On the existence of a priori bounds for positive solutions of elliptic problems, II

Rosa Pardo [*]

Universidad Complutense de Madrid, Departamento de Análisis Matemático y
Matemática Aplicada, Madrid, Spain.

Abstract. We continue studying the existence of uniform L^∞ *a priori* bounds for positive solutions of subcritical elliptic equations

$$(\mathcal{P})_p \qquad -\Delta_p u = f(u), \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega,$$

We provide sufficient conditions for having *a-priori* L^∞ bounds for $C^{1,\mu}(\overline{\Omega})$ positive solutions to a class of subcritical elliptic problems in bounded, convex, C^2 domains. In this part II, we extend our results to Hamiltonian elliptic systems $-\Delta u = f(v), -\Delta v = g(u)$, in Ω, $u = v = 0$ on $\partial\Omega$, when $f(v) = v^p/[\ln(e + v)]^\alpha$, $g(u) = u^q/[\ln(e + u)]^\beta$, with $\alpha, \beta > 2/(N - 2)$, and p, q are lying in the *critical Sobolev hyperbolae* $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. For quasilinear elliptic equations involving the p-Laplacian, there exists *a-priori* bounds for positive solutions of $(\mathcal{P})_p$ when $f(u) = u^{p^\star-1}/[\ln(e + u)]^\alpha$, with $p^* = Np/(N-p)$, and $\alpha > p/(N-p)$. We also study the asymptotic behavior of radially symmetric solutions $u_\alpha = u_\alpha(r)$ of $(\mathcal{P})_2$ as $\alpha \to 0$.

Keywords: *A priori* estimates, subcritical nonlinearity, moving planes method, Pohozaev identity, critical Sobolev hyperbola, biparameter bifurcation.

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Sobre la existencia de cotas a priori para soluciones positivas de problemas elípticos, II

Resumen. Continuamos estudiando la existencia de cotas uniformes *a priori* para soluciones positivas de equaciones elípticas subcríticas

$$(\mathcal{P})_p \qquad -\Delta_p u = f(u), \text{ en } \Omega, \qquad u = 0, \text{ sobre } \partial\Omega,$$

Proporcionamos condiciones suficientes para que las soluciones positivas en $C^{1,\mu}(\overline{\Omega})$ de una clase de problemas elípticos subcríticos tengan cotas *a-priori* L^∞ en dominios acotados, convexos, y de clase C^2.

En esta parte II, extendemos nuestros resultados a sistemas elípticos Hamiltonianos $-\Delta u = f(v), -\Delta v = g(u)$, en Ω, $u = v = 0$ sobre $\partial\Omega$, cuando $f(v) = v^p/[\ln(e+v)]^\alpha$, $g(u) = u^q/[\ln(e+u)]^\beta$, con $\alpha, \beta > 2/(N-2)$, y p, q varían sobre la *hipérbola crítica de Sobolev* $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. Para ecuaciones elípticas cuasilineales que involucran al operador p-Laplacian, existen cotas *a-priori* para soluciones positivas de $(\mathcal{P})_p$ en el espacio $C^{1,\mu}(\overline{\Omega})$, $\mu \in (0,1)$, cuando $f(u) = u^{p^*-1}/[\ln(e+u)]^\alpha$, con $p^* = Np/(N-p)$, y $\alpha > p/(N-p)$.

También estudiamos el comportamiento asintótico de soluciones radialmente simétric $u_\alpha = u_\alpha(r)$ de $(\mathcal{P})_2$ cuando $\alpha \to 0$.

Palabras clave: Estimaciones *a priori*, no-linealidades subcríticas, método de "moving planes", igualdad de Pohozaev, hipérbola crítica de Sobolev, bifurcación biparamétrica.

1.   Introduction

In this part II, we describe our *a priori* bounds results on semilinear elliptic systems, on quasilinear elliptic equations for the p-laplacian, and the asymptotic behavior of radial solutions as $\alpha \to 0^+$ in several subsections. We leave the proofs for the following sections.

In a previous paper containing part I, when $p = 2$, we show the existence of L^∞ *a-priori* bounds for classical, positive solutions of semilinear elliptic equations $-\Delta u = f(u)$ with Dirichlet homogeneous boundary conditions, for

$$f(u) = \frac{u^{2^*-1}}{\left[\,\ln(e+u)\right]^\alpha}, \qquad \text{with } 2^* = \frac{2N}{N-2}, \quad \text{and } \alpha > \frac{2}{N-2}.$$

Appealing to the Kelvin transform, we extend our results to non-convex domains (see [16], [17]).

1.1.   Hamiltonian elliptic systems

We provide a-priori $L^\infty(\Omega)$-bounds for classical positive solutions of Hamiltonian elliptic systems

$$\begin{cases} -\Delta u = f(v), & \text{in } \Omega, \\ -\Delta v = g(u), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded, convex C^3 domain,

$$f(v) = \frac{v^p}{[\ln(e+v)]^\alpha}, \qquad g(u) = \frac{u^q}{[\ln(e+u)]^\beta}, \qquad \alpha, \beta > \frac{2}{N-2},$$

and p, q, are in the so called *critical Sobolev hyperbola*

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \tag{2}$$

(see [71] and Theorem 1.1).

Using these *a priori* bounds, and local and global bifurcation techniques, we prove the existence of positive solutions for a corresponding parametrized semilinear elliptic system (see [71] and Theorem 1.3).

If the exponents $\alpha = \beta = 0$, then we have the *Lane-Emden* system

$$\begin{cases} -\Delta u = v^p, & \text{in } \Omega, \\ -\Delta v = u^q, & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is either a bounded, smooth subset, or a half space, or $\mathbb{R}^N$. The exponents (p, q) play a crucial role for the *existence or nonexistence* of positive solutions of Lane-Emdem system, depending on the boundedness or not of Ω. When Ω is a bounded smooth star-shaped domain, the Sobolev hyperbola divides, on the *pq*-plane, *existence* and *nonexistence* of positive solutions of the Lane-Emdem system (see fig. 1; we include the details in Section 2). When $\Omega = \mathbb{R}^N$ there is a *conjecture*: the hyperbola (2) is also the dividing curve between existence and nonexistence for the Lane-Emdem system.
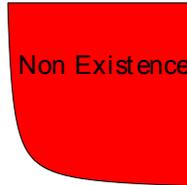
Figure 1. Ω bounded

We now state our main results on elliptic systems. See Section 2, or [71, Theorems 1.1, 1.3] for a proof.

Theorem 1.1 (a-priori L^∞ bounds). *Let us consider the following semilinear elliptic system:*

$$\begin{cases} -\Delta u = \dfrac{v^p}{\big[\ln(e+v)\big]^\alpha}, & \text{in } \Omega, \\ -\Delta v = \dfrac{u^q}{\big[\ln(e+u)\big]^\beta}, & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded, convex domain with boundary $\partial\Omega$ of class C^2. Suppose that $p, q > 1$, $\alpha, \beta > \dfrac{2}{N-2}$, p, q are on the critical Sobolev hyperbola (2), and assume that

$$\min\left\{\frac{p}{\alpha}, \frac{q}{\beta}\right\} \geq \max_{t \geq 0}\left[\frac{t}{(e+t)\ln(e+t)}\right]. \tag{5}$$

Then there exists a uniform constant C, depending only on Ω and p, q, α, β but not on
(u, v), such that

$$\|u\|_{L^\infty(\Omega)} \leq C \quad and \quad \|v\|_{L^\infty(\Omega)} \leq C,$$

for all positive solutions (u, v) of (4).

Remark 1.2. Observe that condition (2) relates the exponents p and q to the Sobolev hyperbola, and condition (5) ensures that our nonlinearities are nondecreasing since it will be needed in the proof. Notice that

$$\max_{t \geq 0} \left[\frac{t}{(e + t)\, \ln(e + t)} \right] = \frac{e}{e + t^*},$$

where t^* is the solution of the logarithmic equation $e \ln(e + t) = t$.

In the next theorem, we state the existence of positive solutions for the semilinear biparameter elliptic systems (4).

Theorem 1.3 (Existence). *Consider the biparameter elliptic system*

$$\begin{cases} -\Delta u = \lambda\, v + \dfrac{v^p}{\left[\, \ln(e + v)\,\right]^\alpha}, & in\ \Omega, \\[2mm] -\Delta v = \mu\, u + \dfrac{u^q}{\left[\, \ln(e + u)\,\right]^\beta}, & in\ \Omega, \\[2mm] u = 0, \quad v = 0 \quad on\ \partial\Omega, \end{cases} \tag{6}$$

where the exponents p, q, α, β are as defined in Theorem 1.1, and the parameters λ and μ are non-negative real parameters.

Then (6) has a positive solution (u, v) if, and only if, $\lambda\mu < \lambda_1^2$, where λ_1 is the principal eigenvalue associated with the linear eigenvalue problem with homogeneous Dirichlet boundary conditions $-\Delta\phi = \lambda\,\phi$ in Ω; $\phi = 0$ on $\partial\Omega$.

Proof. See Section 2, or [71, Theorem 1.3] for a proof.   ☑

The ideas of the proof of Theorem 1.1 are adapted for systems. Moving planes for system provides L^∞ bounds in a neighborhood of the boundary of a convex domain, for classical positive solutions of (1). Rellich-Pohazev identity for systems give us two bounded integrals in Ω. With the help of Morrey's Theorem, we estimate the radius R, (R'), of a ball where the function u, (v), exceeds half of its L^∞ bound. Then, we get a lower bound of the above integrals, deriving the L^∞ bounds for classical positive solutions of (4).

1.2.   The p-**Laplacian**

We consider the Dirichlet problem for positive solutions of the equation $-\Delta_p(u) = f(u)$ in a convex, bounded, smooth domain $\Omega \subset \mathbb{R}^N$, with f locally Lipschitz continuous. We provide sufficient conditions guarantying L^∞ *a priori* bounds for positive solutions of

some elliptic equations involving the p-Laplacian and extend the class of known nonlinearities for which the solutions are L^∞ *a priori* bounded. As a consequence we prove the existence of positive solutions in convex bounded domains.

Specifically, in case of elliptic equations involving the p-Laplacian, we prove the existence of a-priori bounds for $C^{1,\mu}(\overline{\Omega})$ positive solutions of elliptic equations

$$-\Delta_p u = f(u) \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

when

$$f(u) = \frac{u^{p^\star - 1}}{[\ln(e+u)]^\alpha}, \qquad \text{with } p^* = \frac{Np}{N-p}, \quad \text{and } \alpha > \frac{p}{(N-p)},$$

see Example 1 in [31].

Corollary 1.4. *Assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded convex domain, $1 < p < N$, and $u > 0$ is a $C^1(\overline{\Omega})$ solution to*

$$\begin{cases} -\Delta_p u = \dfrac{u^{p^\star - 1}}{[\ln(e+u)]^\alpha}, & in \ \Omega, \\ \qquad u = 0, & on \ \partial\Omega, \end{cases}$$

with $\alpha > p/(N-p)$.

Then, there exists a uniform constant C, depending only on Ω and f but not on the solution, such that $\quad \|u\|_{L^\infty(\Omega)} \le C$.

Proof. It is a Corollary of Theorem 3.5 (see also [31, Theorem 1.7 and Example 1 in p. 491]). ☑

1.3. Final remarks

We finally provide sufficient conditions for a uniform $L^{2^*}(\Omega)$ bound to imply a uniform $L^\infty(\Omega)$ bound, for positive classical solutions to a class of subcritical elliptic problems in bounded C^2 domains. We also establish an equivalent result for sequences of boundary value problems.

We also study the asymptotic behavior of radially symmetric solutions $u_\alpha = u_\alpha(r)$ to the subcritical semilinear elliptic problem

$$(P)_\alpha \quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} \big/ \big[\log(e+u)\big]^\alpha & \text{in } \Omega = B_R(0) \subset \mathbb{R}^N, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

as $\alpha \to 0^+$. We prove that there exists an explicitly defined constant $L(N,R) > 0$, only depending on N and R, such that

$$\limsup_{\alpha \to 0^+} \frac{\alpha u_\alpha(0)^2}{\big[\log(e+u_\alpha(0))\big]^{1+\frac{\alpha(N+2)}{2}}} \ \le \ L(N,R)$$

$$\le \ 2^* \liminf_{\alpha \to 0^+} \frac{\alpha u_\alpha(0)^2}{\big[\log(e+u_\alpha(0))\big]^{\frac{\alpha(N-4)}{2}}},$$
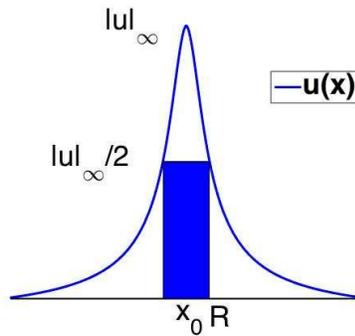
Figure 2. Let (u, v) be a solution of (1); we plot one component u, its L^∞ norm, and the estimate of the radius R such that $u(x) \geq \dfrac{\|u\|_\infty}{2}$ for all $x \in B(x_0, R)$, where x_0 is such that $u(x_0) = \|u\|_\infty$.

see [75].

This paper is organized in the following way. Section 2 is devoted to semilinear elliptic systems. In Section 3 we state an abstract theorem on *a priori* bounds for p-laplacian equations in convex domains. Section 4 contains a result on the equivalence between uniform $L^{2^*}(\Omega)$ *a-priori* bounds and uniform $L^\infty(\Omega)$ *a-priori* bounds for subcritical elliptic equations. In Section 5 we study the asymptotic behavior of radially symmetric solutions $u_\alpha = u_\alpha(r)$ as $\alpha \to 0^+$. In Section 6 we collect some open problems.

2. A priori bounds and existence of positive solutions for semilinear elliptic systems

It is natural to ask whether it is possible to obtain the corresponding *a priori* results for systems. In this Section, we extend the results of [16] from scalar equations to systems. The existence of a-priori bounds for the system (4) is proved in the same lines of [16], that is, using the Rellich-Pohozaev identity and the method of moving planes as in [21], combined with Morrey's Theorem. The moving planes method is used to obtain L^∞ bounds in a neighborhood of the boundary for classical positive solutions of (4), whereas the Rellich-Pohozaev identity is used to get two bounded integrals in Ω. Furthermore, Morrey's Theorem is used to estimate the radius $R, (R')$, of a ball where the function $u, (v)$, exceeds half of its L^∞ bound (see fig. 2), allowing us to reach a contradiction on the lower bounds of the above integrals.

We consider the semilinear elliptic system given by (4) where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded, convex domain with a smooth boundary $\partial\Omega$ (at least of class C^3), and $1 < p, q < \infty$, $\alpha, \beta > 0$. The purpose of this Section is to establish *a-priori* estimates for positive classical solutions of (4) and subsequently prove an existence result for the parametrized version for the system. By a positive classical solution of (4), we mean (u, v) that satisfies (4) and both components are positive. Let us mention that when the

exponents $\alpha = \beta = 0$, we have the system

$$
\begin{cases}
-\Delta u = v^p, & \text{in } \Omega, \\
-\Delta v = u^q, & \text{in } \Omega, \\
u = 0, \quad v = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{7}
$$

that is usually referred to as the Lane-Emden system. This problem arises in modeling spatial phenomena in a variety of biological and chemical problems. Naturally positive solutions of system (7) is of particular interest, and there have been significant studies of positive solutions of (7) where Ω is either a bounded, smooth subset of $\mathbb{R}^N$, a half space, or the entire space $\mathbb{R}^N$ (see [11], [12], [14], [21], [23], [40], [39], [72], [76], [79], [86], [87], [92] and references therein).

It is known that the pair of exponents (p, q) plays a crucial role in the questions of existence and nonexistence of positive solutions of (7). For instance, it has been shown that on a bounded smooth star-shaped domain $\Omega \subset \mathbb{R}^N$, the Sobolev hyperbola (2) is precisely the dividing curve on the pq-plane between existence and nonexistence of positive solutions of (7) (see [21], [72], [76]).

In [21], the authors established *a-priori* estimates and proved the existence of positive solutions of (7) when (p, q) is subcritical (i.e. (p, q) lies below the critical Sobolev hyperbola), that is, $\dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{N-2}{N}$. Moreover, in [72], the author proved that if (p, q) is critical (i.e. (p, q) lies on critical Sobolev hyperbola) or supercritical (i.e. (p, q) lies above the critical Sobolev hyperbola), namely if $\dfrac{1}{p+1} + \dfrac{1}{q+1} \leq \dfrac{N-2}{N}$, then (7) has no positive solution (see fig. 1).

When $\Omega = \mathbb{R}^N$, it has been conjectured that the hyperbola (2) is also the dividing curve between existence and nonexistence for (7). The conjecture has been completely proved for radial positive solutions (see e.g. [72], [85]), that is, if (p, q) is subcritical, then there are no radial positive classical solution to (7) (see [72] for $p > 1$, $q > 1$) and it has been extended in [85] for the case $p > 0$ and $q > 0$. Furthermore, if (p, q) is critical or supercritical, system (7) does admit (bounded) positive radial solutions (see e.g. [72], [85]). In the more general case, i.e. without assuming radial symmetry, the question has not been completely answered yet. Partial answers are known for nonexistence of positive entire solutions of (7) when the pair of exponents are subcritical; for example, it has been proved the nonexistence in certain space dimensions [72], [87] or in certain subregions, below the critical hyperbola in the (p, q)−plane (see e.g. [14], [87]). For $\Omega = \mathbb{R}_+^N$ (i.e. the half space), we refer to [11] for the study of nonexistence of positive solutions. These nonexistence results in $\mathbb{R}^N$ or $\mathbb{R}_+^N$ allow to prove *a-priori* bounds for positive solutions of semilinear elliptic equations in bounded domains via the blow-up method (see e.g. [52], [53], [11], [92]).

In the present Section, we use the method of moving planes and the Rellich-Pohazev identity for systems to establish *a-priori* L^∞ bounds when the pair of exponents (p, q) lies on the critical Sobolev hyperbola (2) and $\alpha, \beta > \dfrac{2}{N-2}$, and then subsequently prove an existence result for the parametrized version for the system (see system (6) below). Problems of type (4) has been considered by several authors (we refer to [37], [22]). In

[37, Theorem 1.3] the authors study the existence of solutions of (4) when the pair of exponents (p, q) lies on the critical Sobolev hyperbola (2) and $\alpha < 0$, $\beta > \dfrac{q+1}{p+1} \, |\alpha|$, that is, one nonlinearity is above the power function. Whereas in [22, Theorem 2.7] the authors study related nonlinearities when the pair of exponents (p, q) lies below the critical Sobolev hyperbola (2), using variational approaches.

Throughout this Section we assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded, convex domain. The hypothesis on convexity of the domain is needed in order to establish *a priori* bounds in a neighborhood of the boundary, via the moving planes method (see Lemma 2.1). In [89, Lemma 4.3] the author develop the moving planes method for systems assuming that both nonlinearities are nondecreasing and do not depend explicitly on the spatial variable x.

Let us first briefly recall the different methods for deriving *a-priori* estimates for elliptic systems in the literature. In [93], the existence of *a-priori* estimates of classical positive solutions of (4) was proved based on scaling or blow-up method which requires the nonlinearities to have a precise asymptotic behavior at infinity, typically $f \sim v^p$, $g \sim u^q$. In [79], [80], the authors derived *a-priori* estimates results using a method close to the Hardy-Sobolev inequalities method. This method requires only upper bounds on the growth of the nonlinearities and allows the nonlinearities to depend on x, u and v, but the growth bounds assumed on f, g are more restrictive than in general.

For elliptic equations on general bounded domains, not necessarily convex, de Figueiredo, Lions and Nussbaum [40] applied the moving planes method on the Kelvin transform in order to avoid the difficulty of an empty *cap* (see [51], [16] for details and the definition of a cap). In that situation, it turns out that the (transformed) nonlinearity depends on the spatial variable x. Then, they obtained *a priori* bounds in a neighborhood of the boundary for classical positive solutions of scalar equations on non-convex domains. To the best of our knowledge, the moving planes method for systems is not yet developed for nonlinearities depending also on the variable x. Hence, we focus on convex domains.

The ideas of the proof of Theorem 1.3 are based on local bifurcation techniques [24], combined with global bifurcation theorem [34], [63], [82], and the *a-priori* estimates. From the seminal works of Crandall and Rabinowitz (see [24], [82]), there are a huge amount of references corresponding to one-parameter bifurcation theory. There are not so many on *multiparameter bifurcation*. Let us mention Alexander and Antman's Theorem [3] on global multiparameter bifurcation techniques, looking for a change of fixed point index, and providing a manifold of solutions of topological dimension at least the number of parameters, [64] on local multiparameter bifurcation techniques on elliptic systems, [48], [63], [64], [65], [66], [67], [68] on combination of local and global multiparameter bifurcation techniques on elliptic systems, and [49] on the multiparameter bifurcation for the p-laplacian.

In this Section, we state two lemmas that are relevant in order to obtain the *a priori* estimates. The first lemma provides L^∞ *a priori* bounds for any positive solution of (4) in a neighborhood of the boundary (see [40]). The hypothesis of convexity of the domain is needed in order to establish these *a priori* bounds in a neighborhood of the boundary. Whereas the second lemma provides a Rellich-Pohozaev-Mitidieri type identity (see [72] for a proof).

Lemma 2.1. *Let (u, v) be a positive classical solution of the system* (4)*. Assume that the hypotheses of Theorem 1.1 are satisfied. Then there exists a constant $\delta > 0$ depending only on Ω and not on p, q, α, β or (u, v), and a constant C depending only on Ω and p, q, α, β, but not on (u, v), such that*

$$\max_{\Omega \setminus \Omega_\delta} u \leq C \quad and \quad \max_{\Omega \setminus \Omega_\delta} v \leq C, \tag{8}$$

where $\Omega_\delta := \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}$.

The proof is done in a similar way as step 2 in the proof of Theorem 1.1 in [40], using the moving planes method for systems [89, Lemma 4.3]. See also step 1 and 2 in the proof of Theorem 2.1 in [21].

Lemma 2.2 (Rellich-Pohozaev-Mitidieri type identity)**.** *Let u and v be in $C^2(\bar{\Omega})$, where Ω is a C^1 domain in $\mathbb{R}^N$, and $u = v = 0$ on $\partial\Omega$. Then,*

$$\int_\Omega \Delta u \, (x \cdot \nabla v) + \Delta v \, (x \cdot \nabla u) = (N - 2) \int_\Omega (\nabla u \cdot \nabla v) + \int_{\partial\Omega} \frac{\partial u}{\partial n} \, (x \cdot \nabla v)$$

$$+ \int_{\partial\Omega} \frac{\partial v}{\partial n} \, (x \cdot \nabla u) - \int_{\partial\Omega} (\nabla u, \nabla v) \, (x \cdot n),$$

where n denotes the exterior normal, and $(x \cdot n)$ denotes the inner product.

The proof of Theorem 1.1 and Theorem 1.3 can be read in [71]; we include it below by completeness.

Proof of Theorem 1.3. Step 1. It follows from (8) and de Giorgi-Nash type Theorems for systems (see [59, Theorem 3.1, p. 397]) that

$$\|(u, v)\|_{C^{0,\alpha}(\Omega_{\delta/8} \setminus \Omega_{7\delta/8})} \leq C, \quad \text{for any} \quad \alpha \in (0, 1),$$

where $\Omega_t := \{x \in \Omega \ : \ d(x, \partial\Omega) > t\}$, and $\|(u, v)\| := \|u\| + \|v\|$. Using Schauder interior estimates (see [54, Theorem 6.2]),

$$\|(u, v)\|_{C^{2,\alpha}(\Omega_{\delta/4} \setminus \Omega_{3\delta/4})} \leq C.$$

Finally, combining L^p estimates with Schauder boundary estimates (see [13], [54], [59]),

$$\|(u, v)\|_{W^{2,p}(\Omega \setminus \Omega_{\delta/2})} \leq C, \quad \text{for any} \quad p \in (1, \infty).$$

By the Sobolev embedding for $p > N$, we have that there exists two constants C, $\delta > 0$, independent of u, such that

$$\|(u, v)\|_{C^{1,\alpha}(\Omega \setminus \Omega_\delta)} \leq C, \quad \text{for any} \quad \alpha \in (0, 1). \tag{9}$$

Step 2. Let $\theta \in (0, 1)$ be such that

$$\frac{1}{p + 1} = \theta \, \frac{N - 2}{N} \quad \text{and} \quad \frac{1}{q + 1} = (1 - \theta) \, \frac{N - 2}{N}, \tag{10}$$

which is possible by (2).

Set $F(t) := \displaystyle\int_0^t f(s)\, ds \qquad$ and $\quad G(t) := \displaystyle\int_0^t g(s)\, ds, \quad$ where

$$f(s) = \frac{s^p}{\big[\ln(e+s)\big]^\alpha}, \qquad g(s) = \frac{s^q}{\big[\ln(e+s)\big]^\beta}.$$

Integrating by parts and taking into account (10) we have that

$$F(t) - \theta\left(\frac{N-2}{N}\right) t\, f(t) = \frac{\alpha}{p+1} \int_0^t \frac{s^{p+1}}{\ln(e+s)^{\alpha+1}} \frac{ds}{e+s}. \tag{11}$$

Likewise

$$G(t) - (1-\theta)\left(\frac{N-2}{N}\right) t\, g(t) = \frac{\beta}{q+1} \int_0^t \frac{s^{q+1}}{\ln(e+s)^{\beta+1}} \frac{ds}{e+s}. \tag{12}$$

Now, if we set $W(s,t) := F(t) + G(s)$, then $W_s = g(s)$ and $W_t = f(t)$. Therefore, for solutions $u > 0$ and $v > 0$ of (4),

$$\int_\Omega -[\Delta u\, (x\cdot\nabla v) + \Delta v\, (x\cdot\nabla u)] = \int_\Omega \sum_j x_j \left(\frac{\partial W}{\partial v}\frac{\partial v_j}{\partial x_j} + \frac{\partial W}{\partial u}\frac{\partial u_j}{\partial x_j}\right)$$

$$= \int_\Omega \sum_j x_j \frac{\partial W}{\partial x_j} = -N\int_\Omega W + \int_\Omega \operatorname{div}(W\,\vec{x})$$

$$= -N\int_\Omega [F(v) + G(u)] + \int_{\partial\Omega} (x\cdot n)\, W(u,v),$$

and

$$\int_\Omega \nabla u \nabla v = (1-\theta)\int_\Omega u g(u) + \theta \int_\Omega v f(v).$$

Applying Lemma 2.2 (Pohozaev-Rellich-Mitidieri type identity) we get that

$$N\int_\Omega [F(v) + G(u)] - (N-2)\int_\Omega [\theta\, v\, f(v) + (1-\theta)\, u\, g(u)]$$

$$= \int_{\partial\Omega} (x\cdot n)\, W(u,v) - \int_{\partial\Omega} (\nabla u \cdot \nabla v)\, (x\cdot n) \tag{13}$$

$$+ \int_{\partial\Omega} \frac{\partial u}{\partial n}\, (x\cdot\nabla v) + \int_{\partial\Omega} \frac{\partial v}{\partial n}\, (x\cdot\nabla u).$$

Therefore, it follows from (13) and (9) that

$$\left| N\int_\Omega [F(v) + G(u)] - (N-2)\int_\Omega [\theta\, v\, f(v) + (1-\theta)\, u\, g(u)] \right| \le C. \tag{14}$$

Using (11), (12), and (14), we obtain that

$$
\left| \frac{\alpha}{p+1} \int_\Omega \left(\int_0^{v(x)} \frac{s^{p+1}}{\ln(e+s)^{\alpha+1}} \frac{ds}{e+s} \right) dx \right.
$$
$$
\left. + \frac{\beta}{q+1} \int_\Omega \left(\int_0^{u(x)} \frac{s^{q+1}}{\ln(e+s)^{\beta+1}} \frac{ds}{e+s} \right) dx \right| \le C.
$$
(15)

Moreover,

$$
\lim_{t\to\infty} \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)} \right)^{\alpha+1} \frac{s^{p+1}}{e+s} ds}{\dfrac{t^{p+1}}{\ln(e+t)^{\alpha+1}}} = \frac{1}{p+1},
$$

and

$$
\lim_{t\to\infty} \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)} \right)^{\beta+1} \frac{s^{q+1}}{e+s} ds}{\dfrac{t^{q+1}}{\ln(e+t)^{\beta+1}}} = \frac{1}{q+1}.
$$

Therefore, for any $\varepsilon > 0$ there exists a constant t_ε, such that if $t > t_\varepsilon$, then

$$
\frac{1}{p+1} - \varepsilon < \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)} \right)^{\alpha+1} \frac{s^{p+1}}{e+s} ds}{\dfrac{t^{p+1}}{\ln(e+t)^{\alpha+1}}}
$$

and

$$
\frac{1}{q+1} - \varepsilon < \frac{\displaystyle\int_0^t \left(\frac{1}{\ln(e+s)} \right)^{\beta+1} \frac{s^{q+1}}{e+s} ds}{\dfrac{t^{q+1}}{\ln(e+t)^{\beta+1}}}.
$$

Let us choose $\varepsilon = \frac{1}{2} \min\left\{ \frac{1}{p+1}, \frac{1}{q+1} \right\}$; then there exists a constant $C > 0$ such that for any $t > 0$,

$$
\frac{t^{p+1}}{\ln(e+t)^{\alpha+1}} \le C \left(1 + \int_0^t \left(\frac{1}{\ln(e+s)} \right)^{\alpha+1} \frac{s^{p+1}}{e+s} ds \right)
$$

and

$$
\frac{t^{q+1}}{\ln(e+t)^{\beta+1}} \le C \left(1 + \int_0^t \left(\frac{1}{\ln(e+s)} \right)^{\beta+1} \frac{s^{q+1}}{e+s} ds \right).
$$

Hence, applying the above inequalities for any $v(x)$ and $u(x)$ solving (4) respectively, integrating in Ω and using (15), we have that

$$\left| \int_\Omega \frac{v^{p+1}}{\ln(e+v)^{\alpha+1}} + \int_\Omega \frac{u^{q+1}}{\ln(e+u)^{\beta+1}} \right| \leq C.$$

This implies that

$$\int_\Omega \frac{vf(v)}{\ln(e+v)} = \int_\Omega \frac{v^{p+1}}{\ln(e+v)^{\alpha+1}} \leq C,$$
$$\int_\Omega \frac{ug(u)}{\ln(e+u)} = \int_\Omega \frac{u^{q+1}}{\ln(e+u)^{\beta+1}} \leq C. \tag{16}$$

As pointed out in Remark 1.2, condition (5) ensures that f and g are nondecreasing.

Step 3. Now, let us fix $r, r' \in (\frac{N}{2}, N)$ such that $r \geq 1 + \dfrac{1}{p}$ and $r' \geq 1 + \dfrac{1}{q}$. Then using (16) and the fact that f and g are nondecreasing for $\|v\|_\infty$ large enough, we get that

$$\int_\Omega |f(v)|^r = \int_\Omega \frac{v^{pr}}{\ln(e+v)^{\alpha r}}$$
$$= \int_\Omega \frac{v^{pr-p-1}}{\ln(e+v)^{\alpha(r-1)-1}} \frac{v^{p+1}}{\ln(e+v)^{\alpha+1}}$$
$$\leq C \frac{\|v\|_\infty^{pr-(p+1)}}{\ln(e+\|v\|_\infty)^{\alpha(r-1)-1}}$$
$$= C \left(\frac{\|v\|_\infty^p}{\ln(e+\|v\|_\infty)^\alpha} \right)^{r-1-\frac{1}{p}} \frac{1}{\ln(e+\|v\|_\infty)^{\frac{\alpha}{p}-1}},$$

and similarly

$$\int_\Omega |g(u)|^{r'} \leq C \left(\frac{\|u\|_\infty^q}{\ln(e+\|u\|_\infty)^\beta} \right)^{r'-1-\frac{1}{q}} \frac{1}{\ln(e+\|u\|_\infty)^{\frac{\beta}{q}-1}}.$$

All the other arguments work as in part I, Theorems 2.1, 2.2 (see also [71, proof of Theorems 1.1 and 1.3]). ☑

Proof of Theorem 1.3. The proof is divided in two parts. We first prove that if there exists a positive solution $\big((\lambda,\mu),(u,v)\big)$ of equation (6) with $\lambda, \mu \geq 0$, then $\lambda\mu < \lambda_1^2$. In the second part, we prove the converse.

Part I. Assume that there exists a positive solution $\big((\lambda,\mu),(u,v)\big)$ of equation (6), and that $\lambda, \mu \geq 0$. Let $\phi_1 > 0$ be the principal eigenfunction associated to λ_1 and normalized in the $L^2(\Omega)$ norm. Multiplying each equation of (6) by ϕ_1, and integrating by parts on Ω, it yields that

$$\lambda_1 \int_\Omega u\phi_1 = \lambda \int_\Omega v\phi_1 + \int_\Omega f(v)\,\phi_1,$$
$$\lambda_1 \int_\Omega v\phi_1 = \mu \int_\Omega u\phi_1 + \int_\Omega g(u)\,\phi_1.$$

Multiplying the first equation by λ_1, the second equation by λ and adding both equations we deduce

$$(\lambda_1^2 - \lambda\mu) \int_\Omega u\,\phi_1 = \int_\Omega [\lambda_1 f(v) + \lambda g(u)]\,\phi_1 > 0.$$

Thus, $\lambda\mu < \lambda_1^2$.

Part II. Assume that $\lambda, \mu \geq 0$ and $\lambda\mu < \lambda_1^2$; we will prove that there is a positive solution (u, v) of (6). The proof is divided in three steps. In step 1, we reformulate problem (6) in abstract (operators) setting. In step 2, we fix one parameter, say, $\lambda = \lambda_0 > 0$, and choosing μ as bifurcation parameter, we use Crandall-Rabinowitz's Theorem to prove that when $\sqrt{\lambda_0\mu} = \lambda_1$ there is a bifurcation phenomena from the trivial solution. Moreover, at least locally, the nontrivial solution pairs (u, v) are strictly positive. In step 3 we use the global bifurcation result stated by Rabinowitz [82] and completed by Dancer [34] (see also [35], [63]) to prove that equation (6) has at least one solution. We conclude with a remark on the fact that varying λ we obtain a whole curve of non-isolated bifurcation points, and using Alexander and Antman's result [3], we can deduce that in a neighborhood of that curve, there is a bifurcating two-dimensional surface of nontrivial solution pairs $\big((\lambda, \mu), (u, v)\big)$ of equation (6).

Step1. We start by reformulating problem (6).

Let $\overline{f}$ and $\overline{g}$ be the extension of f and g, defined by

$$\overline{f}(t) = \frac{|t|^p}{\big[\ln(e + |t|)\big]^\alpha}, \qquad \overline{g}(s) = \frac{|s|^q}{\big[\ln(e + |s|)\big]^\beta}, \qquad \text{for} \quad s, t \leq 0,$$

and denote by $F(w) := \begin{pmatrix} \overline{f}(v) \\ \overline{g}(u) \end{pmatrix}$. Then (6) can be extended to non-positive and changing sign solutions and can be rewritten

$$-\Delta w = Aw + F(w), \quad \text{in } \Omega, \qquad w = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial\Omega, \tag{17}$$

where $A := \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$, $w := \begin{pmatrix} u \\ v \end{pmatrix}$, and any positive solution (u, v) of (17) is a positive solution of (6), and conversely.

Following the same ideas used in the proof of Theorem 1.1, it can be easily checked that for any $[a, b] \times [c, d] \subset \mathbb{R}_+^2$, there exists a constant $C > 0$ such that any positive solution (u, v) of equation (17) satisfies

$$\|u\|_{L^\infty(\Omega)} \leq C, \quad \|v\|_{L^\infty(\Omega)} \leq C, \qquad \forall\,(\lambda, \mu) \in [a, b] \times [c, d]. \tag{18}$$

Assume that $\lambda, \mu > 0$ and consider the Jordan canonical form of the matrix A. We can decompose $A = P^{-1}JP$, where

$$J = \begin{pmatrix} \sqrt{\lambda\mu} & 0 \\ 0 & -\sqrt{\lambda\mu} \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{\mu}{\lambda}} & 1 \\ -\sqrt{\frac{\mu}{\lambda}} & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \sqrt{\frac{\lambda}{\mu}} & -\sqrt{\frac{\lambda}{\mu}} \\ 1 & 1 \end{pmatrix}.$$

Multiplying (17) by P on the left and denoting by $z = Pw$, we obtain

$$-\Delta z = Jz + G(z), \quad \text{in } \Omega, \qquad z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial\Omega,$$

where $G(z) := PF(w) = PF(P^{-1}z)$.

Step 2. We check that the conditions of Crandall-Rabinowitz's Theorem [24] are satisfied. Fix $\lambda = \lambda_0 > 0$, and choose μ as the bifurcation parameter. Let $\sigma \in (0,1)$; define $E_2 = \{u \in C^{2,\sigma}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ equipped with its standard norm; E_2 is a Banach space. Set $E_0 = C^\sigma(\overline{\Omega})$ and define the operator $\mathbf{F} : \mathbb{R} \times (E_2)^2 \to (C^\sigma(\overline{\Omega}))^2$ by

$$\mathbf{F}(\mu, w) \;:=\; -\Delta w - \begin{pmatrix} 0 & \lambda_0 \\ \mu & 0 \end{pmatrix} w - F(w);$$

$$L_0 w \;:=\; D_{(u,v)}\mathbf{F}(\mu_0, 0)\, w = -\Delta w - \begin{pmatrix} 0 & \lambda_0 \\ \mu_0 & 0 \end{pmatrix} w,$$

$$L_1 w \;:=\; D_{\mu,(u,v)}\mathbf{F}(\mu_0, 0)\, w = \begin{pmatrix} 0 \\ -v. \end{pmatrix}, \qquad \text{where} \quad w = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Set $\mu_0 = \frac{\lambda_1^2}{\lambda_0}$, and $P_0 = P(\lambda_0, \mu_0)$. Observe that $w \in N(L_0)$ (where $N(L_0)$ is the kernel of L_0) if, and only if, $z = P_0 w \in N\left(-\Delta - \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}\right) = span\left[\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}\right]$. Therefore, $N(L_0) = span\left[\begin{pmatrix} \lambda_0\phi_1 \\ \lambda_1\phi_1 \end{pmatrix}\right]$.

Now, we claim that $L_1(N(L_0)) \not\subset R(L_0)$, where $R(L_0)$ is the range of L_0.

Indeed, assume that there exist $w \in N(L_0)$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2. \end{pmatrix} \in D(L_0)$ such that $L_1 w = L_0 \psi$ or, equivalently by definition of L_0 and L_1,

$$-\Delta\psi - \begin{pmatrix} 0 & \lambda_0 \\ \mu_0 & 0 \end{pmatrix} \psi = a \begin{pmatrix} 0 \\ -\lambda_1\phi_1 \end{pmatrix}, \qquad \text{for some} \quad a \in \mathbb{R}. \tag{19}$$

Multiplying (19) on the left by P_0, and denoting by $\varphi = P_0\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2. \end{pmatrix}$, we obtain

$$-\Delta\varphi - \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix} \varphi = -\frac{a\lambda_1}{2} \begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix}.$$

Multiplying the first (component) equation by ϕ_1, integrating on Ω, and applying Green's formulae we obtain

$$0 = \int_\Omega (-\Delta\varphi_1 - \lambda_1\varphi_1)\phi_1 = -\frac{a\lambda_1}{2} \int_\Omega \phi_1^2.$$

Therefore $a = 0$. Hence, the hypotheses of Crandall Rabinowitz theorem are satisfied. Thus, there exists a neighborhood of $\left(\frac{\lambda_1^2}{\lambda_0}, (0,0)\right)$ in $\mathbb{R} \times (E_2)^2$, and continuous functions

$\mu(s), \tilde{w}(s), \ s \in (-\epsilon, \epsilon)$, such that $\mu(0) = \mu_0$, $\tilde{w}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}$, with $\int_\Omega \tilde{w}_i \phi_1 = 0$, and the only nontrivial solutions of (6) for $\lambda = \lambda_0$ fixed, are

$$\left(\mu(s), s \begin{pmatrix} \lambda_0 \phi_1 \\ \lambda_1 \phi_1 \end{pmatrix} + s\, \tilde{w}(s) \right).$$

Observe that for $s > 0$ small enough, $w = \begin{pmatrix} u \\ v \end{pmatrix}$ satisfies $u > 0, v > 0$, $\frac{\partial u}{\partial n} < 0$, $\frac{\partial v}{\partial n} < 0$ on $\partial\Omega$, hence $\lambda_0 \mu < \lambda_1^2$.

Step 3. Now, we use the global bifurcation Theorem as stated by Rabinowitz [82] and as completed by Dancer [34]. Let $(-\Delta)^{-1}$ denote the inverse of $(-\Delta)$ with homogeneous Dirichlet boundary conditions. It follows from Schauder estimates that $(-\Delta)^{-1}$ maps bounded subsets of E_0 into bounded subsets of E_2, which in turn are relatively compact in E_0. Thus, $(-\Delta)^{-1} : E_0 \to E_0$ is compact.

Observe that, fixed points of the operator $(-\Delta)^{-1}[J(.)+G(.)]$ corresponds to fixed points of the operator $(-\Delta)^{-1}[A(.) + F(.)]$, that is,

$$z = (-\Delta)^{-1}[Jz + G(z)] \iff w = (-\Delta)^{-1}[Aw + F(w)].$$

Let us keep fixed $\lambda = \lambda_0 > 0$, and allow μ to vary. It follows from Rabinowitz's global bifurcation Theorem [82, Theorem 1.3] that there is a continuum of solutions, emanating from the trivial solution at $(\lambda, \mu) = \left(\lambda_0, \frac{\lambda_1^2}{\lambda_0} \right)$, which is either unbounded, or meets another bifurcation point from the trivial solution. Let

$$\mathcal{C}_{\lambda_0} := \left\{ \left((\lambda_0, \mu), (u_{\lambda_0, \mu}, v_{\lambda_0, \mu}) \right) \in \mathbb{R}^2 \times \left(C^\sigma(\overline{\Omega}) \right)^2 \right\}$$

be the continuum emanating from the trivial solution at $\mu = \mu_0 = \frac{\lambda_1^2}{\lambda_0}$, and solving (6) for $\lambda = \lambda_0$ fixed. By elliptic regularity, it is known that $\mathcal{C}_{\lambda_0} \subset \mathbb{R}^2 \times \left(C^{2,\sigma}(\overline{\Omega}) \right)^2$.

Considering the positive cone $\mathcal{P} := \{ u \in C^{1,\sigma}(\overline{\Omega}) : u > 0, \ \text{in } \Omega, \ \frac{\partial u}{\partial n} < 0, \ \text{on } \partial\Omega \}$, let us denote by $\mathcal{C}_{\lambda_0}^+ := \mathcal{C}_{\lambda_0} \cap \mathbb{R}^2 \times (\mathcal{P})^2 \neq \emptyset$. Since the classical positive solutions are *a priori* bounded (see (18)), we have that $\mathcal{C}_{\lambda_0}^+ \cap \left(\{\lambda_0\} \times \left[0, \frac{\lambda_1^2}{\lambda_0} \right) \times \left(C^\sigma(\overline{\Omega}) \right)^2 \right)$ is bounded. Assume that there exists $\mu = \mu^* \in \left[0, \frac{\lambda_1^2}{\lambda_0} \right)$ such that $\left((\lambda_0, \mu^*), (u^*, v^*) \right) \in \overline{\mathcal{C}_{\lambda_0}^+} \setminus \mathcal{C}_{\lambda_0}^+$; then, either $(u^*, v^*) = (0, 0)$ or $u^* \geq 0, v^* \geq 0$ in Ω, $\frac{\partial u^*}{\partial n} \leq 0, \frac{\partial v^*}{\partial n} \leq 0$ on $\partial\Omega$, with $(u^*, v^*) \neq (0, 0)$. If $(u^*, v^*) = (0, 0)$ then $\left((\lambda_0, \mu^*), (u^*, v^*) \right)$ is a bifurcation point from the trivial solution to positive solutions. Due to the unique bifurcation point from the trivial solution to positive solutions at $\lambda = \lambda_0$ is attained at $\mu = \frac{\lambda_1^2}{\lambda_0}$, if $(u^*, v^*) = (0, 0)$ then we reach a contradiction. On the other hand, if $u^* \geq 0, v^* \geq 0$ in Ω, $(u^*, v^*) \neq (0, 0)$, from the Maximum Principle and the Hopf Maximum Principle $u^* > 0$, $v^* > 0$ in Ω, and $\frac{\partial u^*}{\partial n^*} < 0$, $\frac{\partial v^*}{\partial n} < 0$; therefore $\left((\lambda_0, \mu^*), (u^*, v^*) \right) \in \mathcal{C}_{\lambda_0}^+$, which contradicts the hypothesis. ☑

Remark 2.3. Let us mention that when moving λ we obtain a whole curve of non-isolated bifurcation points. If $\mathcal{S}$ denote the closure of the set of nontrivial solutions pairs

$\big((\lambda,\mu),w\big)$ of (17), and $\mathcal{F}$ denote the set of bifurcation points of (17) from the trivial solution, we proved in Step 2 that the set

$$\mathcal{F}_1 := \left\{ \left(\left(\lambda, \frac{\lambda_1^2}{\lambda}\right), (0,0) \right) : \lambda > 0 \right\}$$

is a set of bifurcation points of (17) from the trivial solution. All points in $\mathcal{F}_1$ are non-isolated bifurcation points. Using Alexander and Antman's result [3], we can deduce that in a neighborhood of that curve, there is a bifurcating two-dimensional surface of nontrivial solution pairs $\big((\lambda,\mu),(u,v)\big)$ of equation (6).

3. A priori estimates for quasilinear elliptic equations involving the p-Laplacian

Let Ω be a smooth, bounded, and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$. We prove $L^\infty(\Omega)$ *a priori* bounds for $C^1(\overline{\Omega})$ weak solutions of the problem

$$\begin{cases} -\Delta_p(u) = f(u) & \text{in } \Omega, \\ \quad\;\; u > 0 & \text{in } \Omega, \\ \quad\;\; u = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

where $\Delta_p(u) = \operatorname{div}(|Du|^{p-2} Du)$ is the p-Laplace operator, $1 < p < \infty$, and

(H_1) $f : [0,\infty) \to \mathbb{R}$ is a locally Lipschitz continuous function with $f(0) \geq 0$.

   If $p > 2$ we also assume that $f(s) > 0$ if $s > 0$.

The equation $-\Delta_p u = f(u)$ is the L^p counterpart to the classical semilinear elliptic equation $-\Delta u = f(u)$, and appears for instance in the theory of non-Newtonian fluids, dilatant fluids in the case $p \geq 2$, pseudo-plastic fluids in the case $1 < p < 2$ (see [6], [69], [70]).

If $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of the problem (20), then $u \in C^{1,\tau}(\overline{\Omega})$ with $\tau < 1$ (see [41], [60]), so we assume from the beginning a C^1 regularity for the solution, which is in general only a *weak* solution. A standard setting in the applications of *a priori* estimates to existence of solutions to elliptic problems, is the space of continuous functions. If $u \in C^0(\overline{\Omega})$, then also $f(u)$ is continuous and the solution u belongs to the space $C^{1,\tau}(\overline{\Omega})$, by the cited regularity results.

Mitidieri and Pohozaev in [73], [74] proved Liouville theorems for quasilinear elliptic inequalities in $\mathbb{R}^N$ involving the p-laplacian. Later, Serrin and Zou [86], and also Farina and Serrin [42], [43], proved Liouville theorems for more general operators.

With the help of the blow-up procedure, Azizieh and Clement in [9] proved *a priori* estimates for the equation (20) when Ω is strictly convex, $1 < p < 2$ and $f(u)$ growing not faster than a power u^ϑ at infinity, with

$$1 < \vartheta < p_* - 1 \,, \quad p_* = \frac{(N-1)p}{N-p}$$

(see [10] for the case of systems). The exponent $p_* = \frac{(N-1)p}{N-p}$ is the optimal exponent for Liouville theorems for elliptic *inequalities*; observe that

$$p_* = \frac{(N-1)p}{N-p} \ < \ p^* = \frac{Np}{N-p},$$

where p^* is the critical exponent for the Sobolev's embeddings. The restriction $1 < p < 2$ depends on the fact that using a blow-up procedure and Liouville theorems on the whole space, they need to exclude concentration of maximum points of the solutions at the boundary, and they use some result proved in [29], [30] on the symmetry and monotonicity of solutions to p-Laplace equations in the singular case $1 < p < 2$; these results were later extended to the case $p > 2$ in the papers [32], [33].

Ruiz [83] proved *a priori* estimates for equation more general than (20), using a different technique based among other tools on Harnack type inequalities; therein $f = f(x, u, Du)$ can depend on x and on the gradient, for any $1 < p < N$ and for general domains. Once again the growth at infinity with respect to u must be less than powers with exponent $\vartheta < p_* - 1$.

In both papers, there is also a general discussion on how the existence of solutions follows from the *a priori* estimates, using some abstract results by Krasnoselskij already used in [40].

Later, Zou [92] proved Liouville theorems in half spaces that, together with the results in [84], allow him to use the blow-up method and prove *a priori* estimates for equations more general than (20), in case $1 < p < N$; therein $f = f(x, u, Du)$ can depend on x and on the gradient, and under various hypotheses on the nonlinearities. In particular, it is assumed that $f = f(x, u, Du)$ grows with respect to u as a subcritical power at infinity and zero.

For monotonicity and Liouville type theorems in half spaces many other papers appeared recently (see in particular the papers of Farina, Montoro, and Sciunzi [44], [45], [46], and Farina, Montoro, Riey, and Sciunzi [47]).

In recent years, related *a priori* estimates for general operators were established by D'Ambrosio and Mitidieri in [26], [27].

The aim of this Section is to state *a priori* estimates for solutions of (20) in the case of Ω being a smooth bounded strictly convex domain, for any value of $p > 1$. In case $1 < p < N$, $f(u)$ is assumed to have a subcritical grow at infinity, but allowing more general functions than merely subcritical powers (see for example Corollary 1.4).

We adapt the technique introduced in [40] for the case $p = 2$, that allows to give the same proof in cases $1 < p < N$, $p = N$ and $p > N$. Of course, in the latter cases, much weaker hypotheses are needed in order to obtain the desired estimates (in particular in case $p > N$ we only need that $f(u)$ grows faster than u^{p-1} at infinity, condition that for $p = 2$ is the superlinearity at infinity). This proofs rely deeply on the C^2 regularity of solutions, which are then classical solutions, and on the $W^{2,q}$ estimates based on the Calderón-Zygmund and Agmon-Douglas-Nirenberg estimates [1], [2]. These estimates are not available in the singular/degenerate case $p \neq 2$. We use instead regularity results for the p-laplacian (see [41], [60], [88], and gradient estimates [57], [19], [20]). There are many other tools

that we had to adapt to handle in our case. We use Strong Maximum Principle and Hopf's Lemma for the p-Laplacian (see [91]), weak and strong comparison principles for the p-Laplacian (see [25], [28], [32], [33], [55], [78]), Picone's identity for the p-Laplacian ([4]), Pohozaev's identity for the p-Laplacian ([55]). Although the general theory of eigenvalues for the p-Laplacian is far from complete (see [50], [77]), the properties of the first eigenvalue are known and are the same as in the case $p = 2$ (see [5], [77], [61]).

All the following results are already known (see [31]):

Theorem 3.1 (Case $p > N$). *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex. Assume that $p > N$, the condition (H_1) holds and*

(H_2) *there exist $\tau > 0$ and $C_1 > 0$ such that*

$$\liminf_{s \to +\infty} \frac{f(s)}{s^{p-1+\tau}} > C_1 > 0.$$

Then, the solutions of (20) are a priori bounded in L^∞: there exists a constant C, depending on p, Ω and f, but independent of the solution u, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (20).

Proof. See [31, Proof of Theorem 1.1] for a proof.   ☑

Remark 3.2. For the ordinary laplacian ($p = 2$), the above theorem corresponds to the case of dimension $N = 1$. In [40, Remark 1.3] it was observed that if $N = 1$, solutions are uniformly bounded under the only hypothesis of superlinearity at infinity, which corresponds for $p = 2$ to the hypothesis (H_2) with $\tau = 0$, and the bound from below strictly bigger than λ_1, the first eigenvalue for the Laplacian operator.

We need the slightly stronger form (with $\tau > 0$ but arbitrarily small) for technical reasons (use of the Picone's Identity for the p-laplacian).

Theorem 3.3 (Case $p = N$). *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex. Let $p = N$ and assume that (H_1) and (H_2) hold, as well as*

(H_3) *there exists $C_2 > 0$ such that* $\displaystyle \liminf_{s \to +\infty} \frac{F(s)}{sf(s)} > C_2 > 0$, *where $F(t) = \int_0^t f(s)\,ds$ is a primitive of the function f;*

(H_4) *there exists $\theta > 0$ such that* $\displaystyle \limsup_{s \to +\infty} \frac{|f(s)|}{s^\theta} < +\infty$ *(of course this is equivalent to:*

   there exists $\eta > 0$ such that $\displaystyle \lim_{s \to +\infty} \frac{|f(s)|}{s^\eta} = 0$ *).*

Then, the solutions of (20) are a priori bounded in L^∞: there exists a constant C, depending on p, Ω and f, but independent of the solution u, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (20).

Proof. See [31, Proof of Theorem 1.3] for a proof.   ☑

Remark 3.4. If $p = 2$ (the ordinary laplacian), the above theorem corresponds to the case of dimension $N = 2$, and in that case solutions are uniformly bounded under the only hypotheses of superlinearity together with polynomial growth at infinity (cf. [40, Theorem 1.1]). All the functions f growing polynomially at infinity are included in hypotheses (H_3), and (H_4).

Nevertheless, when $p = N$ the critical embedding is of exponential type, and those hypotheses are not optimal (neither are the hypotheses in [40]), and we think that they can be improved.

The following result can be seen as the counterpart for $p \neq 2$ to the results above (see [31, Proof of Theorem 1.7] for a proof).

Theorem 3.5 (Case $1 < p < N$)**.** *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is strictly convex.*

Let $1 < p < N$, and assume that (H_1) and (H_2) hold, as well as

$(H_3")$ *there exist a nonincreasing positive function $H : [0, +\infty) \to \mathbb{R}$ such that*

$$\liminf_{s \to +\infty} \frac{p^* F(s) - s f(s)}{s f(s) H(s)} > 0,$$

     where $p^* = \frac{Np}{N-p}$, and

$(H_4")$ $\displaystyle \lim_{s \to +\infty} \frac{f(s)}{s^{p^*-1} [H(s)]^{\frac{p}{N-p}}} = 0;$

(H_5) *there exist $C_4 > 0$, $C_5 > 0$ such that*

$$\liminf_{s \to +\infty} \frac{\min_{[\frac{s}{2}, s]} f}{f(s)} \geq C_4 > 0, \qquad \limsup_{s \to +\infty} \frac{\max_{[0, s]} f}{f(s)} \leq C_5.$$

Then, the solutions of (20) are a priori bounded in L^∞: there exists a constant C, depending on p, Ω and the function f, but independent of the solution u, such that $\|u\|_{L^\infty(\Omega)} \leq C$ for any solution of (20).

The existence of solutions for (20) follows from the *a priori* estimates, with a further hypothesis about the behavior of the nonlinearity at zero.

This was proved in [40] (with the hypothesis (H_0) below) for $p = 2$, using some variants of topological arguments, connected with theorems of Krasnoselskii [58] and Rabinowitz [81] based on degree theory. It was extended to the case of p-Laplace equations in [9], [83], [92].

It also can be adapted to our hypotheses. More precisely, the following result holds (see [31, Proof of Theorem 1.8] for a proof).

Theorem 3.6. *Let us assume that the hypotheses of one of the previous theorem hold, and assume also that*

(H_0) $\limsup\limits_{s\to 0^+} \dfrac{f(s)}{s^{p-1}} < \lambda_{1,p}$,

where $\lambda_{1,p}$ is the first eigenvalue for the p-Laplacian (see [31, Section 2]). (Since $f(0) \geq 0$ by (H_1), this hypothesis implies that $f(0) = 0$).

Then, there exists a positive solution of (20).

4. Equivalence between uniform $L^{2^\star}(\Omega)$ a-priori bounds and uniform $L^\infty(\Omega)$ a-priori bounds for subcritical elliptic equations

We consider the existence of $L^\infty(\Omega)$ *a-priori* bounds for classical positive solutions to the boundary value problem

$$-\Delta u = f(u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{21}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with C^2 boundary $\partial\Omega$. Trudinger [90] proved that any weak solution in $H^1(\Omega)$ is in fact in $L^\infty(\Omega)$ and consequently in $C^\infty(\Omega)$.

We provide sufficient conditions on f for uniform $L^{2^*}(\Omega)$ *a-priori* bounds to imply uniform $L^\infty(\Omega)$ a-priori bounds, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. The converse is obviously true without any additional hypotheses.

The ideas for the proof of the following theorem are similar to those used in [16, Theorem 1.1] (see also part I, Theorems 2.1, 2.2). Unlike the proof in [16], here we do not use Pohozaev or moving planes arguments.

Our main result is the following theorem.

Theorem 4.1. *Assume that the nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}$ is a locally Lipschitzian function that satisfies:*

(H1) *There exists a constant $C_0 > 0$ such that* $\liminf\limits_{s\to\infty} \dfrac{\min\limits_{[s/2,s]} f}{f(s)} \geq C_0$.

(H2) *There exists a constant $C_1 > 0$ such that* $\limsup\limits_{s\to\infty} \dfrac{\max\limits_{[0,s]} f}{f(s)} \leq C_1$.

(F) $\lim\limits_{s\to+\infty} \dfrac{f(s)}{s^{2^\star-1}} = 0$; *that is, f is subcritical.*

Then the following conditions are equivalent:

(i) *there exists a uniform constant C (depending only on Ω and f) such that for every positive classical solution u of* (21),

$$\|u\|_{L^\infty(\Omega)} \leq C;$$

(ii) *there exists a uniform constant C (depending only on Ω and f) such that for every positive classical solution u of (21),*

$$\int_\Omega |f(u)|^{\frac{2N}{N+2}}\,dx \leq C;$$

(iii) *there exists a uniform constant C (depending only on Ω and f) such that for every positive classical solution u of (21),*

$$\|u\|_{L^{2^*}(\Omega)} \leq C.$$

Proof. All the arguments work in a similar way as in part I, Theorems 2.1, 2.1 (see also [15, proof of Theorem 1.1]). ☑

In this Section, we also provide sufficient conditions for the equivalence of the existence of a uniform $L^{2^*}(\Omega)$ *a priori* bound with that of a uniform $L^\infty(\Omega)$ *a priori* bound for sequences of boundary value problems. In fact, we have the following theorem, which is [15, Theorem 1.2]; we include the proof by the sake of completeness.

Theorem 4.2. *Consider the following sequence of BVPs*

$$-\Delta v = g_k(v) \quad in\ \Omega, \qquad v = 0 \quad on\ \partial\Omega, \tag{21}_k$$

with $g_k : \mathbb{R}^+ \to \mathbb{R}$ locally Lipschitzian. We assume that the following hypotheses are satisfied:

$(H1)_k$ *there exists a uniform constant $C_1 > 0$ such that* $\displaystyle\liminf_{s\to+\infty} \frac{\min\limits_{[s/2,s]} g_k}{g_k(s)} \geq C_1;$

$(H2)_k$ *there exists a uniform constant $C_2 > 0$ such that* $\displaystyle\limsup_{s\to+\infty} \frac{\max\limits_{[0,s]} g_k}{g_k(s)} \leq C_2.$

Let $\{v_k\}$ be a sequence of classical positive solutions to $(21)_k$ for $k \in \mathcal{N}$. If

$(F)_k$ $\displaystyle\lim_{k\to+\infty} \frac{g_k(\|v_k\|)}{\|v_k\|^{2^*-1}} = 0,$

then, the following two conditions are equivalent:

(i) *there exists a uniform constant C, depending only on Ω and the sequence $\{g_k\}$, but independent of k, such that for every $v_k > 0$, classical solution to $(21)_k$,*

$$\limsup_{k\to\infty} \|v_k\|_{L^\infty(\Omega)} \leq C;$$

(ii) *there exists a uniform constant C_0, depending only on Ω and the sequence $\{g_k\}$, but independent of k, such that for every $v_k > 0$, classical solution to $(21)_k$,*

$$\limsup_{k\to\infty} \int_\Omega |g_k(v_k)|^{\frac{2N}{N+2}}\,dx \leq C_0; \tag{22}$$

(iii) there exists a uniform constant C (depending only on Ω and the sequence $\{g_k\}$) such that for every positive classical solution v_k of $(21)_k$,

$$\|v_k\|_{L^{2^*}(\Omega)} \leq C.$$

Hypothesis $(H1)_k$, and $(H2)_k$, are not sufficient for the existence of a uniform L^∞ *a priori* bound. Atkinson and Peletier in [8] show that for $f_\varepsilon(s) = s^{2^\star-1-\varepsilon}$ and Ω a ball in $\mathbb{R}^3$, there exists $x_0 \in \Omega$ and a sequence of solutions u_ε such that $\lim_{\varepsilon\to 0} u_\varepsilon = 0$ in $C^1(\Omega \setminus \{x_0\})$ and $\lim_{\varepsilon\to 0} u_\varepsilon(x_0) = +\infty$. See also Han [56], for minimum energy solutions in general domains.

Furthermore, hypotheses $(H1)_k$, $(H2)_k$, and $(F)_k$, are not sufficient for the existence of an L^∞ *a priori* bound. In fact, at the end of this Section, we construct a sequence of BVP satisfying $(H1)_k$, $(H2)_k$, and $(F)_k$, and a sequence of solutions v_k such that $\lim_{k\to\infty} \|v_k\|_\infty = +\infty$. Our example also shows the non-uniqueness of positive solutions.

Remark 4.3. One can easily see that condition (22), elliptic regularity and Sobolev embeddings imply that there exists a uniform constant $C_4 > 0$ such that

$$\limsup_{k\to\infty} \int_\Omega v_k\, g_k(v_k)\, dx \leq C_4, \tag{23}$$

for all classical positive solutions $\{v_k\}$ to $(21)_k$.

The ideas for the proof of the above theorem are similar to those used in [16, Theorem 1.1] (see also Theorems 2.1, 2.1 in part I). But, unlike the proof in [16], here we do not use Pohozaev or moving planes arguments; therefore, the structure of the proof described in Subsection 1.1 of part I will start in one adaptation of Step 3.

Proof of Theorem 4.2. Clearly, condition (i) implies (ii) and (iii). By the elliptic regularity and condition (22), we have that $\|v_k\|_{W^{2,\frac{2N}{N+2}}} \leq C$. Therefore, $\|v_k\|_{H^1(\Omega)} \leq C$. So, by the Sobolev embedding, we deduce that

$$\|v_k\|_{L^{2^*}(\Omega)} \leq C \quad \text{for all } k.$$

Using similar arguments as in Theorems 2.1, 2.1 of part I and condition $(F)_k$, one can show that (ii) and (iii) are equivalent. We shall concentrate our attention in proving that (ii) implies (i). All throughout this proof C denotes several constants independent of k.

Observe that $1 + \frac{1}{2^\star-1} = \frac{2N}{N+2}$. From hypothesis (ii) (see (22)), there exists a fixed constant $C > 0$ (independent of k) such that

$$
\begin{aligned}
\int_\Omega \left|g_k\big(v_k(x)\big)\right|^q dx &\leq \int_\Omega \left|g_k\big(v_k(x)\big)\right|^{1+\frac{1}{2^\star-1}} \left|g_k\big(v_k(x)\big)\right|^{q-1-\frac{1}{2^\star-1}} dx \\
&\leq C \left\|g_k\big(v_k(\cdot)\big)\right\|_\infty^{q-1-\frac{1}{2^\star-1}},
\end{aligned}
\tag{24}
$$

for k big enough, and for any $q > N/2$.

Therefore, from elliptic regularity, (see [54, Lemma 9.17]),

$$\|v_k\|_{W^{2,q}(\Omega)} \le C \|\Delta v_k\|_{L^q(\Omega)} \le C \; \left\|g_k\big(v_k(\cdot)\big)\right\|_\infty^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}} \tag{25}$$

for k big enough.

Let us restrict $q \in (N/2, N)$. From Sobolev embeddings, for $1/q^* = 1/q - 1/N$ with $q^* > N$, we can write

$$\|v_k\|_{W^{1,q^*}(\Omega)} \le C \|v_k\|_{W^{2,q}(\Omega)} \le C \; \left\|g_k\big(v_k(\cdot)\big)\right\|_\infty^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}} \, ,$$

for k big enough.

From Morrey's Theorem (see [13, Theorem 9.12 and Corollary 9.14]), there exists a constant C only dependent on Ω, q and N such that

$$|v_k(x_1) - v_k(x_2)| \le C|x_1 - x_2|^{1-N/q^*}\|v_k\|_{W^{1,q^*}(\Omega)}, \quad \forall x_1, x_2 \in \Omega,$$

for any k.

Therefore, for all $x \in B(x_1, R) \subset \Omega$,

$$|v_k(x) - v_k(x_1)| \le C \; R^{2-\frac{N}{q}}\|v_k\|_{W^{2,q}(\Omega)}, \tag{26}$$

for any k.

From now on, we argue by contradiction. Let $\{v_k\}$ be a sequence of classical positive solutions to $(21)_k$ and assume that

$$\lim_{k\to\infty} \|v_k\| = +\infty, \qquad \text{where} \quad \|v_k\| := \|v_k\|_\infty.$$

Let $x_k \in \Omega$ be such that

$$v_k(x_k) = \max_\Omega v_k.$$

Let us choose R_k such that $B_k := B(x_k, R_k) \subset \Omega$, and

$$v_k(x) \ge \frac{1}{2}\, \|v_k\| \qquad \text{for any} \quad x \in B_k,$$

and there exists $y_k \in \partial B_k$ such that

$$v_k(y_k) = \frac{1}{2}\, \|v_k\|. \tag{27}$$

Let us denote by

$$m_k := \min_{[\|v_k\|/2, \|v_k\|]} g_k, \qquad M_k := \max_{[0, \|v_k\|]} g_k.$$

Therefore, we obtain

$$m_k \le g_k\big(v_k(x)\big) \quad \text{if } x \in B_k, \qquad g_k\big(v_k(x)\big) \le M_k \quad \forall x \in \Omega. \tag{28}$$

Then, reasoning as in (24), we obtain

$$\int_\Omega \left| g_k(v_k) \right|^q \, dx \;\; \leq \;\; C \, M_k^{\, q-1-\frac{1}{2^\star-1}}.$$

From elliptic regularity (see (25)), we deduce

$$\|v_k\|_{W^{2,q}(\Omega)} \leq C \, M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}.$$

Therefore, from Morrey's Theorem (see (26)), for any $x \in B_k$),

$$|v_k(x) - v_k(x_k)| \leq C \, (R_k)^{2-\frac{N}{q}} \, M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}.$$

Particularizing $x = y_k$ in the above inequality, and from (27), we obtain

$$C \, (R_k)^{2-\frac{N}{q}} \, M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}} \geq |v_k(y_k) - v_k(x_k)| = \frac{1}{2}\|v_k\|,$$

which implies

$$(R_k)^{2-\frac{N}{q}} \geq \frac{1}{2C} \, \frac{\|v_k\|}{M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}},$$

or equivalently,

$$R_k \geq \left(\frac{1}{2C} \, \frac{\|v_k\|}{M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{1/\left(2-\frac{N}{q}\right)}. \tag{29}$$

Consequently, taking into account (28),

$$\int_{B_k} v_k |g_k(v_k)| \, dx \;\; \geq \;\; \frac{1}{2}\|v_k\| m_k \, \omega \, (R_k)^N,$$

where $\omega = \omega_N$ is the volume of the unit ball in $\mathbb{R}^N$.

Due to $B_k \subset \Omega$, substituting inequality (29), and rearranging terms, we obtain

$$\int_\Omega v_k |g_k(v_k)| \, dx \geq \frac{1}{2}\|v_k\| m_k \, \omega \, \left(\frac{1}{2C} \, \frac{\|v_k\|}{M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{N}{2-\frac{N}{q}}}$$

$$= C \, m_k \, \left(\left[\|v_k\| \right]^{\frac{2}{N}-\frac{1}{q}} \frac{\|v_k\|}{M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}}$$

$$= C \, m_k \, \left(\frac{\|v_k\|^{1+\frac{2}{N}-\frac{1}{q}}}{M_k^{\, 1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}}$$

$$= C \, \frac{m_k}{M_k} \, \left(\frac{\|v_k\|^{1+\frac{2}{N}-\frac{1}{q}}}{M_k^{\, 1-\frac{2}{N}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}}.$$

At this moment, let us observe that from hypothesis $(H1)_k$ and $(H2)_k$,

$$\frac{m_k}{M_k} \geq C, \qquad \text{for all} \quad k \quad \text{big enough.}$$

Hence, taking again into account hypothesis $(H2)_k$, and rearranging exponents, we can assert that

$$
\begin{aligned}
\int_\Omega v_k |g_k(v_k)| \, dx &\geq C \left(\frac{\|v_k\|^{1+\frac{2}{N}-\frac{1}{q}}}{M_k^{1-\frac{2}{N}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} \\
&\geq C \left(\frac{\|v_k\|^{1+\frac{2}{N}-\frac{1}{q}}}{\left[g_k\left(\|v_k\|\right) \right]^{1-\frac{2}{N}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} \\
&= C \left(\frac{\|v_k\|^{(N+2)\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]}}{\left[g_k\left(\|v_k\|\right) \right]^{(N-2)\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}}.
\end{aligned}
$$

Finally, from hypothesis $(F)_k$ we deduce

$$\int_\Omega v_k |g_k(v_k)| \, dx \geq C \left(\frac{\|v_k\|^{2^\star-1}}{g_k\left(\|v_k\|\right)} \right)^{\frac{(N-2)\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]}{\frac{2}{N}-\frac{1}{q}}} \to \infty, \qquad \text{as} \quad k \to \infty,$$

which contradicts (23).   ☑

4.1.   Radial problems with almost critical exponent

In this Section, we build an example of a sequence of functions $\{g_k\}$ growing subcritically, and satisfying the hypotheses $(H1)_k$, $(H2)_k$, and $(F)_k$, such that the corresponding sequence of BVP

$$\begin{cases} \Delta w_k + g_k(w_k) = 0 & \text{in} \quad |x| \leq 1, \\ w_k(x) = 0 & \text{for} \quad |x| = 1. \end{cases} \tag{30}$$

has an unbounded (in the $L^\infty(\Omega)$-norm) sequence $\{w_k\}$ of positive solutions. As a consequence of Theorem 4.2, this sequence $\{w_k\}$ is also unbounded in the $L^{2^\star}(\Omega)$-norm.

Let $N \geq 3$ be an integer. For each positive integer $k > 2$ let $g_k(s) = 0$ for $s < 0$; $g_k(s) = s^{\frac{N+2}{N-2}}$ for $s \in [0, k]$; $g_k(s) = k^{\frac{N+2}{N-2}}$ for $s \in \left[k, k^{\frac{N+2}{N-2}}\right]$, and

$$g_k(s) = k^{\frac{N+2}{N-2}} + \left(s - k^{\frac{N+2}{N-2}}\right)^{\frac{N+1}{N-2}}, \qquad \forall s > k^{\frac{N+2}{N-2}}.$$

For the sake of simplicity in notation, we write $g_k := g$.

Let $u_k := u$ denote the solution to

$$\begin{cases} u'' + \dfrac{N-1}{r}u' + g(u) = 0, & r \in (0,1], \\ \qquad\qquad u(0) = k^{\frac{N}{N-2}}, & u'(0) = 0. \end{cases}$$

Let $r_1 = \sup\left\{r > 0; \ u_k(s) \geq k \text{ on } [0,r]\right\}$. Since $g \geq 0$, u is decreasing, consequently for $r \in [0, r_1]$, $k \leq u(r) \leq k^{\frac{N}{N-2}}$, and

$$-r^{N-1}u'(r) = \int_0^r s^{N-1}g(u(s))\,ds = \int_0^r s^{N-1}k^{\frac{N+2}{N-2}}\,ds = \frac{k^{\frac{N+2}{N-2}}}{N}\,r^N,$$

so

$$-u'(r) = \frac{k^{\frac{N+2}{N-2}}}{N}\,r. \tag{31}$$

Hence,

$$u(r) = k^{\frac{N}{N-2}} - \frac{k^{\frac{N+2}{N-2}}}{2N}\,r^2, \quad \text{for} \quad r \in [0, r_1]. \tag{32}$$

Thus, $u(r) \geq k^{\frac{N}{N-2}}/2$, for all $0 \leq r \leq r_0 := \sqrt{N}\Big/k^{\frac{1}{N-2}}$, and $u(r_0) = k^{\frac{N}{N-2}}/2$.

By well established arguments based on the Pohozaev identity (see [18]), we have

$$P(r) := r^N E(r) + \frac{N-2}{2}r^{N-1}u(r)\,u'(r) = \int_0^r s^{N-1}\Gamma\big(u(s)\big)\,ds, \tag{33}$$

where $E(r) = \frac{1}{2}(u'(r))^2 + G(u(r))$, $\Gamma(s) = NG(s) - \frac{N-2}{2}sg(s)$, and $G(s) = \int_0^s g(t)dt$. For $s \in \left[k, k^{\frac{N}{N-2}}\right]$,

$$\Gamma(s) = \frac{N+2}{2}k^{\frac{N+2}{N-2}}(s-k) \geq 0. \tag{34}$$

Hence,

$$\Gamma(u(r)) \geq \frac{N+2}{2}k^{\frac{2N+2}{N-2}} \quad \text{for all} \ \ r \leq r_0, \ \ k \geq 4^{\frac{N-2}{2}}. \tag{35}$$

Due to $\Gamma(s) = 0$ for all $s \leq k$, (33) and (34), for $r \geq r_0$,

$$P(r) \geq P(r_0) \geq \frac{N+2}{2N}k^{\frac{2N+2}{N-2}}r_0^N = \frac{N+2}{2}N^{\frac{N-2}{2}}k^{\frac{N+2}{N-2}}. \tag{36}$$

From definition of r_1 and (32),

$$u(r_1) = k, \quad \text{with}$$

$$\begin{aligned} r_1 &= \sqrt{2N\left[\left(\frac{1}{k}\right)^{\frac{2}{N-2}} - \left(\frac{1}{k}\right)^{\frac{4}{N-2}}\right]} \\ &= \sqrt{2N}\left(\frac{1}{k}\right)^{\frac{1}{N-2}} + o\left(\left(\frac{1}{k}\right)^{\frac{1}{N-2}}\right). \end{aligned} \tag{37}$$

From definition of g, $-u'(r_1) = \frac{1}{N} k^{\frac{N+2}{N-2}} r_1$ (see (31)), which implies

$$P(r_1) \geq r_1^{N+2} O\left(k^{\frac{2(N+2)}{N-2}}\right) - r_1^N O\left(k^{\frac{2N}{N-2}}\right)$$

$$\geq O\left(k^{\frac{N+2}{N-2}}\right) - O\left(k^{\frac{N}{N-2}}\right)$$

$$\geq O\left(k^{\frac{N+2}{N-2}}\right).$$

For $r \geq r_1$,

$$-\frac{N-2}{2} r^{N-1} u(r) u'(r) \geq \frac{(N-2)r^N}{2N} u(r) u(r)^{\frac{N+2}{N-2}}$$

$$= \frac{(N-2)r^N}{2N} u(r)^{\frac{2N}{N-2}}$$

$$= r^N G(u(r)).$$

This, Pohozaev's identity, and (36) imply

$$\left[(u'(r))\right]^2 \geq O\left(k^{\frac{N+2}{N-2}}\right) \frac{1}{r^N}, \quad \text{or} \quad -u'(r) \geq O\left(k^{\frac{N+2}{2(N-2)}}\right) \frac{1}{r^{N/2}}.$$

Integrating on $[r_1, r]$ we have

$$u(r) \leq k - O\left(k^{\frac{N+2}{2(N-2)}}\right) \left(\frac{1}{r_1^{(N-2)/2}} - \frac{1}{r^{(N-2)/2}}\right),$$

which implies that there exists k_0 such that if $k \geq k_0$ then $u(r) = 0$ for some $r \in (r_1, 2r_1]$. Since (37), $r_1 = r_1(k) \to 0$ as $k \to \infty$.

Let $v := v_k$ denote the solution to

$$\begin{cases} v'' + \dfrac{N-1}{r} v' + g(v) = 0, & r \in (0,1], \\ \quad v(0) = k^{\frac{N+2}{N-2}}, & v'(0) = 0. \end{cases}$$

Let $r_1' = \sup\left\{r > 0; \ v_k(s) \geq k \text{ on } [0,r]\right\}$. For $v(r) \geq k$, integrating (31) we deduce

$$v(r) = k^{\frac{N+2}{N-2}} - \frac{k^{\frac{N+2}{N-2}}}{2N} r^2, \quad \text{for} \quad r \in [0, r_1']$$

and

$$v(r_1') = k^{\frac{N+2}{N-2}} - \frac{k^{\frac{N+2}{N-2}}}{2N} r_1^2 = k;$$

therefore,

$$r_1' = \sqrt{2N\left(1 - \left(\frac{1}{k}\right)^{\frac{4}{N-2}}\right)} > 1,$$

so $v(r) \geq k$ for all $r \in [0,1]$.

Therefore, by continuous dependence on initial conditions, there exists $d_k \in \left(k^{\frac{N}{N-2}}, k^{\frac{N+2}{N-2}} \right)$ such that the solution $w = w_k$ to

$$\begin{cases} w'' + \dfrac{N-1}{r} w' + g_k(w) = 0, & r \in (0,1], \\ \qquad\qquad w(0) = d_k, & w'(0) = 0, \end{cases}$$

satisfies $w(r) \geq 0$ for all $r \in [0,1]$, and $w(1) = 0$. Since k may be taken arbitrarily large, and as a consequence of Theorem 4.2, we have established the following result.

Corollary 4.4. *There exists a sequence of functions $g_k : \mathbb{R} \to \mathbb{R}$ and a sequence $\{w_k\}$ of positive solutions to (30), such that each function g_k grows subcritically and satisfies the hypotheses (H1)$_k$, (H2)$_k$ and (F)$_k$ of Theorem 4.2, and the sequence $\{w_k\}$ of positive solutions to (30) is unbounded in the $L^\infty(\Omega)$-norm.*

Moreover, this sequence $\{w_k\}$ is also unbounded in the $L^{2^*}(\Omega)$-norm.

Let now $v := v_k$ denote the solution to

$$\begin{cases} v'' + \dfrac{N-1}{r} v' + g(v) = 0, & r \in (0,1], \\ \qquad\qquad v(0) = k, & v'(0) = 0. \end{cases} \tag{38}$$

Since $\Gamma(s) = 0$ for all $s \leq k$, and the solution is decreasing, by Pohozaev's identity,

$$\frac{1}{2} r(v'(r))^2 + \frac{N-2}{2N} \, r \, v(r)^{\frac{2N}{N-2}} + \frac{N-2}{2} \, v(r) v'(r) = 0, \qquad \forall r \in [0,1].$$

Hence, if $v(\hat{r}) = 0$ for some $\hat{r} \in (0,1]$, then $v'(\hat{r}) = 0$ and the uniqueness of the solution of the IVP (38), implies $v(r) = 0$ for all $r \in [0,1]$. Since this contradicts $v(0) = k > 0$ we conclude that $v(r) > 0$ for all $r \in [0,1]$. Therefore, by continuous dependence on initial conditions, there exists $d'_k \in \left(k, k^{\frac{N}{N-2}} \right)$ such that the solution $z = z_k$ to

$$\begin{cases} z'' + \dfrac{N-1}{r} z' + g_k(z) = 0, & r \in (0,1], \\ \qquad\qquad z(0) = d'_k, & z'(0) = 0, \end{cases}$$

satisfies $z(r) \geq 0$ for all $r \in [0,1]$, and $z(1) = 0$.

Corollary 4.5. *For any $k \in \mathcal{N}$, the BVP (30) has at least two positive solutions.*

5. Asymptotics for positive radial solutions of elliptic equations approaching critical growth

Consider the classical Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ u = 0 & \text{in } \partial B_R, \end{cases} \tag{39}$$

for $u \in C^2(\overline{B_R})$, in which $B_R = B_R(0) \subset \mathbb{R}^N$, $N > 2$, is the open ball of radius R, and f is locally-Lipschitz in $[0, \infty)$ and superlinear at infinity (i.e. $\liminf f(u)/u > \lambda_1$ as $u \to \infty$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions). Denote by $2^* := 2N/(N-2)$ the critical Sobolev exponent; $H^1(\Omega)$ is compactly embedded in $L^p(\Omega)$ if, and only if, $p < 2^*$. The extended real number $f^\star := \lim\limits_{u \to \infty} f(u)/u^{2^*-1}$ discriminates the problem (39) into three types: *critical* if $f^\star \in (0, \infty)$, *supercritical* if $f^* = \infty$ and *subcritical* if $f^\star = 0$.

Assume the nonlinearity is a pure subcritical power $f(u) = u^{2^*-1-\varepsilon}$, $\varepsilon > 0$, and Ω is a ball. Atkinson and Peletier in [8] studied the assymptotic behavior as $\varepsilon \to 0^+$ of solutions to (39), and proved that

$$\lim_{\varepsilon \to 0^+} \varepsilon u_\varepsilon(0)^2 = L(N, R),$$

and $\forall r \neq 0$,

$$\lim_{\varepsilon \to 0^+} \frac{u_\varepsilon(r)}{\sqrt{\varepsilon}} = \widetilde{L}(N, R) \left(\frac{1}{r^{N-2}} - \frac{1}{R^{N-2}} \right),$$

where $L(N, R)$, $\widetilde{L}(N, R)$ are constants only dependent on N, and R, defined by

$$L(N, R) := \frac{4}{N-2} \left[N(N-2) \right]^{\frac{N-2}{2}} \frac{\Gamma(N)}{\Gamma(N/2)^2} \frac{1}{R^{N-2}}, \tag{40}$$

$$\widetilde{L}(N, R) := \frac{(N-2)^{\frac{1}{2}}}{2} \left[N(N-2) \right]^{\frac{N-2}{4}} \frac{\Gamma(N/2)}{\Gamma(N)^{1/2}} R^{\frac{N-2}{2}} = \frac{\left[N(N-2) \right]^{\frac{N-2}{2}}}{L(N, R)^{1/2}}, \tag{41}$$

where Γ denotes the Gamma function. See also [56] with similar results for least energy solutions on general domains.

The above sections lead to a natural question: Is the lower bound on $\alpha > \frac{2}{N-2}$ a technical or an intrinsic condition?

In the present Section, we focus our attention on nonlinearities

$$f(u) = f_\alpha(u) := \frac{|u|^{2^*-2}u}{\left[\log(e + |u|) \right]^\alpha}, \qquad \text{with } \alpha \in \left(0, \frac{2}{N-2} \right). \tag{42}$$

We analyze the asymptotic behavior as $\alpha \to 0^+$ of solutions to (39). Firstly, we prove that for each $\alpha \in \left(0, \frac{2}{N-2} \right)$ fixed, the set of positive solutions to (39) is *a priori* bounded. Henceforth, the bound from below on α in [16], [17], [31], [71] are technical rather than intrinsic, at least when Ω is the open ball of radius R. Secondly, we provide estimates for the growth of $u_\alpha(0)$ as $\alpha \to 0^+$. We adapt for our nonlinearities, techniques introduced by Atkinson and Peletier for the case of subcritical powers in [7], [8].

Our first result is on the existence of solutions to (39), and of uniform L^∞ *a priori* bounds for each α fixed (see [75, Proof of Theorem 1.1] for a proof).

Theorem 5.1. *Fix* $\alpha \in \left(0, \frac{2}{N-2} \right)$; *let* $f = f_\alpha$ *be as in* (42) *and assume* Ω *is the open ball of radius* R. *Then the following results hold:*

(i) *There exists a radially symmetric solution to* (39), $u = u_\alpha(r) > 0$.

(ii) *There are constants* $A = A_\alpha(N, R)$, $B = B_\alpha(N, R) > 0$ *depending only on* α, N *and* R, *such that for every* $u = u_\alpha > 0$, *radially symmetric solution to* (39), *we have*

$$A_\alpha(N, R) \leq \|u_\alpha\|_{L^\infty(\Omega)} \leq B_\alpha(N, R), \qquad for\ each\ \alpha \in \left(0, \frac{2}{N-2}\right).$$

Our second result is an estimate of the asymptotic behavior of $u_\alpha(0) = \|u_\alpha\|_{L^\infty(\Omega)}$ as $\alpha \to 0^+$.

Theorem 5.2. *Let* $f = f_\alpha$ *be as in* (42) *with* $\alpha \in \left(0, \frac{2}{N-2}\right)$, *and* Ω *be the open ball of radius* R. *Then, there exists a constant* $L(N, R) > 0$ *only depending on* N *and* R, *such that for any* $u_\alpha = u_\alpha(r)$, *radially symmetric positive solution to* (39), *we have*

$$\limsup_{\alpha \to 0^+} \frac{\alpha u_\alpha(0)^2}{\left[\log(e + u_\alpha(0))\right]^{1 + \frac{\alpha(N+2)}{2}}} \quad \leq \quad L(N, R), \tag{43}$$

$$\liminf_{\alpha \to 0^+} \frac{\alpha u_\alpha(0)^2}{\left[\log(e + u_\alpha(0))\right]^{\frac{\alpha(N-4)}{2}}} \quad \geq \quad \frac{1}{2^*} L(N, R), \tag{44}$$

where $L(N, R)$ *is defined by* (40).

Proof. See [75, Proof of Theorem 1.2].   ☑

6. Some Open Problems

- When $N = 2$, which is the critical nonlinearity f for having *a priori* bounds?
  (See [38], [39]).

- When $N > 2$, which is the more general subcritical nonlinearity f for having *a priori* bounds?

- When $p = N$, which is the critical nonlinearity f for having *a priori* bounds?
  (See [36]).

- When $N > p$, which is the more general subcritical nonlinearity f for having *a priori* bounds?
  (See [36]).

- The conjecture on Lane-Emdem systems when $\Omega = \mathbb{R}^N$.
  See the discussion corresponding to the Lane-Emdem system (7).

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