r=0}^{+\infty} rp^{-r} = \frac{p}{(p-1)^2}$*, one may show that*

$$\int_{\mathbb{Z}_p} \ln\left(|x|_p\right)\ dx = -\frac{\ln p}{p-1}.$$

Example 3.12. *We compute*

$$Z(s, x^2-1) = \int_{\mathbb{Z}_p} \left|x^2-1\right|_p^s\ dx, \quad \textit{for} \quad \operatorname{Re}(s) > -1,\, p \neq 2.$$

Let us take $\{0, 1, \ldots, p-1\} \subset \mathbb{Z} \subset \mathbb{Z}_p$ *as a system of representatives of* $\mathbb{F}_p \simeq \mathbb{Z}_p/p\mathbb{Z}_p$*. Then,*

$$\mathbb{Z}_p = \bigsqcup_{j=0}^{p-1} (j + p\mathbb{Z}_p),$$

and

$$\begin{aligned} Z(s, x^2-1) &= \sum_{j=0}^{p-1} \int_{j+p\mathbb{Z}_p} |(x-1)(x+1)|_p^s\, dx \\ &= p^{-1} \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} |(j-1+py)(j+1+py)|_p^s\, dy, \quad (x = j+py). \end{aligned}$$

Let us consider first the integrals in which $j \mp 1 + py \in \mathbb{Z}_p^\times$*, i.e., the reduction mod* p *of* $j \mp 1$ *is a nonzero element of* $\mathbb{F}_p$*; in this case,*

$$\int_{\mathbb{Z}_p} |(j-1+py)(j+1+py)|_p^s\, dy = 1,$$

and since $p \neq 2$*, there are exactly* $p-2$ *of those* j*'s; then,*

$$\begin{aligned} Z(s, x^2-1) &= (p-2)\, p^{-1} + p^{-1} \int_{\mathbb{Z}_p} |py\,(2+py)|_p^s\, dy + p^{-1} \int_{\mathbb{Z}_p} |(-2+py)\, py|_p^s\, dy \\ &= (p-2)\, p^{-1} + 2p^{-1-s} \int_{\mathbb{Z}_p} |y|_p^s\, dy = (p-2)\, p^{-1} + 2p^{-1-s} \frac{1-p^{-1}}{1-p^{-1-s}}. \end{aligned}$$

Lemma 3.13. *Take* $q(x) = \prod_{i=1}^{r} (x-\alpha_i)^{e_i} \in \mathbb{Z}_p[x]$*,* $\alpha_i \in \mathbb{Z}_p$*,* $e_i \in \mathbb{N} \smallsetminus \{0\}$*. Assume that* $\alpha_i \not\equiv \alpha_j \mod p$*. Then by using the methods presented in examples 3.9 and 3.12, one can compute the integral*

$$Z(s, q(x)) = \int_{\mathbb{Z}_p} |q(x)|_p^s\ dx.$$

3.5. Change of variables (general case)

A function $h : U \to \mathbb{Q}_p$ is said to be *analytic on an open subset* $U \subseteq \mathbb{Q}_p$, if there exists a convergent power series $\sum_i a_i x^i$ for $x \in \widetilde{U} \subset U$, with $\widetilde{U}$ open, such that $h(x) = \sum_i a_i x^i$ for $x \in \widetilde{U}$. In this case, $h'(x) = \sum_i i a_i x^{i-1}$ is a convergent power series. A function f is said to be *bi-analytic* if f and f^{-1} are analytic.

Let $K_0, K_1 \subset \mathbb{Q}_p$ be open compact subsets. Let $\sigma : K_1 \to K_0$ be a bi-analytic function such that $\sigma'(y) \neq 0$, $y \in K_1$. Then, if f is a continuous function over K_0,

$$\int\limits_{K_0} f(x)\ dx = \int\limits_{K_1} f(\sigma(y)) \left|\sigma'(y)\right|_p\ dy, \quad (x = \sigma(y)).$$

4. Implicit Function Theorems on $\mathbb{Q}_p$

Let us denote by $\mathbb{Q}_p[[x_1, \ldots, x_n]]$, the *ring of formal power series* with coefficients in $\mathbb{Q}_p$. An element of this ring has the form

$$\sum c_i x^i = \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

A formal series $\sum c_i x^i$ is said to be *convergent* if there exists $r \in \mathbb{Z}$ such that $\sum c_i a^i$ converges for any $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$ satisfying $\|a\|_p = \max_i |a_i|_p < p^r$. The convergent series form a subring of $\mathbb{Q}_p[[x_1, \ldots, x_n]]$, which will be denoted as $\mathbb{Q}_p\langle\langle x_1, \ldots, x_n\rangle\rangle$.

If for $\sum c_i x^i$ there exists $\sum c_i^{(0)} x^i \in \mathbb{R}\langle\langle x_1, \ldots, x_n\rangle\rangle$ such that $|c_i|_p \leq c_i^{(0)}$ for all $i \in \mathbb{N}^n$, we say that $\sum c_i^{(0)} x^i$ is a *dominant series* for $\sum c_i x^i$ and write

$$\sum c_i x^i << \sum c_i^{(0)} x^i.$$

Proposition 4.1. *A formal power series is convergent if, and only if, it has a dominant series.*

Proof. Set $|i| := i_1 + \ldots + i_n$, for $(i_1, \ldots, i_n) \in \mathbb{N}^n$. Assume that $\sum c_i x^i << \sum c_i^{(0)} x^i$, then

$$\lim_{|i|\to\infty} |c_i|_p \leq \lim_{|i|\to\infty} c_i^{(0)} = 0,$$

and thus $\sum c_i x^i$ is convergent by Lemma 2.10.

If $\sum c_i x^i \in \mathbb{Q}_p\langle\langle x_1, \ldots, x_n\rangle\rangle$, then there exists $r \in \mathbb{Z}$ such that $\sum c_i a^i$ converges for any $\|a\|_p < p^r$. Choose $r_0 \in \mathbb{Z}$ such that $0 < p^{r_0} < p^r$. Then for every $a \in \mathbb{Q}_p^n$ satisfying $\|a\|_p < p^{r_0}$, we have

$$\left|c_i a^i\right|_p \leq |c_i|_p\, p^{|i| r_0} < |c_i|_p\, p^{|i| r},$$

and thus $\lim_{|i|\to\infty} |c_i|_p\, p^{|i| r_0} = 0$. Hence, $|c_i|_p\, p^{|i| r_0} \leq M$, for some positive constant M. Finally,

$$\sum c_i x^i << \sum \left(\frac{M}{p^{|i| r_0}}\right) x^i.$$

☑

We say that $f(x) = \sum c_i x^i \in \mathbb{Q}_p[[x_1, \ldots, x_n]]$ is *a special restricted power series*, abbreviated SRP, if $f(0) = 0$, i.e., $c_0 = 0$, and $c_i \equiv 0 \mod p^{|i|-1}$, for any $i \in \mathbb{N}^n$, $i \neq 0$.

Lemma 4.2. *Assume that $f(x)$ is a SRP; then the following assertions hold: (i) $f(x) \in \mathbb{Z}_p[[x_1, \ldots, x_n]]$; (ii) $f(x)$ is convergent at every a in $\mathbb{Z}_p^n$; (iii) $f(a) \in \mathbb{Z}_p$.*

Theorem 4.3 (First Version of the Implicit Function Theorem). *(i) Take $F(x,y) = (F_1(x,y), \ldots, F_m(x,y))$, with $F_i(x,y) \in \mathbb{Q}_p[[x,y]] := \mathbb{Q}_p[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$ such that $F_i(0,0) = 0$, and*

$$\det\left[\frac{\partial F_i}{\partial y_j}(0,0)\right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \neq 0.$$

Then there exists a unique $f(x) = (f_1(x), \ldots, f_m(x))$, with $f_i(x) \in \mathbb{Q}_p[[x_1, \ldots, x_n]]$, $f_i(0) = 0$, satisfying $F(x, f(x)) = 0$, i.e., $F_i(x, f(x)) = 0$ for all i.

(ii) If each $F_i(x,y)$ is a convergent power series, then every $f_i(x)$ is a convergent power series. Furthermore, if a is near 0 in $\mathbb{Q}_p^n$, then $f(a)$ is near 0 in $\mathbb{Q}_p^m$ and $F(a, f(a)) = 0$; and if (a,b) is near $(0,0)$ in $\mathbb{Q}_p^n \times \mathbb{Q}_p^m$ and $F(a,b) = 0$, then $b = f(a)$.

For a proof of this result the reader may consult [31, Thm. 2.1.1].

Corollary 4.4 ([31, Cor. 2.1.1]). *(i) If $g_i(x) \in \mathbb{Q}_p[[x_1, \ldots, x_n]]$, $g_i(0) = 0$ for $1 \leq i \leq n$, and*

$$\det\left[\frac{\partial g_i}{\partial x_j}(0)\right] \neq 0,$$

then there exists a unique $f(x) = (f_1(x), \ldots, f_n(x))$ with $f_i(x) \in \mathbb{Q}_p[[x_1, \ldots, x_n]]$, $f_i(0) = 0$, for all i, such that $g(f(x)) = x$.

(ii) If $g_i(x) \in \mathbb{Q}_p\langle\langle x_1, \ldots, x_n \rangle\rangle$, then $f_i(x) \in \mathbb{Q}_p\langle\langle x_1, \ldots, x_n \rangle\rangle$ for all i. Furthermore, if b is near 0 in $\mathbb{Q}_p^n$ and $a = g(b)$, then a is also near 0 in $\mathbb{Q}_p^n$ and $b = f(a)$. Therefore, $y = f(x)$ gives rise to a bi-continuous map from a small neighborhood of 0 in $\mathbb{Q}_p^n$ to another neighborhood of 0 in $\mathbb{Q}_p^n$.

Remark 4.5. (i) Take $U_1 \subset \mathbb{Q}_p^n$, $U_2 \subset \mathbb{Q}_p^m$, open subsets containing the origin. Assume that each $F_i(x,y) : U_1 \times U_2 \to \mathbb{Q}_p$ is a convergent power series. A set of the form

$$V := \{(x,y) \in U_1 \times U_2 : F_i(x,y) = 0,\, i = 1, \ldots, m\}$$

is called an *analytic set*. In the case in which all the $F_i(x,y)$ are polynomials and $U_1 = \mathbb{Q}_p^n$, $U_2 = \mathbb{Q}_p^m$, V is called an *algebraic set*. If all the $F_i(x,y) \in \mathbb{Q}_p\langle\langle x, y \rangle\rangle$ satisfy the hypotheses of the implicit function theorem, V has a parametrization, possible after shrinking U_1, U_2, i.e. there exist open subsets containing the origin $\widetilde{U}_1 \subset U_1$, $\widetilde{U}_2 \subset U_2$, such that

$$\begin{aligned} V &= \left\{(x,y) \in \widetilde{U}_1 \times \widetilde{U}_2 : F_i(x,y) = 0,\, i = 1, \ldots, m\right\} \\ &= \left\{(x,y) \in \widetilde{U}_1 \times \widetilde{U}_2 : y = f(x)\right\}. \end{aligned}$$

(ii) If we now use as coordinates

$$x_1, \ldots, x_n, z_1 = y_1 - f_1(x), \ldots, z_m = y_m - f_m(x),$$

we have
$$V := \left\{(x,z) \in \widetilde{U}_1 \times \widetilde{U}_2 : z_1 = \cdots = z_m = 0\right\}.$$

We say that such V is *a closed analytic submanifold of* $\widetilde{U}_1 \times \widetilde{U}_2 \subset \mathbb{Q}_p^n \times \mathbb{Q}_p^m$ *of codimension* m. The word 'closed' means that V is closed in the p-adic topology.

In the next version of the implicit function theorem we can control the radii of the balls involved in the theorem.

Theorem 4.6 (Second Version of the Implicit Function Theorem). *(i) If* $F_i\,(x,y) \in \mathbb{Z}_p\,[[x,y]] := \mathbb{Z}_p\,[[x_1,\ldots,x_n,y_1,\ldots,y_m]]$, $F_i\,(0,0) = 0$ *for all* i *and*

$$\det\left[\frac{\partial F_i}{\partial y_j}(0,0)\right]_{\substack{1\leq i\leq m\\ 1\leq j\leq m}} \not\equiv 0 \mod p,$$

then there exists a unique solution $f\,(x) = (f_1\,(x),\ldots,f_m\,(x))$, *with* $f_i\,(x) \in \mathbb{Z}_p\,[[x_1,\ldots,x_n]]$, $f_i\,(0) = 0$, *of* $F\,(x,f\,(x)) = 0$, *i.e.* $F_i\,(x,f\,(x)) = 0$ *for all* i.

(ii) If every $F_i\,(x,y)$ *is an SRP in* $x_1,\ldots,x_n,y_1,\ldots,y_m$, *then every* $f_i\,(x)$ *is an SRP in* $x_1,\ldots,x_n$. *Furthermore, if* $a \in \mathbb{Z}_p^n$, *then* $f(a) \in \mathbb{Z}_p$ *and* $F\,(a,f(a)) = 0$, *and if* $(a,b) \in \mathbb{Z}_p^n \times \mathbb{Z}_p^m$ *satisfies* $F(a,b) = 0$, *then* $b = f(a)$.

For a proof of this result the reader may consult [31, Thm. 2.2.1].

Corollary 4.7 ([31, Cor. 2.2.1]). *(i) If* $g_i(x) \in \mathbb{Z}_p\,[[x_1,\ldots,x_n]]$, $g_i(0) = 0$ *for all* i, *and further*
$$\det\left[\frac{\partial g_i}{\partial x_j}(0,0)\right]_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \not\equiv 0 \mod p,$$
then every $f_j(x)$ *in the unique solution of* $g_i\,(f_1(x),\ldots,f_n(x)) = x$ *satisfying* $f_j(0) = 0$ *is also in* $\mathbb{Z}_p\,[[x_1,\ldots,x_n]]$.

(ii) If every $g_i(x)$ *is a SRP in* $x_1,\ldots,x_n$, *then every* $f_j(x)$ *is also a SRP in the same variables, and,* $y = f(x)$ *gives rise to a bi-continuous map from* $\mathbb{Z}_p^n$ *to itself.*

Remark 4.8. Assume that every $F_i(x,y)$ is a SRP in x,y. Take

$$V = \left\{(x,y) \in \mathbb{Z}_p^n \times \mathbb{Z}_p^m : F_i(x,y) = 0, i = 1,\ldots,m\right\}.$$

Under the hypotheses of the second version of the implicit function theorem, we have

$$V = \left\{(x,y) \in \mathbb{Z}_p^n \times \mathbb{Z}_p^m : y = f(x), x \in \mathbb{Z}_p^n\right\}.$$

By using the coordinate system

$$x_1,\ldots,x_n, z_1 = y_1 - f_1(x),\ldots,z_m = y_m - f_m(x),$$

V takes the form

$$V = \left\{(x,z) \in \mathbb{Z}_p^n \times \mathbb{Z}_p^m : z_1 = \cdots = z_m = 0\right\},$$

and we will say V is a closed analytic submanifold of $\mathbb{Z}_p^n \times \mathbb{Z}_p^m$ of codimension m.

4.1. General change of variables formula

Theorem 4.9 ([31, Prop. 7.4.1]). *Let $K_0, K_1 \subset \mathbb{Q}_p^n$ be open compact subsets, and let $\sigma = (\sigma_1, \ldots, \sigma_n) : K_1 \to K_0$ be a bi-analytic map such that*

$$\det \left[\frac{\partial \sigma_i}{\partial y_j}(z) \right] \neq 0, \quad z \in K_1.$$

If f is a continuous function on K_0, then

$$\int_{K_0} f(x) \, d^n x = \int_{K_1} f(\sigma(y)) \left| \det \left[\frac{\partial \sigma_i}{\partial y_j}(y) \right] \right|_p d^n y, \qquad (x = \sigma(y)).$$

5. The Igusa local zeta functions

Let p be a fixed prime number. Set

$$A_m := \mathbb{Z}/p^m\mathbb{Z}, \, m \in \mathbb{N} \smallsetminus \{0\},$$

the ring of *integers modulo* p^m. Recall that any integer can be written in a unique form as

$$a_0 + a_1 p + \ldots + a_k p^k, \, a_i \in \{0, 1, \ldots, p-1\}.$$

Thus we can identify, as sets, A_m with

$$\left\{ a_0 + a_1 p + \ldots + a_{m-1} p^{m-1}, \, a_i \in \{0, 1, \ldots, p-1\} \right\}.$$

Take $f(x) \in \mathbb{Z}[x_1, \ldots, x_n] \smallsetminus \mathbb{Z}$, and define

$$N_m = \begin{cases} \#\left\{ x \in (\mathbb{Z}/p^m\mathbb{Z})^n : f(x) \equiv 0 \mod p^m \right\} & \text{if} \quad m \geq 1, \\ 1 & \text{if} \quad m = 0. \end{cases}$$

A basic problem is to study the behavior of the sequence N_m as $m \to \infty$.

More generally, we can take $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_n] \smallsetminus \mathbb{Z}_p$ (recall that $\mathbb{Z} \subset \mathbb{Z}_p$ and that $\mathbb{Z}/p^m\mathbb{Z} \simeq \mathbb{Z}_p/p^m\mathbb{Z}_p$), and

$$N_m = \begin{cases} \#\left\{ x \in (\mathbb{Z}_p/p^m\mathbb{Z}_p)^n : f(x) \equiv 0 \mod p^m \right\} & \text{if} \quad m \geq 1, \\ 1 & \text{if} \quad m = 0, \end{cases}$$

where $x \equiv y \mod p^m$ means $x - y \in p^m\mathbb{Z}_p$. To study the sequence $\{N_m\}_{m \in \mathbb{N}}$ we introduce the following Poincaré series:

$$P(t) := \sum_{m=0}^{\infty} N_m \left(p^{-n} t\right)^m, \; t \in \mathbb{C} \text{ with } |t| < 1. \tag{7}$$

We expect that the analytic properties of $P(t)$ provide information about the asymptotic behavior of the sequence $\{N_m\}_{m \in \mathbb{N}}$. A key question is the following:

Is $P(t)$ a rational function of t?

In what follows we will use the convention: given $a > 0$ and $s \in \mathbb{C}$, we set $a^s = e^{s \ln a}$.

Definition 5.1. Let $f(x) \in \mathbb{Q}_p[x_1, \ldots, x_n] \smallsetminus \mathbb{Q}_p$ and let φ be a locally constant function with compact support, i.e., an element of $\mathcal{D}\left(\mathbb{Q}_p^n\right)$.The local zeta function (also called Igusa's local zeta function) attached to (f, φ) is

$$Z_\varphi(s, f) := \int_{\mathbb{Q}_p^n \smallsetminus f^{-1}(0)} \varphi(x) \left|f(x)\right|_p^s \; d^n x, \quad s \in \mathbb{C}, \operatorname{Re}(s) > 0,$$

where $d^n x$ is the Haar measure of $\left(\mathbb{Q}_p^n, +\right)$ normalized such that $\int_{\mathbb{Z}_p^n} d^n x = 1$.

Remark 5.2. $Z_\varphi(s, f)$ is an holomorphic function on the half-plane $\operatorname{Re}(s) > 0$. For the proof of this fact the reader may consult [31, Lemma 5.3.1].

Given $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_n] \smallsetminus \mathbb{Z}_p$, we set

$$Z(s, f) := Z(s) = \int_{\mathbb{Z}_p^n \smallsetminus f^{-1}(0)} \left|f(x)\right|_p^s \; d^n x, \quad s \in \mathbb{C}, \operatorname{Re}(s) > 0.$$

Proposition 5.3. *With the above notation,*

$$P(t) = \frac{1 - tZ(s)}{1 - t}, \; t = p^{-s}, \; \textit{for} \; \operatorname{Re}(s) > 0,$$

where $P(t)$ is the Poincaré series defined in (7).

Proof. We first note that

$$Z(s) = \int_{\mathbb{Z}_p^n \smallsetminus f^{-1}(0)} \left|f(x)\right|_p^s d^n x = \sum_{j=0}^{\infty} p^{-js} \int_{\left\{x \in \mathbb{Z}_p^n : |f(x)| = p^{-j}\right\}} d^n x.$$

On the other hand,

$$\left\{x \in \mathbb{Z}_p^n : |f(x)| = p^{-j}\right\} = \left\{x \in \mathbb{Z}_p^n : ord\left(f(x)\right) = j\right\}$$
$$= \left\{x \in \mathbb{Z}_p^n : ord\left(f(x)\right) \geq j\right\} \smallsetminus \left\{x \in \mathbb{Z}_p^n : ord\left(f(x)\right) \geq j + 1\right\}.$$

Now, take $x_0 \in \mathbb{Z}_p^n$ satisfying $ord\left(f(x_0)\right) \geq j$, then, by using Taylor expansion,

$$f\left(x_0 + p^j z\right) = f(x_0) + p^j \sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(x_0)\left(x_i - x_{0,i}\right) + p^{2j}\left(\text{higher order terms}\right),$$

we have $ord\left(f\left(x_0 + p^j z\right)\right) \geq j$, for all $z \in \mathbb{Z}_p^n$, i.e. $ord\left(f\left(x_0 + p^j \mathbb{Z}_p^n\right)\right) \geq j$. This fact implies:

(i) $x_0 \in \left(\mathbb{Z}_p / p^j \mathbb{Z}_p\right)^n$ satisfies $f(x_0) \equiv 0 \mod p^j$;

(ii) $A_j := \left\{x \in \mathbb{Z}_p^n : ord\left(f(x)\right) \geq j\right\} = \bigsqcup_{f(x_0) \equiv 0 \mod p^j} x_0 + p^j \mathbb{Z}_p^n$;

(iii) $\int_{A_j} d^n x = N_j p^{-jn}$.

Therefore,

$$
\begin{aligned}
Z(s) &= \sum_{j=0}^{\infty} p^{-js}\left(N_j p^{-jn} - N_{j+1} p^{-(j+1)n}\right) \\
&= \sum_{j=0}^{\infty} N_j p^{-js-jn} - \sum_{j=0}^{\infty} N_{j+1} p^{-js-(j+1)n} \\
&= \sum_{j=0}^{\infty} N_j \left(p^{-n}t\right)^{j} - t^{-1}\sum_{j'=1}^{\infty} N_{j'}\left(p^{-n}t\right)^{j'} \quad (t=p^{-s}) \\
&= P(t) - t^{-1}\left(P(t)-1\right),
\end{aligned}
$$

i.e., $P(t) = \frac{1-tZ(s)}{1-t}$, for $\mathrm{Re}(s) > 0$. ☑

Theorem 5.4 (Igusa, [31, Thm. 8.2.1]). *Let $f(x)$ be a non-constant polynomial in $\mathbb{Q}_p[[x_1,\ldots,x_n]]$. There exist a finite number of pairs $(N_E, v_E) \in (\mathbb{N}\smallsetminus\{0\}) \times (\mathbb{N}\smallsetminus\{0\})$, $E \in T$, such that*

$$\prod_{E\in T}\left(1-p^{v_E - sN_E}\right) Z_{\varphi}(s,f)$$

is a polynomial in p^{-s} with rational coefficients.

The proof of this theorem will be given in Section 8. From Theorem 5.4 and Proposition 5.3, we get:

Corollary 5.5. *$P(t)$ is a rational function of t.*

The rationality of $P(t)$ was conjectured in the sixties by Borevich and Shafarevich. Igusa proved this result at middle of the seventies. The rationality of $Z_{\varphi}(s,f)$ also allows us to find bounds for the N_m's (see e.g. [28] and [31]).

The proof of Theorem 5.4 given by Igusa depends on a deep result in algebraic geometry known as Hironaka's resolution of singularities theorem. Now we introduce the *stationary phase formula*, which is an elementary method for computing p-adic integrals like $Z_{\varphi}(s,f)$, Igusa has conjectured in [29] that this method will conduct to a new elementary proof of the rationality of $Z_{\varphi}(s,f)$.

6. The Stationary Phase Formula

Let us identify $\mathbb{F}_p$, set-theoretically, with $\{0,1,\ldots,p-1\}$. Let '-' denote the reduction mod p map, i.e.,

$$
\begin{array}{ccc}
\mathbb{Z}_p & \to & \mathbb{F}_p \\
x_0 + p(\ldots) & \mapsto & x_0.
\end{array}
$$

This map can be extended to $\mathbb{Z}_p^n \to \mathbb{F}_p^n$. The reduction mod p of a subset $E \subset \mathbb{Z}_p^n$ will be denoted as $\overline{E} \subset \mathbb{F}_p^n$. If $f(x) \in \mathbb{Z}_p[x_1,\ldots,x_n] \smallsetminus p\mathbb{Z}_p[x_1,\ldots,x_n]$, $\overline{f}$ denotes its reduction mod p.

Proposition 6.1 (Stationary Phase Formula). *Take $\overline{E} \subset \mathbb{F}_p^n$ and denote by $\overline{S}$ the subset consisting of all $\overline{a} \in \overline{E}$ such that $\overline{f}(\overline{a}) \equiv \frac{\partial \overline{f}}{\partial x_i}(\overline{a}) \equiv 0 \mod p$, for $1 \leq i \leq n$. Denote by E, S the preimages of $\overline{E}, \overline{S}$ under reduction $\mod p$ map $\mathbb{Z}_p^n \to \mathbb{F}_p^n$, and by N the number of zeros of $\overline{f}(x)$ in $\overline{E}$. Then*

$$\int_E |f(x)|_p^s \, d^n x = p^{-n}\left(\#\overline{E} - N\right) + \frac{p^{-n-s}\left(1-p^{-1}\right)\left(N - \#\overline{S}\right)}{1-p^{-1-s}} + \int_S |f(x)|_p^s \, d^n x.$$

Proof. By definition $E = \bigsqcup_{\overline{a} \in \overline{E}} a + p\mathbb{Z}_p^n$; then,

$$\begin{aligned}\int_E |f(x)|_p^s \, d^n x &= \sum_{\overline{a}\in\overline{E}} \int_{a+(p\mathbb{Z}_p)^n} |f(x)|_p^s \, d^n x = p^{-n} \sum_{\overline{a}\in\overline{E}} \int_{\mathbb{Z}_p^n} |f(a+px)|_p^s \, d^n x \\ &= p^{-n} \sum_{\overline{a}\in\overline{E}\setminus\overline{S}} \int_{\mathbb{Z}_p^n} |f(a+px)|_p^s \, d^n x + p^{-n} \sum_{\overline{a}\in\overline{S}} \int_{\mathbb{Z}_p^n} |f(a+px)|_p^s \, d^n x \\ &= p^{-n} \sum_{\overline{a}\in\overline{E}\setminus\overline{S}} \int_{\mathbb{Z}_p^n} |f(a+px)|_p^s \, d^n x + \int_S |f(x)|_p^s \, d^n x.\end{aligned}$$

Take $\overline{a} \in \overline{E} \setminus \overline{S}$ such that $\overline{f}(\overline{a}) \neq 0$, i.e., $|f(a+px)|_p = 1$; in this case,

$$\int_{\mathbb{Z}_p^n} |f(a+px)|_p^s \, d^n x = 1,$$

and the contribution of these $\overline{a}$'s is $p^{-n}\left(\#\overline{E} - N\right)$.

Take now

$$\overline{a} \in \overline{E} \setminus \overline{S} \text{ such that } \overline{f}(\overline{a}) = 0, \frac{\partial \overline{f}}{\partial x_i}(\overline{a}) \neq 0 \text{ for some } i, \tag{8}$$

say $i = 1$. Define

$$y_i = \begin{cases} \dfrac{f(a+px) - f(a)}{p} & \text{if} \quad i = 1, \\ \\ x_i & \text{if} \quad i > 1. \end{cases}$$

Then y_i's are SRP's and $\det\left[\frac{\partial y_i}{\partial x_j}(0)\right] = \frac{\partial f}{\partial x_1}(0) \not\equiv 0 \mod p$; hence, the map $x \mapsto y$ gives rise to a measure-preserving map from $\mathbb{Z}_p^n$ to itself (cf. Corollary 4.7). Therefore,

$$\begin{aligned}\int_{\mathbb{Z}_p^n} |f(a+px)|_p^s \, d^n x &= \int_{\mathbb{Z}_p} |py_1 + f(a)|_p^s \, dy_1 = p^{-s} \int_{\mathbb{Z}_p} \left| y_1 + \frac{f(a)}{p} \right|_p^s \, dy_1 \\ &= p^{-s} \int_{\mathbb{Z}_p} |y_1|_p^s \, dy_1 = p^{-s} \frac{1-p^{-1}}{1-p^{-1-s}},\end{aligned}$$

and the contribution of the points of the form (8) is

$$\frac{p^{-n-s}\left(1-p^{-1}\right)\left(N-\#\overline{S}\right)}{1-p^{-1-s}}. \qquad ☑$$

Set $f\left(a+px\right):=p^{e_a}\widetilde{f}\left(x\right)$ with $\widetilde{f}\left(x\right)\in\mathbb{Z}_p\left[x_1,\ldots,x_n\right]\smallsetminus p\mathbb{Z}_p\left[x_1,\ldots,x_n\right]$. The stationary phase formula, abbreviated SPF, can be re-written as

$$\begin{aligned}\int_E \left|f\left(x\right)\right|_p^s\ d^n x &= \frac{L\left(p^{-s}\right)}{1-p^{-1-s}}+\int_S \left|f\left(x\right)\right|_p^s\ d^n x\\ &= \frac{L\left(p^{-s}\right)}{1-p^{-1-s}}+p^{-n}\sum_{\overline{a}\in\overline{S}}\int_{\mathbb{Z}_p^n}\left|f\left(a+px\right)\right|_p^s\ d^n x\\ &= \frac{L\left(p^{-s}\right)}{1-p^{-1-s}}+p^{-n-e_a s}\sum_{\overline{a}\in\overline{S}}\int_{\mathbb{Z}_p^n}\left|\widetilde{f}\left(x\right)\right|_p^s\ d^n x.\end{aligned}$$

We can now apply SPF to each $\int_{\mathbb{Z}_p^n}\left|\widetilde{f}\left(x\right)\right|_p^s\ d^n x$. Igusa has conjectured that by applying recursively SPF, it is possible to establish the rationality of integrals of type $\int_E \left|f\left(x\right)\right|_p^s d^n x$, in the case in which the polynomial f has coefficients in a non-Archimedean complete field of arbitrary characteristic.

The arithmetic of the Laurent formal series field

$$\mathbb{F}_p\left(\left(T\right)\right)=\left\{\sum_{k=k_0}^{\infty}a_k T^k : a_k\in\mathbb{F}_p, k_0\in\mathbb{Z}\right\}$$

is completely analog to that of $\mathbb{Q}_p$. In particular, given a polynomial with coefficients in $\mathbb{F}_p\left(\left(T\right)\right)$, we can attach to it a local zeta function, which is defined like in the p-adic case. The rationality of such local zeta functions is an open problem. The main difficulty here is the lack of a theorem of resolution of singularities in positive characteristic. The above-mentioned conjecture can be re-stated saying that the rationality of local zeta functions for polynomials with coefficients in $\mathbb{F}_p\left(\left(T\right)\right)$ should follow by applying recursively SPF.

Remark 6.2. Take $f\left(x\right)\in\mathbb{Z}_p\left[x_1,\ldots,x_n\right]\smallsetminus p\mathbb{Z}_p\left[x_1,\ldots,x_n\right]$. If the system of equations

$$\overline{f}\left(\overline{a}\right)\equiv\frac{\partial\overline{f}}{\partial x_i}\left(\overline{a}\right)\equiv 0 \mod p,\ 1\leq i\leq n$$

has no solutions in $\mathbb{F}_p^n$, then $S=\emptyset$, and by SPF,

$$Z(s,f)=p^{-n}\left(\#\overline{E}-N\right)+\frac{p^{-n-s}\left(1-p^{-1}\right)N}{1-p^{-1-s}}.$$

Example 6.3. *Let $f\left(x\right)\in\mathbb{Z}_p\left[x_1,\ldots,x_n\right]\smallsetminus p\mathbb{Z}_p\left[x_1,\ldots,x_n\right]$ be a homogeneous polynomial of degree d, such that $\overline{f}\left(\overline{a}\right)\equiv\frac{\partial\overline{f}}{\partial x_i}\left(\overline{a}\right)\equiv 0 \mod p,\ 1\leq i\leq n$, implies $\overline{a}=0$. We now* *compute $Z(s,f)$. We use SPF with $E=\mathbb{Z}_p^n$, $\overline{E}=\mathbb{F}_p^n$, $S=p\mathbb{Z}_p^n$, $\overline{S}=\{0\}$.*

$$\begin{aligned}Z(s,f)&=p^{-n}\left(p^n-N\right)+\frac{p^{-n-s}\left(1-p^{-1}\right)(N-1)}{1-p^{-1-s}}+\int\limits_{p\mathbb{Z}_p^n}|f(x)|_p^s\ d^nx\\&=p^{-n}\left(p^n-N\right)+\frac{p^{-n-s}\left(1-p^{-1}\right)(N-1)}{1-p^{-1-s}}+p^{-n-ds}\int\limits_{\mathbb{Z}_p^n}|f(y)|_p^s\ d^ny\\&=p^{-n}\left(p^n-N\right)+\frac{p^{-n-s}\left(1-p^{-1}\right)(N-1)}{1-p^{-1-s}}+p^{-n-ds}Z(s,f);\end{aligned}$$

therefore,

$$Z(s,f)=\frac{1}{1-p^{-n-ds}}\left\{p^{-n}\left(p^n-N\right)+\frac{p^{-n-s}\left(1-p^{-1}\right)(N-1)}{1-p^{-1-s}}\right\}.$$

6.1. Singular points of hypersurfaces

Take

$$f(x)\in\mathbb{Z}_p\left[x_1,\ldots,x_n\right]\smallsetminus p\mathbb{Z}_p\left[x_1,\ldots,x_n\right]$$

and define

$$V_f\left(\mathbb{Q}_p\right)=\left\{z\in\mathbb{Q}_p^n:f(z)=0\right\}.$$

$V_f\left(\mathbb{Q}_p\right)$ is the set of $\mathbb{Q}_p$-rational points of the hypersurface defined by f. This notion can be formulated on an arbitrary field K. Set

$$V_{\overline{f}}\left(\mathbb{F}_p\right):=\left\{z\in\mathbb{F}_p^n:\overline{f}(z)=0\right\}.$$

$V_{\overline{f}}\left(\mathbb{F}_p\right)$ is the set of $\mathbb{F}_p$-rational points of the hypersurface defined by $\overline{f}$. If

$$V_f\left(\mathbb{Z}_p\right):=V_f\left(\mathbb{Q}_p\right)\cap\mathbb{Z}_p^n,$$

then $\overline{V_f\left(\mathbb{Z}_p\right)}=V_{\overline{f}}\left(\mathbb{F}_p\right)$. A point $a\in V_f\left(\mathbb{Q}_p\right)$ is said to be *singular* if $\frac{\partial f}{\partial x_i}(a)=0$ for $1\leq i\leq n$. The set of singular points of $V_f\left(\mathbb{Q}_p\right)$ is denoted as $Sing_f\left(\mathbb{Q}_p\right)$. We define $Sing_f\left(\mathbb{Z}_p\right)=Sing_f\left(\mathbb{Q}_p\right)\cap\mathbb{Z}_p^n$. In a similar form we define $Sing_{\overline{f}}\left(\mathbb{F}_p\right)$.

Note that $\overline{Sing_f\left(\mathbb{Z}_p\right)}\neq Sing_{\overline{f}}\left(\mathbb{F}_p\right)$. In fact, it may occur that $Sing_f\left(\mathbb{Z}_p\right)=\emptyset$ and that $Sing_{\overline{f}}\left(\mathbb{F}_p\right)\neq\emptyset$. For instance, if $f(x,y)=px+x^2-y^3$, then $Sing_f\left(\mathbb{Z}_p\right)=\emptyset$, but $Sing_{\overline{f}}\left(\mathbb{F}_p\right)=\{(0,0)\}$.

Example 6.4. *We compute $Z(s,f)$ for $f(x,y)=px+x^2-y^3$ by using SPF. Note that $E=\mathbb{Z}_p^2$, $\overline{E}=\mathbb{F}_p^2$, $S=p\mathbb{Z}_p\times p\mathbb{Z}_p$, $\overline{S}=\{(0,0)\}$,*

$$N=\#\left\{(u,v)\in\mathbb{F}_p^2:u^2-v^3=0\right\};$$

then, by applying SPF,

$$Z(s,f)=p^{-2}\left(p^2-N\right)+\frac{p^{-2-s}\left(1-p^{-1}\right)(N-1)}{1-p^{-1-s}}+\int\limits_{p\mathbb{Z}_p\times p\mathbb{Z}_p}\left|px+x^2-y^3\right|_p^s\ dxdy.$$

By changing variables in the last integral as $x = pu, y = pv$, we have

$$Z(s,f) = \frac{L(p^{-s})}{1-p^{-1-s}} + p^{-2}\int_{\mathbb{Z}_p^2} \left|p^2u + p^2u^2 - p^3v^3\right|_p^s \, dudv = \frac{L(p^{-s})}{1-p^{-1-s}}$$

$$+ p^{-2-2s}\int_{\mathbb{Z}_p^2} \left|u + u^2 - pv^3\right|_p^s \, dudv.$$

We now apply SPF to the last integral. Take $g(u,v) = u + u^2 - pv^3$, $\overline{g}(u,v) = u + u^2$. Since the system of equations

$$u + u^2 = 0, \; 1 + 2u = 0, \;\; u, v \in \mathbb{F}_p$$

has no solutions, $\overline{S} = Sing_{\overline{g}}(\mathbb{F}_p) = \emptyset$; by applying SPF we get

$$Z(s,f) = \frac{L(p^{-s})}{1-p^{-1-s}} + p^{-2-2s}\left\{p^{-2}\left(p^2 - 2p\right) + \frac{2p^{-1-s}\left(1-p^{-1}\right)}{1-p^{-1-s}}\right\}.$$

Remark 6.5. Take $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_n] \smallsetminus p\mathbb{Z}_p[x_1, \ldots, x_n]$. If $Sing_f(\mathbb{Q}_p) = \emptyset$, but $Sing_{\overline{f}}(\mathbb{F}_p) \neq \emptyset$; then,

$$Z(s,f) = \frac{L(p^{-s})}{1-p^{-1-s}},$$

where $L(p^{-s})$ is a polynomial in p^{-s} with rational coefficients (see [60]-[61]). The denominator of $Z(s,f)$ is controlled by $Sing_f(\mathbb{Q}_p)$. Nowadays the numerator is not fully understood, but it depends strongly on $Sing_{\overline{f}}(\mathbb{F}_p)$. The lack of $\mathbb{Q}_p$-singular point, i.e. $Sing_f(\mathbb{Q}_p) = \emptyset$, makes the denominator of $Z(s,f)$ 'trivial': 1 or $1 - p^{-1-s}$.

Example 6.6. *We now compute*

$$Z(s, x^2 + y^3) = \int_{\mathbb{Z}_p^2} \left|x^2 + y^3\right|_p^s \, dxdy,$$

by using SPF. Note that $E = \mathbb{Z}_p^2$, $\overline{E} = \mathbb{F}_p^2$, $S = p\mathbb{Z}_p \times p\mathbb{Z}_p$, $\overline{S} = \{(0,0)\}$, and that

$$N = \#\left\{(u,v) \in \mathbb{F}_p^2 : u^2 + v^3 = 0\right\} = p,$$

because the set $\left\{(u,v) \in \mathbb{F}_p^2 : u^2 + v^3 = 0\right\}$ can be parametrized as $u = \alpha^3$, $v = \alpha^2$, with $\alpha \in \mathbb{F}_p$. By applying SPF we have

$$Z(s, x^2 + y^3) = p^{-2}\left(p^2 - p\right) + \frac{p^{-2-s}(p-1)\left(1-p^{-1}\right)}{1-p^{-1-s}} + \int_{p\mathbb{Z}_p \times p\mathbb{Z}_p} \left|x^2 + y^3\right|_p^s \, dxdy$$

$$= \left(1 - p^{-1}\right)\frac{1-p^{-2-s}}{1-p^{-1-s}} + \int_{p\mathbb{Z}_p \times p\mathbb{Z}_p} \left|x^2 + y^3\right|_p^s \, dxdy.$$

By changing variables in the last integral as $x = pu$, $y = pv$, $dxdy = p^{-2}dudv$,

$$\begin{aligned} Z(s, x^2 + y^3) &= \left(1 - p^{-1}\right) \frac{1 - p^{-2-s}}{1 - p^{-1-s}} + p^{-2-2s} \int\limits_{\mathbb{Z}_p^2} \left|u^2 + pv^3\right|_p^s \, dudv \\ &=: \left(1 - p^{-1}\right) \frac{1 - p^{-2-s}}{1 - p^{-1-s}} + p^{-2-2s} Z_1(s). \end{aligned}$$

We now apply SPF to $Z_1(s)$: $E = \mathbb{Z}_p^2$, $\overline{E} = \mathbb{F}_p^2$; since $\overline{g}(u,v) = \overline{u^2 + pv^3} = u^2$, the solution set of $u^2 = 2u = 0$ is $\overline{S} = \{0\} \times \mathbb{F}_p$, then $S = p\mathbb{Z}_p \times \mathbb{Z}_p$. In addition, $\#\left\{(u,v) \in \mathbb{F}_p^2 : u^2 = 0\right\} = p$. Therefore,

$$\begin{aligned} Z_1(s) &= p^{-2}\left(p^2 - p\right) + \int\limits_{p\mathbb{Z}_p \times \mathbb{Z}_p} \left|u^2 + pv^3\right|_p^s \, dudv \\ &= \left(1 - p^{-1}\right) + p^{-1-s} \int\limits_{\mathbb{Z}_p^2} \left|pu^2 + v^3\right|_p^s \, dudv \\ &=: \left(1 - p^{-1}\right) + p^{-1-s} Z_2(s), \end{aligned}$$

and

$$Z(s, x^2 + y^3) = \left(1 - p^{-1}\right) \frac{1 - p^{-2-s}}{1 - p^{-1-s}} + p^{-2-2s}\left(1 - p^{-1}\right) + p^{-3-3s} Z_2(s).$$

We now apply SPF to $Z_2(s)$: $E = \mathbb{Z}_p^2$, $\overline{E} = \mathbb{F}_p^2$, $S = \mathbb{Z}_p \times p\mathbb{Z}_p$, $\overline{S} = \mathbb{F}_p \times \{0\}$; then,

$$\begin{aligned} Z_2(s) &= p^{-2}\left(p^2 - p\right) + \int\limits_{\mathbb{Z}_p \times p\mathbb{Z}_p} \left|pu^2 + v^3\right|_p^s \, dudv \\ &= \left(1 - p^{-1}\right) + p^{-1-s} \int\limits_{\mathbb{Z}_p^2} \left|u^2 + p^2v^3\right|_p^s \, dudv \\ &=: \left(1 - p^{-1}\right) + p^{-1-s} Z_3(s), \end{aligned}$$

and

$$Z(s, x^2 + y^3) = \left(1 - p^{-1}\right) \frac{1 - p^{-2-s}}{1 - p^{-1-s}} + p^{-2-2s}\left(1 - p^{-1}\right) + p^{-3-3s}\left(1 - p^{-1}\right) + p^{-4-4s} Z_3(s).$$

Finally, we apply SPF to $Z_3(s)$: $E = \mathbb{Z}_p^2$, $\overline{E} = \mathbb{F}_p^2$, $S = p\mathbb{Z}_p \times \mathbb{Z}_p$, $\overline{S} = \{0\} \times \mathbb{F}_p$; then,

$$\begin{aligned} Z_3(s) &= p^{-2}\left(p^2 - p\right) + \int\limits_{p\mathbb{Z}_p \times p\mathbb{Z}_p} \left|u^2 + p^2v^3\right|_p^s \, dudv \\ &= \left(1 - p^{-1}\right) + p^{-1-2s} \int\limits_{\mathbb{Z}_p^2} \left|u^2 + v^3\right|_p^s \, dudv \\ &= \left(1 - p^{-1}\right) + p^{-1-2s} Z(s, x^2 + y^3), \end{aligned}$$

and

$$Z(s,x^2+y^3)=\left(1-p^{-1}\right)\frac{1-p^{-2-s}}{1-p^{-1-s}}+p^{-2-2s}\left(1-p^{-1}\right)+p^{-3-3s}\left(1-p^{-1}\right)$$
$$+p^{-4-4s}\left(1-p^{-1}\right)+p^{-5-6s}Z(s,x^2+y^3),$$

i.e.,

$$\begin{aligned}Z(s,x^2+y^3)&=\frac{\left(1-p^{-1}\right)}{1-p^{-5-6s}}\left\{\frac{1-p^{-2-s}}{1-p^{-1-s}}+p^{-2-2s}+p^{-3-3s}+p^{-4-4s}\right\}\\&=\frac{\left(1-p^{-1}\right)}{\left(1-p^{-1-s}\right)\left(1-p^{-5-6s}\right)}\left\{1-p^{-2-s}+p^{-2-2s}-p^{-5-6s}\right\}.\end{aligned}$$

6.2. Quasi-homogenous singularities

Take $w=(w_1,\ldots,w_n)\in(\mathbb{N}\smallsetminus\{0\})^n$ and $f(x)\in\mathbb{Z}_p[x_1,\ldots,x_n]$. We say that $f(x)$ is a *quasi-homogeneous polynomial of degree d with respect to w* if: (1) $f(\lambda^{w_1}x_1,\ldots,\lambda^{w_n}x_n)=\lambda^d f(x)$, $\lambda\in\mathbb{Q}_p^{\times}$; (2) $Sing_f(\mathbb{Q}_p)=\{0\}\subset\mathbb{Q}_p^n$.

Set $|w|:=w_1+\ldots+w_n$, and

$$Z(s,f)=\int_{\mathbb{Z}_p^n}|f(x)|_p^s\ d^nx.$$

Proposition 6.7. *With the above notation and hypotheses,*

$$Z(s,f)=\frac{L\left(p^{-s}\right)}{\left(1-p^{-1-s}\right)\left(1-p^{-ds-|w|}\right)},$$

where $L(p^{-s})$ is a polynomial in p^{-s} with rational coefficients.

Proof. Set

$$\begin{aligned}A&=\left\{(x_1,\ldots,x_n)\in\mathbb{Z}_p^n: ord\,(x_i)\geq w_i \text{ for } i=1,\ldots,n\right\}\\&=p^{w_1}\mathbb{Z}_p\times\ldots\times p^{w_n}\mathbb{Z}_p,\end{aligned}$$

$$A^c:=\mathbb{Z}_p^n\smallsetminus A.$$

Then,

$$Z(s,f)=\int_A|f(x)|_p^s\ d^nx+\int_{A^c}|f(x)|_p^s\ d^nx.$$

By changing variables in the first integral as $x_i=p^{w_i}u_i$, $i=1,\ldots,n$, $d^nx=p^{-|w|}d^nu$, we have

$$\int_A|f(x)|_p^s\ d^nx=p^{-ds-|w|}Z(s,f),$$

and

$$Z(s,f)=\frac{1}{1-p^{-ds-|w|}}\int\limits_{A^c}|f(x)|_p^s\ d^nx.$$

We now note that $Sing_f(\mathbb{Q}_p)\cap A^c=\emptyset$, but it may occur that $Sing_{\overline{f}}(\mathbb{F}_p)\cap\overline{A^c}\neq\emptyset$; this makes the computation of the integral on A^c not simple. By using SPF recursively and some ideas on Néron p-desingularization, one can show that $\int_{A^c}|f(x)|_p^s\ d^nx=\frac{L(p^{-s})}{1-p^{-1-s}}$. For a detailed proof, including the most general case of the semiquasi-homogeneous singularities, the reader may consult [60]. ☑

7. p-adic Analytic Manifolds and Resolution of Singularities

This section is based on [31, Sec. 2.4]. Let $U\subset\mathbb{Q}_p^n$ be a non-empty open set, and let $f:U\to\mathbb{Q}_p$ be a function. If at every point $a=(a_1,\ldots,a_n)$ of U there exists an element $f_a(x)\in\mathbb{Q}_p\langle\langle x-a\rangle\rangle=\mathbb{Q}_p\langle\langle x_1-a_1,\ldots,x_n-a_n\rangle\rangle$ such that $f(x)=f_a(x)$ for any point x near to a, we say that f is an *analytic function on* U. It is not hard to show that all the partial derivatives of f are analytic on U.

Let U be as above and let $h=(h_1,\ldots,h_m):U\to\mathbb{Q}_p^m$ be a mapping. If each h_i is an analytic function on U, we say that h is *an analytic mapping on* U.

Let X denote a Hausdorff space and n a fixed non–negative integer. A pair (U,ϕ_U), where U is a nonempty open subset of X and $\phi_U:U\to\phi_U(U)$ is a bi-continuous map (i.e., a homeomorphism) from U to an open set $\phi_U(U)$ of $\mathbb{Q}_p^n$, is called *a chart*. Furthermore $\phi_U(x)=(x_1,\ldots,x_n)$, for a variable point x of U are called *the local coordinates* of x. A set of charts $\{(U,\phi_U)\}$ is called an *atlas* if the union of all U is X and for every U, U' such that $U\cap U'\neq\emptyset$ the map

$$\phi_{U'}\circ\phi_U^{-1}:\phi_U(U\cap U')\to\phi_{U'}(U\cap U')$$

is analytic. Two atlases are considered equivalent if their union is also an atlas. This is an equivalence relation and any equivalence class is called an *n-dimensional p-adic analytic structure on X*. If $\{(U,\phi_U)\}$ is an atlas in the equivalence class, we say that X is an *n-dimensional p-adic analytic manifold*, and we write $n=\dim(X)$.

Suppose that X, Y are p-adic analytic manifolds respectively, defined by $\{(U,\phi_U)\}$, $\{(V,\psi_V)\}$, and $f:X\to Y$ is a map. If for every U,V such that $U\cap f^{-1}(V)\neq\emptyset$ the map

$$\psi_V\circ f\circ\phi_U^{-1}:\phi_U\left(U\cap f^{-1}(V)\right)\to\mathbb{Q}_p^{\dim(Y)}$$

is analytic, then we say that f is an *analytic map*. This notion does not depend on the choice of atlases.

Suppose that X is a p-adic analytic manifold defined by $\{(U,\phi_U)\}$ and Y is a nonempty open subset of X. If for every $U'=Y\cap U\neq\emptyset$ we put $\phi_{U'}=\phi_U\mid_{U'}$, then $\{(U',\phi_{U'})\}$ gives an atlas on Y, which makes Y a p-adic analytic *open submanifold of* X, with $\dim(X)=\dim(Y)$.

If U, U' are neighborhoods of an arbitrary point a of X, and f, g are p-adic analytic functions respectively on U, U' such that $f\mid_W=g\mid_W$ for some neighborhood W of a contained in $U\cap U'$, then we say that f, g are *equivalent at* a. An equivalence class is said to be a *germ of analytic functions at a*. The set of germs of analytic functions at a form a local ring denoted by $\mathcal{O}_{X,a}$, or simply $\mathcal{O}_a$.

Suppose that Y is a nonempty closed subset of X, a p-adic analytic manifold as before, and $0 < m \leq n$ such that an atlas $\{(U, \phi_U)\}$ defining X can be chosen with the following property: If $\phi_U(x) = (x_1, \ldots, x_n)$ and $U' = Y \cap U \neq \emptyset$, there exist p-adic analytic functions $F_1, \ldots, F_m$ on U such that firstly U' becomes the set of all x in U satisfying $F_1(x) = \cdots = F_m(x) = 0$, and secondly,

$$\det \left[\frac{\partial F_i}{\partial x_j}\right]_{\substack{1\leq i\leq m\\1\leq j\leq m}} (a) \neq 0 \text{ at every } a \text{ in } U'.$$

Then by Corollary 4.4-(ii) the mapping $x \mapsto (F_1(x), \ldots, F_m(x), x_{m+1}, \ldots, x_n)$ is a bi-analytic mapping from a neighborhood of a in U to its image in $\mathbb{Q}_p^n$. If we denote by V the intersection of such neighborhood of a an Y, and put $\psi_V(x) = (x_{m+1}, \ldots, x_n)$ for every x in V, then $\{(V, \psi_V)\}$ gives an atlas on Y. Therefore Y becomes a p-adic analytic manifold with $\dim(Y) = n - m$. We call Y a *closed submanifold of X of codimension m*.

Let μ_n denote the normalized Haar measure of $\mathbb{Q}_p^n$. Take X and $\{(U, \phi_U)\}$ as before. Set α a differential form of degree n on X; then $\alpha\,|_U$ has an expression of the form

$$\alpha(x) = f_U(x)\ dx_1 \wedge \cdots \wedge dx_n,$$

in which f_U is an analytic function on U. If A is an open and compact subset of X contained in U, then we define its measure $\mu_\alpha(A)$ as

$$\mu_\alpha(A) = \int\limits_A |f_U(x)|_p\ \mu_n(\phi_U(x)) = \sum_{e\in\mathbb{Z}} p^{-e}\mu_n\left(\phi_U\left(f_U^{-1}\left(p^e\mathbb{Z}_p^\times\right)\cap A\right)\right). \tag{9}$$

We note that the above series converges because $f_U(A)$ is a compact subset. If $(U', \phi_{U'})$ is another chart and $A \subset U'$, then we will have the same $\mu_\alpha(A)$ relative to that chart. In fact, if $\phi_{U'}(x) = (x_1', \ldots, x_n') = x'$, then

$$f_{U'}(x)\det\left[\frac{\partial x_i'}{\partial x_i}\right] = f_U(x), \quad \text{and} \quad \mu_n(\phi_{U'}(x)) = \left|\det\left[\frac{\partial x_i'}{\partial x_i}\right]\right|_p \mu_n(\phi_U(x)).$$

Actually, the previous equations just give account of the change of variables rule as $x \mapsto x'$, in the integral (9), that is

$$\int\limits_A |f_U(x)|_p\ \mu_n(\phi_U(x)) = \int\limits_A |f_{U'}(x)|_p\ \mu_n(\phi_{U'}(x)$$

(see [31, pg. 112 and Proposition 7.4.1]). Note that if $X = U \subset \mathbb{Q}_p^n$ and

$$\alpha = dx_1 \wedge \cdots \wedge dx_n,$$

then μ_α is the normalized Haar measure of $\mathbb{Q}_p^n$.

Theorem 7.1 (Hironaka). *Take $f(x)$ a nonconstant polynomial in $\mathbb{Q}_p[x_1,\ldots,x_n]$, and put $X=\mathbb{Q}_p^n$. Then there exist an n-dimensional p-adic analytic manifold Y, a finite set $T=\{E\}$ of closed submanifolds of Y of codimension 1 with a pair of positive integers (N_E,v_E) assigned to each E, and a p-adic analytic proper mapping $h:Y\to X$ satisfying the following conditions: (i) h is the composition of a finite number of monoidal transformations each one with a smooth center; (ii)*

$$(f\circ h)^{-1}(0)=\bigcup_{E\in T}E$$

and h induces a p-adic bianalytic map

$$Y\smallsetminus h^{-1}\left(Sing_f\left(\mathbb{Q}_p\right)\right)\to X\smallsetminus Sing_f\left(\mathbb{Q}_p\right);$$

(iii) at every point b of Y, if $E_1,\ldots,E_m$ are all the E in T containing b with local equations $y_1,\ldots,y_m$ around b and $(N_i,v_i)=(N_E,v_E)$ for $E=E_i$, then there exist local coordinates of Y around b of the form $(y_1,\ldots,y_m,y_{m+1},\ldots,y_n)$ such that

$$(f\circ h)(y)=\varepsilon(y)\prod_{i=1}^{m}y_i^{N_i},\quad h^*\left(\bigwedge_{1\le i\le n}dx_i\right)=\eta(y)\left(\prod_{i=1}^{m}y_i^{v_i-1}\right)\bigwedge_{1\le i\le n}dy_i \tag{10}$$

on some neighborhood of b, in which $\varepsilon(y)$, $\eta(y)$ are units of the local ring $\mathcal{O}_b$ of Y at b. In particular, $\cup_{E\in T}E$ has normal crossings.

8. Proof of Theorem 5.4

We want to end this notes by proving Igusa's Theorem about the meromorphic continuation of $Z_\varphi(s,f)$ (see Theorem 5.4 in Section 5). We follow the proof given by Igusa in [31, Thm. 8.2.1].

Let $\left|\bigwedge_{1\le i\le n}dx_i\right|$ denote the measure induced by the differential form $\bigwedge_{1\le i\le n}dx_i$ on $\mathbb{Q}_p^n$, which agrees with the Haar measure of $\mathbb{Q}_p^n$. Then

$$Z_\varphi(s,f)=\int_{\mathbb{Q}_p^n\smallsetminus f^{-1}(0)}\varphi(x)\,|f(x)|_p^s\left|\bigwedge\nolimits_{1\le i\le n}dx_i\right|.$$

Pick a resolution of singularities $h:Y\to X$ for $f^{-1}(0)$ as in Theorem 7.1; we use all the notation introduced there. Then $Y\smallsetminus h^{-1}\left(f^{-1}(0)\right)\to X\smallsetminus f^{-1}(0)$ is a p-adic bianalytic proper map, i.e., a proper analytic coordinate change; then,

$$Z_\varphi(s,f)=\int_{Y\smallsetminus h^{-1}(f^{-1}(0))}\varphi(h(y))\,|f(h(y))|_p^s\left|h^*\left(\bigwedge\nolimits_{1\le i\le n}dx_i\right)(y)\right|.$$

At every point b of $Y\smallsetminus h^{-1}\left(f^{-1}(0)\right)$ we can choose a chart (U,ϕ_U) such that (10) holds. Since h is proper and the support of φ, say A, is compact, we see that $h^{-1}(A):=B$ is compact. Then we can cover B by a finite disjoint union of open compact balls B_α such that each of these balls is contained in some U above. Since φ is locally constant, after subdividing B_α we may assume that $(\varphi\circ h)\Big|_{B_\alpha}=\varphi(h(b)), |\varepsilon(y)|_p\Big|_{B_\alpha}=|\varepsilon(b)|_p, |\eta(y)|_p\Big|_{B_\alpha}=|\eta(b)|_p$, and further that $\phi_U(B_\alpha)=c+p^e\mathbb{Z}_p^n$, for some $c=(c_1,\dots,c_n)$ in $\mathbb{Q}_p^n$ and $e\in\mathbb{N}$. Then,

$$Z_\varphi(s,f)=\sum_\alpha \varphi(h(b))\,|\varepsilon(b)|_p\,|\eta(b)|_p\cdot\prod_{1\le i\le n}\int_{c_i+p^e\mathbb{Z}_p}|y_i|_p^{N_is+v_i-1}\,dy_i,$$

with the understanding that $N_i=0, v_i=1$ in the case E_i is not crossing through b. Finally one has by [31, Lemma 8.2.1] that

$$\int_{c_i+p^e\mathbb{Z}_p}|y_i|_p^{N_is+v_i-1}\,dy_i=\begin{cases} p^{-(N_is+v_i)e}\left(\frac{1-p^{-1}}{1-p^{-N_is-v_i}}\right) & \text{if} \quad c_i\in p^e\mathbb{Z}_p\\ \\ p^{-e}\,|c_i|_p^{N_is+v_i-1} & \text{if} \quad c_i\notin p^e\mathbb{Z}_p.\end{cases}$$

Acknowledgements. The authors were partially supported by CONACYT grants 286445 and 250845.

References

[1] Albeverio S., Khrennikov A. and Shelkovich V.M., "Theory of p-adic distributions: linear and nonlinear models", in *London Mathematical Society, Lecture Note Series, 370*, Cambridge University Press, Cambridge (2010), xvi+351 pp.

[2] Aref'eva I., Dragović B.G. and Volovich I.V., "On the adelic string amplitudes", *Phys. Lett. B* 209 (1988), No. 4, 445–450.

[3] Arnold V.I., Gusein-Zade S.M. and Varchenko A.N., *Singularities of differentiable maps*. Vol. II. Monodromy and asymptotics of integrals. Translated from the Russian by Hugh Porteous. Translation revised by the authors and James Montaldi. Monographs in Mathematics, 83. Birkhauser Boston, Inc., Boston, MA, 1988.

[4] Atiyah M.F., "Resolution of Singularities and Division of Distributions", *Comm. pure Appl. Math.* 23 (1970), 145–150.

[5] Belkale P. and Brosnan P., "Periods and Igusa local zeta functions", *Int. Math. Res. Not.* 49 (2003), 2655–2670.

[6] Bernstein I.N., "Modules over the ring of differential operators; the study of fundamental solutions of equations with constant coefficients", *Funct. Anal. Appl.* 5 (1972), No. 2, 1–16.

[7] Bleher P.M., "Analytic continuation of massless Feynman amplitudes in the Schwartz space $\mathcal{S}'$", *Rep. Math. Phys.* 19 (1984), No. 1, 117–142.

[8] Bocardo-Gaspar M., García-Compeán H. and Zúñiga-Galindo W.A., "Regularization of p-adic String Amplitudes, and Multivariate Local Zeta Functions", *Lett. Math. Phys.*(2018), 1–38. https://doi.org/10.1007/s11005-018-1137-1.

[9] Bocardo-Gaspar M., García-Compeán H. and Zúñiga-Galindo W.A., " On p-Adic string amplitudes in the limit p approaches to one", *J. High Energ. Phys.* (2018), No. 8, 043. https://doi.org/10.1007/JHEP08(2018)043.

[10] Bogner C. and Weinzierl S., "Periods and Feynman integrals", *J. Math. Phys.* 50 (2009), No. 4, 042302, 16 pp.

[11] Bollini C.G., Giambiagi J.J. and González Domínguez A., "Analytic regularization and the divergences of quantum field theories", *Il Nuovo Cimiento* 31 (1964), No. 3, 550–561.

[12] Brekke L., Freund P.G.O., Olson M. and Witten E., "Non-Archimedean string dynamics", *Nuclear Phys. B* 302 (1988), No. 3, 365–402.

[13] Brekke L. and Freund P.G.O., "p-adic numbers in physics", *Phys. Rep.* 233 (1993), No. 1, 1–66.

[14] Denef J., "Report on Igusa's Local Zeta Function", Séminaire Bourbaki 43 (1990–1991), exp. 741; Astérisque 201–202–203, 359–386 (1991).

[15] Denef J. and Loeser F., "Caractéristiques D'Euler-Poincaré, Fonctions Zeta locales et modifications analytiques", *J. Amer. Math. Soc.* 5 (1992), No. 4, 705–720.

[16] Denef J. and Loeser F., "Motivic Igusa zeta functions", *J. Algebraic Geom.* 7 (1998), No. 3, 505–537.

[17] Denef J. and Loeser F., "On some rational generating series occuring in arithmetic geometry", in *Geometric aspects of Dwork theory Vol I, II*, Walter de Gruyter, Berlin, (2004), 509–526.

[18] Frampton P.H. and Okada Y., "p-adic string N-point function", *Phys. Rev. Lett.* 60 (1988), No. 6, 484–486.

[19] Freund P.G.O. and Olson M., "Non-Archimedean strings", *Phys. Lett. B* 199 (1987), No. 2, 186–190.

[20] Freund P.G.O. and Witten E., "Adelic string amplitudes", *Phys. Lett. B* 199 (1987), No. 2, 191–194.

[21] Gel'fand I.M. and Shilov G.E., *Generalized Functions*, Academic Press, New York and London, 1977.

[22] Ghoshal D., "Noncommutative p-tachyon", *Proc. Steklov Inst. Math.* 245 (2004), No. 2, 83–90.

[23] Ghoshal D., "p-adic string theories provide lattice discretization to the ordinary string worldsheet", *Phys. Rev. Lett.* 97 (2006), 151601.

[24] Ghoshal D. and Sen A., "Tachyon condensation and brane descent relations in p-adic string theory", *Nucl. Phys. B* 584 (2000), No. 1-2, 300–312.

[25] Gouvêa F., *p-adic numbers: An introduction*, Universitext, Springer-Verlag, Berlin, 1997.

[26] Halmos P., *Measure Theory*, D. Van Nostrand Company Inc., New York, 1950.

[27] Hloušek Z. and Spector D., "p-adic string theories", *Ann. Physics* 189 (1989), No. 2, 370–431.

[28] Igusa J.-I., *Lectures on forms of higher degree*, Tata Institute of Fundamental Research, Lectures on Mathematics and Physics vol. 59, The Narosa Publishing House, New Delhi, 1978.

[29] Igusa J.-I., "A stationary phase formula for p-adic integrals and its applications", in *Algebraic geometry and its applications* (West Lafayette, IN, 1990), , Springer, New York (1994), 175–194.

[30] Igusa J.-I., "On local zeta functions" [translation of Sūgaku 46 (1994), no. 1, 2338], in *Selected papers on number theory and algebraic geometry*, 1–20, *Amer. Math. Soc. Transl. Ser.* 2, 172, Amer. Math. Soc., Providence, RI, 1996.

[31] Igusa J.-I., "An introduction to the theory of local zeta functions", in *AMS/IP Studies in Advanced Mathematics, 14*, American Mathematical Society, Providence, RI; International Press, Cambridge, MA (2000), xii+232 pp.

[32] Katok S., "p-adic analysis compared with real", in *Student Mathematical Library, 37*, American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA (2007), xiv+152 pp.

[33] Khrennikov A., Kozyrev S. and Zúñiga-Galindo W.A., "Ultrametric Pseudodifferential Equations and Applications", in *Encyclopedia of Mathematics and its Applications* vol. 168, Cambridge University Press (2018), xv+237 pp.

[34] Kimura T., *Introduction to prehomogeneous vector spaces*, Translated from the 1998 Japanese original by Makoto Nagura and Tsuyoshi Niitani and revised by the author, Translations of Mathematical Monographs, 215. American Mathematical Society, Providence, RI, 2003.

[35] Koblitz N., "p-adic Numbers, p-adic Analysis and Zeta functions", Second edition, in *Graduate Texts in Mathematics, 58*, Springer-Verlag, New York (1984), xii+150 pp.

[36] Lang S., "Real and functional analysis", Third edition, *in Graduate Texts in Mathematics, 142*, Springer-Verlag, New York (1993), xiv+580 pp.

[37] Lerner È.Y., "Feynman integrals of a p-adic argument in a momentum space. I. Convergence", *Theoret. and Math. Phys.* 102 (1995), No. 3, 267–274.

[38] Lerner È. Y., "Feynman integrals of a p-adic argument in a momentum space. II. Explicit formulas", *Theoret. and Math. Phys.* 104 (1995), No. 3, 1061–1077.

[39] Lerner È. Y. and Missarov M. D., "p-adic Feynman and string amplitudes", *Comm. Math. Phys.* 121 (1989), No. 1, 35–48.

[40] Lin S., "Ideal-theoretic strategies for asymptotic approximation of marginal likelihood integrals", *J. Algebr. Stat.* 8 (2017), No. 1, 22–55.

[41] Loeser F., "Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants", *Ann. Sci. École Norm. Sup.* 22 (1989), No. 3, 435–471.

[42] Marcolli M., *Feynman motives*, World Scientific Publishing, NJ, 2010.

[43] Mendoza-Martínez M.L., Vallejo J.A. and Zúñiga-Galindo W.A., "Acausal quantum theory for non-Archimedean scalar fields", arXiv:1805.08613.

[44] Meuser D., "A survey of Igusa's local zeta function", *Amer. J. Math.* 138 (2016), No. 1, 149–179.

[45] Nicaise J., "An introduction to p-adic and motivic zeta functions and the monodromy conjecture", in *Algebraic and analytic aspects of zeta functions and L-functions*, MSJ Mem., 21, Math. Soc. Japan, Tokyo (2010), 141–166.

[46] Robert A., *A course in p-adic Analysis*, Springer-Verlag, New York, 2000.

[47] Schikhof W.H., "Ultrametric calculus: An introduction to p-adic analysis", in *Cambridge Studies in Advanced Mathematics, 4*, Cambridge University Press, Cambridge (2006), xii+306 pp.

[48] Smirnov V.A., "Renormalization in p-adic quantum field theory", *Modern Phys. Lett. A* 6 (1991), No. 15, 1421–1427.

[49] Smirnov V.A., "Calculation of general p-adic Feynman amplitude", *Comm. Math. Phys.* 149 (1992), No. 3, 623–636.

[50] Speer E.R., "Generalized Feynman amplitudes", in *Annals of Mathematics Studies* vol. 62, Princeton University Press (1969), vi+120 pp.

[51] Speer E.R., "Ultraviolet and infrared singularity structure of generic Feynman amplitudes", *Ann. Inst. H. Poincaré Sect. A (N.S.)* 23 (1975), No. 1, 1–21.

[52] Spokoiny B.L., "Quantum Geometry of Non-archimedean Particles and Strings", *Phys. Lett. B* 208 (1988), No. 3-4, 401–406.

[53] Taibleson M.H., *Fourier analysis on local fields*, Princeton University Press, Tokyo, 1975.

[54] Varadarajan V.S., *Reflections on quanta, symmetries, and supersymmetries*, Springer, New York, 2011.

[55] Veys W., "Arc spaces, motivic integration and stringy invariants", in *Singularity theory and its applications*, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo (2006), 529–572.

[56] Vladimirov V.S., Volovich I.V., and Zelenov E.I., "p-adic analysis and mathematical physics", in *Series on Soviet and East European Mathematics 1,* World Scientific, River Edge, NJ (1994), xx+319 pp.

[57] Volovich I.V., "p-adic string", *Classical Quantum Gravity* 4 (1987), No. 4, L83–L87.

[58] Watanabe S., "Algebraic geometry and statistical learning theory", in *Cambridge Monographs on Applied and Computational Mathematics, 25*, Cambridge University Press, Cambridge (2009), viii+286 pp.

[59] Weil A., "Sur la formule de Siegel dans la théorie des groupes classiques", *Acta Math.* 113 (1965), 1–87.

[60] Zúñiga-Galindo W.A., "Igusa's local zeta functions of semiquasihomogeneous polynomials", *Trans. Amer. Math. Soc.* 353 (2001), No. 8, 3193–3207.

[61] Zúñiga-Galindo W.A., "Local zeta functions and Newton polyhedra", *Nagoya Math. J.* 172 (2003), 31–58.

[62] Zúñiga-Galindo W.A., "Pseudodifferential equations over non-Archimedean spaces", in *Lectures Notes in Mathematics* vol. 2174, Springer, Cham. (2016), xvi+175 pp.