



On some Chebyshev type inequalities for the complex integral

SILVESTRU SEVER DRAGOMIR*

Victoria University, College of Engineering and Science, Melbourne, Australia.

University of the Witwatersrand, School of Computer Science and Applied Mathematics, Johannesburg, South Africa.

Abstract. Assume that f and g are continuous on γ , $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$, and the complex Chebyshev functional is defined by

$$\mathcal{D}_\gamma(f, g) := \frac{1}{w-u} \int_\gamma f(z)g(z) dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma g(z) dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f, g)$ under Lipschitzian assumptions for the functions f and g , and provide a complex version for the well known Chebyshev inequality.

Keywords: Complex integral, Continuous functions, Holomorphic functions, Chebyshev inequality.

MSC2010: 26D15, 26D10, 30A10, 30A86.

Sobre algunas desigualdades tipo Chebyshev para la integral compleja

Resumen. Sean f y g funciones continuas sobre γ , siendo $\gamma \subset \mathbb{C}$ un camino suave por partes parametrizado por $z(t)$, $t \in [a, b]$ con $z(a) = u$ y $z(b) = w$, $w \neq u$, y el funcional de Chebyshev complejo definido por

$$\mathcal{D}_\gamma(f, g) := \frac{1}{w-u} \int_\gamma f(z)g(z) dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma g(z) dz.$$

En este artículo establecemos algunas cotas para la magnitud del funcional $\mathcal{D}_\gamma(f, g)$ bajo condiciones de lipschitzianidad para las funciones f y g , y damos una versión compleja para la conocida desigualdad de Chebyshev.

Palabras clave: Integral compleja, funciones continuas, funciones holomorfas, desigualdad de Chebyshev.

*E-mail: sever.dragomir@vu.edu.au

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1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Chebyshev functional* defined by

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \frac{1}{b-a} \int_a^b g(t) dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f, g)| \leq \frac{1}{4} (M - m)(N - n), \quad (1)$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b]. \quad (2)$$

The constant $\frac{1}{4}$ in (1) is sharp.

Another, however less known result, even though it was obtained by Chebyshev in 1882, [8], states that

$$|C(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2, \quad (3)$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Chebyshev inequality (3) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be *absolutely continuous* and $f', g' \in L_\infty[a, b]$, while $\|f'\|_\infty = \text{esssup}_{t \in [a, b]} |f'(t)|$.

For other inequality of Grüss' type see [1]-[16] and [18]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex valued function which is continuous on γ . Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z) dz = \int_{\gamma_{u,w}} f(z) dz := \int_a^b f(z(t)) z'(t) dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$; then, assuming that f is continuous on γ , we define

$$\int_{\gamma_{u,w}} f(z) dz := \int_{\gamma_{u,v}} f(z) dz + \int_{\gamma_{v,w}} f(z) dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length:

$$\int_{\gamma_{u,w}} f(z) |dz| := \int_a^b f(z(t)) |z'(t)| dt,$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)| dt.$$

Let f and g be holomorphic in G , an open domain, and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz. \tag{4}$$

We recall also the *triangle inequality* for the complex integral, namely,

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma, \infty} \ell(\gamma), \tag{5}$$

where $\|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the p -norm with $p \geq 1$ by

$$\|f\|_{\gamma, p} := \left(\int_{\gamma} |f(z)|^p |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma, 1} := \int_{\gamma} |f(z)| |dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölder's inequality, we have

$$\|f\|_{\gamma, 1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma, p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ , we consider the *complex Chebyshev functional* defined by

$$\mathcal{D}_{\gamma}(f, g) := \frac{1}{w-u} \int_{\gamma} f(z) g(z) dz - \frac{1}{w-u} \int_{\gamma} f(z) dz \frac{1}{w-u} \int_{\gamma} g(z) dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_{\gamma}(f, g)$ under various assumptions for the functions f and g , and provide a complex version for the Chebyshev inequality (3).

2. Chebyshev type results

We start with the following identity of interest:

Lemma 2.1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ , then*

$$\begin{aligned} \mathcal{D}_\gamma(f, g) &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right) dz & (6) \\ &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dz \right) dw \\ &= \frac{1}{2(w-u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w)) dz dw. \end{aligned}$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw$ exists and

$$\begin{aligned} I(z) &:= \int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \\ &= \int_\gamma (f(z)g(z) + f(w)g(w) - g(z)f(w) - f(z)g(w)) dw \\ &= f(z)g(z) \int_\gamma dw + \int_\gamma f(w)g(w) dw - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw \\ &= (w-u)f(z)g(z) + \int_\gamma f(w)g(w) dw - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw. \end{aligned}$$

The function $I(z)$ is also continuous on γ , then the integral $\int_\gamma I(z) dz$ exists and

$$\begin{aligned} \int_\gamma I(z) dz &= \int_\gamma \left[(w-u)f(z)g(z) + \int_\gamma f(w)g(w) dw \right. \\ &\quad \left. - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw \right] dz \\ &= (w-u) \int_\gamma f(z)g(z) dz + (w-u) \int_\gamma f(w)g(w) dw \\ &\quad - \int_\gamma f(w) dw \int_\gamma g(z) dz - \int_\gamma g(w) dw \int_\gamma f(z) dz \\ &= 2(w-u) \int_\gamma f(z)g(z) dz - 2 \int_\gamma f(z) dz \int_\gamma g(z) dz \\ &= 2(w-u)^2 \mathcal{D}_\gamma(f, g), \end{aligned}$$

which proves the first equality in (6).

The rest follows in a similar manner and we omit the details. \square

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $h : \gamma \rightarrow \mathbb{C}$ a continuous function on γ . Define the quantity:

$$\begin{aligned} \mathcal{P}_\gamma (h, \bar{h}) &= \frac{1}{\ell(\gamma)} \int_\gamma |h(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_\gamma h(z) |dz| \right|^2 \\ &= \frac{1}{\ell(\gamma)} \int_\gamma \left| h(v) - \frac{1}{\ell(\gamma)} \int_\gamma h(z) |dz| \right|^2 |dv| \geq 0. \end{aligned} \tag{7}$$

We say that the function $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ is L - h -Lipschitzian on the subset G if

$$|f(z) - f(w)| \leq L |h(z) - h(w)|$$

for any $z, w \in G$. If $h(z) = z$, we recapture the usual concept of L -Lipschitzian functions on G .

Theorem 2.2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$, $h : \gamma \rightarrow \mathbb{C}$ is continuous, f and g are L_1 , L_2 - h -Lipschitzian functions on γ ; then*

$$|\mathcal{D}_\gamma (f, g)| \leq L_1 L_2 \frac{\ell^2(\gamma)}{|w - u|^2} \mathcal{P}_\gamma (h, \bar{h}). \tag{8}$$

Proof. Taking the modulus in the first equality in (6), we get

$$\begin{aligned} |\mathcal{D}_\gamma (f, g)| &= \frac{1}{2|w - u|^2} \left| \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right) dz \right| \\ &\leq \frac{1}{2|w - u|^2} \int_\gamma \left| \int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right| |dz| \\ &\leq \frac{1}{2|w - u|^2} \int_\gamma \left(\int_\gamma |(f(z) - f(w))(g(z) - g(w))| |dw| \right) |dz| \\ &\leq \frac{L_1 L_2}{2|w - u|^2} \int_\gamma \left(\int_\gamma |h(z) - h(w)|^2 |dw| \right) |dz| =: A. \end{aligned} \tag{9}$$

Now, observe that

$$\begin{aligned}
 & \int_{\gamma} \left(\int_{\gamma} |h(z) - h(w)|^2 |dw| \right) |dz| \tag{10} \\
 &= \int_{\gamma} \left(\int_{\gamma} (|h(z)|^2 - 2\operatorname{Re}(h(z)\overline{h(w)}) + |h(w)|^2) |dw| \right) |dz| \\
 &= \int_{\gamma} \left(\ell(\gamma) |h(z)|^2 - 2\operatorname{Re} \left(h(z) \int_{\gamma} \overline{h(w)} |dw| \right) + \int_{\gamma} |h(w)|^2 |dw| \right) |dz| \\
 &= \ell(\gamma) \int_{\gamma} |h(z)|^2 |dz| - 2\operatorname{Re} \left(\int_{\gamma} h(z) |dz| \int_{\gamma} \overline{h(w)} |dw| \right) \\
 &+ \ell(\gamma) \int_{\gamma} |h(w)|^2 |dw| \\
 &= 2\ell(\gamma) \int_{\gamma} |h(z)|^2 |dz| - 2\operatorname{Re} \left(\int_{\gamma} h(z) |dz| \overline{\left(\int_{\gamma} h(w) |dw| \right)} \right) \\
 &= 2 \left[\ell(\gamma) \int_{\gamma} |h(z)|^2 |dz| - \left| \int_{\gamma} h(z) |dz| \right|^2 \right] = 2\ell^2(\gamma) \mathcal{P}_{\gamma}(h, \overline{h}).
 \end{aligned}$$

Therefore, by (10) we get

$$A = L_1 L_2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_{\gamma}(h, \overline{h}),$$

and by (9) we get the desired result (8). \square

Further, for $\gamma \subset \mathbb{C}$ a piecewise smooth path parametrized by $z(t)$, and by taking $h(z) = z$ in (7), we can consider the quantity

$$\begin{aligned}
 \mathcal{P}_{\gamma} &:= \frac{1}{\ell(\gamma)} \int_{\gamma} |z|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} z |dz| \right|^2 \tag{11} \\
 &= \frac{1}{\ell(\gamma)} \int_{\gamma} \left| v - \frac{1}{\ell(\gamma)} \int_{\gamma} z |dz| \right|^2 |dv| \\
 &= \frac{1}{2\ell^2(\gamma)} \int_{\gamma} \left(\int_{\gamma} |z-w|^2 |dw| \right) |dz| \geq 0.
 \end{aligned}$$

Corollary 2.3. Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and f and g are L_1 , L_2 -Lipschitzian functions on γ ; then

$$|\mathcal{D}_{\gamma}(f, g)| \leq L_1 L_2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_{\gamma}. \tag{12}$$

Remark 2.4. Assume that f is L -Lipschitzian on γ . For $g = f$ we have

$$\mathcal{D}_{\gamma}(f, f) = \frac{1}{w-u} \int_{\gamma} f^2(z) dz - \left(\frac{1}{w-u} \int_{\gamma} f(z) dz \right)^2, \tag{13}$$

and by (8) we get

$$|\mathcal{D}_\gamma(f, f)| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma(h, \bar{h}). \tag{14}$$

For $g = \bar{f}$ we have

$$\mathcal{D}_\gamma(f, \bar{f}) = \frac{1}{w-u} \int_\gamma |f(z)|^2 dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma \overline{f(z)} dz, \tag{15}$$

and by (8) we get

$$|\mathcal{D}_\gamma(f, \bar{f})| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma(h, \bar{h}). \tag{16}$$

If f is L -Lipschitzian on γ , then

$$|\mathcal{D}_\gamma(f, f)| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma \tag{17}$$

and

$$|\mathcal{D}_\gamma(f, \bar{f})| \leq L^2 \frac{\ell^2(\gamma)}{|w-u|^2} \mathcal{P}_\gamma. \tag{18}$$

If the path γ is a segment $[u, w]$ connecting two distinct points u and w in \mathbb{C} , then we write $\int_\gamma f(z) dz$ as $\int_u^w f(z) dz$.

Now, if f and g are L_1, L_2 -Lipschitzian functions on $[u, w] := \{(1-t)u + tw, t \in [0, 1]\}$, then by (12) we have

$$|\mathcal{D}_\gamma(f, g)| \leq L_1 L_2 \mathcal{P}_{[u,w]},$$

where

$$\begin{aligned} \mathcal{P}_{[u,w]} &= \frac{|w-u|^2}{2|w-u|^2} \int_0^1 \left(\int_0^1 |(1-t)u + tw - (1-s)u - sw|^2 dt \right) ds \\ &= \frac{1}{2} |w-u|^2 \int_0^1 \left(\int_0^1 (t-s)^2 dt \right) ds = \frac{1}{12} |w-u|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| \frac{1}{w-u} \int_\gamma f(z)g(z) dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma g(z) dz \right| \tag{19} \\ &\leq \frac{1}{12} |w-u|^2 L_1 L_2, \end{aligned}$$

if f and g are L_1, L_2 -Lipschitzian functions on $[u, w]$.

If f is L -Lipschitzian on $[u, w]$, then

$$\left| \frac{1}{w-u} \int_\gamma f^2(z) dz - \left(\frac{1}{w-u} \int_\gamma f(z) dz \right)^2 \right| \leq \frac{1}{12} |w-u|^2 L^2 \tag{20}$$

and

$$\begin{aligned} &\left| \frac{1}{w-u} \int_\gamma |f(z)|^2 dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma \overline{f(z)} dz \right| \tag{21} \\ &\leq \frac{1}{12} |w-u|^2 L^2. \end{aligned}$$

3. Examples for circular paths

Let $[a, b] \subseteq [0, 2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R > 0$:

$$z(t) = R \exp(it) = R(\cos t + i \sin t), \quad t \in [a, b].$$

If $[a, b] = [0, \pi]$, then we get a half circle, while for $[a, b] = [0, 2\pi]$ we get the full circle.

Since

$$\begin{aligned} |e^{is} - e^{it}|^2 &= |e^{is}|^2 - 2\operatorname{Re}(e^{i(s-t)}) + |e^{it}|^2 \\ &= 2 - 2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right) \end{aligned}$$

for any $t, s \in \mathbb{R}$, then

$$|e^{is} - e^{it}|^r = 2^r \left| \sin\left(\frac{s-t}{2}\right) \right|^r \quad (22)$$

for any $t, s \in \mathbb{R}$ and $r > 0$. In particular,

$$|e^{is} - e^{it}| = 2 \left| \sin\left(\frac{s-t}{2}\right) \right|$$

for any $t, s \in \mathbb{R}$.

If $u = R \exp(ia)$ and $w = R \exp(ib)$, then

$$\begin{aligned} w - u &= R[\exp(ib) - \exp(ia)] = R[\cos b + i \sin b - \cos a - i \sin a] \\ &= R[\cos b - \cos a + i(\sin b - \sin a)]. \end{aligned}$$

Since

$$\cos b - \cos a = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b - \sin a = 2 \sin\left(\frac{b-a}{2}\right) \cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned} w - u &= R \left[-2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{b-a}{2}\right) + 2i \sin\left(\frac{b-a}{2}\right) \cos\left(\frac{a+b}{2}\right) \right] \\ &= 2R \sin\left(\frac{b-a}{2}\right) \left[-\sin\left(\frac{a+b}{2}\right) + i \cos\left(\frac{a+b}{2}\right) \right] \\ &= 2Ri \sin\left(\frac{b-a}{2}\right) \left[\cos\left(\frac{a+b}{2}\right) + i \sin\left(\frac{a+b}{2}\right) \right] \\ &= 2Ri \sin\left(\frac{b-a}{2}\right) \exp\left[\left(\frac{a+b}{2}\right)i\right]. \end{aligned}$$

If $\gamma = \gamma_{[a,b],R}$, then the *circular complex Chebyshev functional* is defined by

$$\begin{aligned} \mathcal{C}_{[a,b],R}(f, g) & := \mathcal{D}_{\gamma_{[a,b],R}}(f, g) \\ & = \frac{1}{2 \sin\left(\frac{b-a}{2}\right) \exp\left[\left(\frac{a+b}{2}\right) i\right]} \int_a^b f(R \exp(it)) g(R \exp(it)) \exp(it) dt \\ & \quad - \frac{1}{4 \sin^2\left(\frac{b-a}{2}\right) \exp\left[2\left(\frac{a+b}{2}\right) i\right]} \\ & \quad \times \int_a^b f(R \exp(it)) \exp(it) dt \int_a^b g(R \exp(it)) \exp(it) dt. \end{aligned} \tag{23}$$

If $\gamma = \gamma_{[a,b],R}$, then

$$\mathcal{P}_\gamma := \frac{1}{2\ell^2(\gamma)} \int_\gamma \left(\int_\gamma |z-w|^2 |dw| \right) |dz| \tag{24}$$

$$= \frac{R^4}{2R^2(b-a)^2} \int_a^b \left(\int_a^b |e^{is} - e^{it}|^2 dt \right) ds \tag{25}$$

$$= \frac{R^4}{2R^2(b-a)^2} \int_a^b \left(\int_a^b [2 - 2 \cos(s-t)] dt \right) ds$$

$$= \frac{R^2}{(b-a)^2} \int_a^b \left(\int_a^b [1 - \cos(s-t)] dt \right) ds$$

$$= \frac{R^2}{(b-a)^2} \int_a^b (b-a - \sin(b-s) - \sin(s-a)) ds$$

$$= \frac{R^2}{(b-a)^2} \left[(b-a)^2 - 1 + \cos(b-a) + \cos(b-a) - 1 \right]$$

$$= \frac{R^2}{(b-a)^2} \left[(b-a)^2 - 2(1 - \cos(b-a)) \right]$$

$$= \frac{R^2}{(b-a)^2} \left[(b-a)^2 - 4 \sin^2\left(\frac{b-a}{2}\right) \right]$$

$$= \frac{4R^2}{(b-a)^2} \left[\left(\frac{b-a}{2}\right)^2 - \sin^2\left(\frac{b-a}{2}\right) \right].$$

We have the following result:

Proposition 3.1. *Let $\gamma_{[a,b],R}$ be a circular path centered in 0, with radius $R > 0$ and $[a, b] \subset [0, 2\pi]$. If f and g are L_1, L_2 -Lipschitzian functions on $\gamma_{[a,b],R}$, then*

$$|\mathcal{C}_{[a,b],R}(f, g)| \leq \frac{R^2}{\sin^2\left(\frac{b-a}{2}\right)} \left[\left(\frac{b-a}{2}\right)^2 - \sin^2\left(\frac{b-a}{2}\right) \right] L_1 L_2. \tag{26}$$

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