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# Continuous images of hereditarily indecomposable continua

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**Abstract.** The theorem proven here is that every compact metric continuum is a continuous image of some hereditarily indecomposable metric continuum.
**Keywords**: Continuous maps, continuum, hereditarily indecomposable.
**MSC2010**: 54F15, 54F45, 54E45, 54C60.

## Imágenes continuas de continuos hereditariamente indescomponibles

**Resumen.** El teorema demostrado es que todo continuo métrico es imagen continua de algún continuo métrico hereditariamente indescomponible.
**Palabras clave**: Funciones continuas, continuo, hereditariamente indescomponible.

## 1. Introduction

These definitions are needed in what follows and may or may not be familiar to everyone. A *continuum* $X$ is a compact, connected metric space. A continuum $X$ is *indecomposable* provided that whenever $A$ and $B$ are proper subcontinua of $X$, $A \cup B$ is a proper subset of $X$; $X$ is *hereditarily indecomposable* if, and only if, every subcontinuum of $X$ is indecomposable. A *map* is a continuous function. A map $f$ from a continuum $X$ to a continuum $Y$ is *weakly confluent* provided that given any continuum $M \subseteq Y$ there exists a continuum $W \subseteq X$ such that $f(W) = M$. When $X$ is a continuum, $C(X)$ is the hyperspace of subcontinua of $X$. If $a$ and $b$ are points in $\mathbb{R}^n$ with $a \neq b$, $[a, b]$ denotes the line segment from $a$ to $b$. Let $S^n$ denote the $n$ dimensional sphere. An arc $A \subseteq S^3$ is *tame* if and only if there is a homeomorphism $h\colon S^3 \to S^3$ such that $h(A)$ is an arc of a great circle in $S^3$.

In [4] J. W. Rogers, Jr. asked whether every continuum is a continuous image of some indecomposable continuum. The author [1] gave an affirmative answer to this question.

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Some time later, in conversation, Rogers asked whether every continuum is a continuous image of some hereditarily indecomposable continuum. This article provides a proof that the answer to this question is also yes.

The author first announced this result in [1] but has not published it previously. It has come to my attention that in [4] this result has been extended to the non-metric case, building on the metric result.

## 2. Necessary Lemmas

**Lemma 2.1.** *Let $X$ and $Y$ be continua. Then $f\colon X \to Y$ is weakly confluent if, and only if, the hyperspace map induced by $f$, $C(f)\colon C(X) \to C(Y)$, is surjective.*

*Proof.* This is just a restatement of the definition of weakly confluent. ☑

**Lemma 2.2.** *There exists a hereditarily indecomposable subcontinuum of $\mathbb{R}^4$ which separates $\mathbb{R}^4$ .*

*Remark on proof.* R. H. Bing [2] proved this not just for $n = 4$, but for every $n > 1$.

**Lemma 2.3.** *Each homotopically essential map from a continuum $X$ to the three sphere, $S^3$, is weakly confluent.*

*Proof.* This was essentially proven, although in a different context, by S. Mazurkiewicz in [5, Theoreme I, p. 328]. This argument gives the necessary details. Let $X$ be a continuum, and suppose $g\colon X \to S^3$ be a homotopically essential map. To prove that $g$ is weakly confluent, it suffices to prove that every tame arc in $S^3$ is equal to $g(M)$ for some continuum $M \subseteq X$. This follows from Lemma 2.1 because the set of tame arcs is dense in $C(S^3)$.

First, set up some machinery and notation, as follows. Let $J$ be a tame arc in $S^3$; let $D_n$ be the closed disk in the complex plane with radius $(1/n)$ centered at 0. Let $E_n$ be the corresponding open disk, and let $T_n$ be the circle $D_n \setminus E_n$. Let $C_n$ be the solid cylinder $D_n \times [0, 1]$. Since $J$ is a tame, there exists an embedding $h$ of $C_1$ into $S^3$ such that $h(\{0\} \times [0, 1]) = J$. Consider $C_n$ as a subset of $S^3$ by identifying $C_1$ with $h(C_1)$, and for each $t \in [0, 1]$ let $t$ denote the point $h(0, t) \in J$.

Let $F_n$ denote the manifold boundary of $C_n$ , that is, $F_n = (D_n \times \{0, 1\}) \cup (T_n \times [0, 1])$. Note that given any $n$ and any $a, b \in J$ there is an isotopy $H\colon C_n \times [0, 1] \to C_n$ satisfying the following:

- (i) for each $s \in [0, 1], H(J \times \{s\}) = J$;
- (ii) for each $x \in F_n$ and each $t \in [0, 1], H(x, t) = x$;
- (iii) for every $x \in C_n, H(x, 0) = x$; and
- (iv) $H(b, 1) = a$.

By setting $H(x,t) = x$ for every $x \in S^3 \setminus C_n$, and every $t \in [0,1]$, $H$ can be considered to be a function (hence an isotopy) from $S^3 \times [0,1]$ to $S^3$.

Now, suppose $X$ is a continuum and let $g\colon X \to S^3$ be a homotopically essential map. To prove that $g$ is weakly confluent, it suffices to prove that there exists a continuum $M \subseteq X$ such that $g(M) = J$.

Proceed by contradiction; assume there is no such $M$. Then no component of $g^{-1}(J)$ intersects both $g^{-1}(0)$ and $g^{-1}(1)$. By compactness, there is a separation, $R_0 \cup R_1$ of $g^{-1}(J)$ satisfying $g^{-1}(0) \subseteq R_0$ and $g^{-1}(1) \subseteq R_1$. Since $R_0$ and $R_1$ are disjoint closed sets in $X$, there exist open subsets $S_0$ and $S_1$ of $X$ such that $R_0 \subseteq S_0$ and $R_1 \subseteq S_1$ and $Cl(S_0) \cap Cl(S_1) = \emptyset$. There exists $n$ such that $g^{-1}(Cn) \subseteq S_0 \cup S_1$. Let $p = \inf g(R_1)$ and let $q = \sup g(R_0)$, and let $a, b \in J$ be such that $0 < a < p$ and $q < b < 1$. If $p > q$, then $g$ is not surjective and hence not essential, so $0 < a < p \leq q < b < 1$. Using the number $n$ and the points $a$ and $b$ just chosen, let $H\colon S^3 \times [0,1] \to S^3$ be the isotopy described above. Define a homotopy $G\colon X \times [0,1] \to S^3$ by $G(x,t) = g(x)$ if $x \in X \setminus S_0$ and $G(x,t) = H(g(x),t)$ if $x \in Cl(S_0)$. Define $f\colon X \to S^3$ by $f(x) = G(x,1)$.

Then, note that if $y \in J$ and $a < y < p$, then there does not exist $z \in X$ such that $f(z) = y$, so $f$ is nonsurjective. Hence, $f$ is inessential. Since $g$ is homotopic to $f$, $g$ is inessential also, a contradiction, which completes the proof. ☑

**Lemma 2.4.** *A continuum $X \subseteq R^4$ admits a homotopically essential map onto $S^3$ if, and only if, $R^4$ $X$ is not connected $S^3$.*

*Remark on Proof.* This is a special case of the Borsuk separation theorem. I do not have a reference to the original proof, but a proof can be found in almost any advanced topology or algebraic topology book.

**Lemma 2.5.** *Given any continuum $Y$, there is a continuum $X \subseteq S^3$ that admits a continuous surjection $f\colon X \to Y$.*

*Proof.* Let $Y$ be a continuum and let $C$ and $D$ be Cantor sets in $R^3$ such that $C$ and $D$ lie on lines skew to each other. Then, whenever $a, p \in C$ and $b, q \in D$, and $a, p, b$, and $q$ are all different, the line segments $[a,b]$ and $[p,q]$ are disjoint. Let $g\colon C \cup D \to Y$ be a map such that $g|C\colon C \to Y$ and $g|D\colon D \to Y$ are both onto. Such a $g$ exists since a Cantor set can be mapped onto every compact metric space. Define $X = \bigcup\{[a,b] : a \in C; b \in D \text{ and } g(a) = g(b)\}$. Then $X$ is a continuum in $R^3$. For each $x \in X$, let $[a(x), b(x)]$ be a segment in $X$ satisfying $a(x) \in C$; $b(x) \in D$, and $x \in [a(x), b(x)]$. (This segment is unique unless $x = a(x)$ or $x = b(x)$.) Define $f\colon X \to Y$ by $f(x) = g(a(x)) = g(b(x))$. It is straightforward to verify that $f\colon X \to Y$ is continuous and onto. Since for any point $p \in S^3$, $S^3 \setminus \{p\}$ is a copy of $R^3$, $X$ can be treated as a subcontinuum of $S^3$. ☑

## 3. Main Result

**Theorem 3.1.** *Let $Y$ be an arbitrary continuum. There exists a hereditarily indecomposable continuum $K$ that admits a surjective map $f\colon K \to Y$.*

*Proof.* Let $Y$ be a continuum. By Lemma 2.5, there is a continuum $T \subseteq S^3$ and an onto map $g\colon T \to Y$. By Lemma 2.2, there exists a hereditarily indecomposable continuum $L \subseteq R^4$ that separates $R^4$. Thus by Lemma 2.4, there is a homotopically essential map $h\colon L \to S^3$. By Lemma 2.3, $h$ is weakly confluent, so there exists a continuum $K \subseteq L$ such that $h(K) = T$. Let $f = g \circ (h|K)$. Then $f\colon K \to Y$ is the desired map; $K$ is hereditarily indecomposable since it is a subcontinuum of $L$. ☑

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