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On the Behaviour of Solutions of Second Order Linear Differential Equations

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Abstract

In this paper we study boundedness and certain stability properties of solution of Second Order Linear Differential Equations (2). Some relations with the limit point circle classification are also presented.

1. Introduction

The study of the various boundedness and stability properties associated with solutions of differential equations may be proved by constructing a positive definide function E with

$$\frac{dE}{dx} \leq k_1(x)\, w(E) + k_2(x); \tag{1}$$

where $k_1, k_2 \in L^1(0,\infty)$ and w is a function belonging to a class G defined bellow (see [4], [11], [16] and their references). In this paper we shall apply this idea to obtain certain results on the boundedness and stability of second order linear differential equation:

$$\left(p(x)\, y'(x)\right)' + q(x)\, y(x) = 0, \quad a \leq x < \infty, \tag{2}$$

under suitable conditions on the coefficient functions p and q.

Our approach differs from those of the earlier investigations as all the earlier authors constructed energy functions (a partial survey of this area may be found un [14]). We have adopted the generalized idea mentioned above; our results differ significantly from those obtained previously.

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Definition 1. (See [16]) *Let G be the set of all real functions $w(x) \geq 0$ for $x \geq a$ wich are continuous and non–decreasing for $x \in [a, \infty)$ and have the property that $w(x) = O(x)$ as x tend to infinity.*

It is obvious that:

$$\Omega(u) = \int_{u_0}^{u} \frac{dx}{w(x)} \to \infty, \quad \text{as } u \to \infty.$$

Furthermore, if w_1 and w_2 are both in G, then $w_1 + w_2 \in G$.
We requiere the following results, wich can be found in [3] and [16], respectively.

Lemma 1. *Let the functions $F(x)$ and $Y(x)$ be continuous, $F(x) \geq 0$ for all $x \geq x_0$ and $k \geq 0$. If $w \in G$, then the inequality:*

$$Y(x) \leq k + \int_{x_0}^{x} F(t)\, w(Y(t))\, dt, \tag{3}$$

implies

$$Y(x) \leq \Omega^{-1}\left(\Omega(k) + \int_{x_0}^{x} F(t)\, dt\right). \tag{4}$$

Lemma 2. *Suposse then exists a positive–definite continuous function E of $x \in \mathbb{R}^n$ $(n \geq 1)$ such that along trajectory of*

$$\frac{dy_i}{dx} = y_i(x, y), \quad i = 1, \ldots, n, \tag{5}$$

the total derivate $\frac{dE}{dx}$ satisfies the inequality (1) where $w \in G$ and k_1 and k_2 are non–negative functions in $L^1(0, \infty)$. Suppose also that $E \to \infty$ as $y_r \to \infty$ for some r. Then y_r es bounded.

Throughout this paper we consider the following assumptions:

i) $p \geq \delta > 0$; $p^{-1} \in L^1(a, \infty)$.

ii) $q > 0$; $q \in L^1(a, \infty)$.

Equation (2) is equivalent to the following system:

$$\begin{aligned} y' &= \frac{z}{p}, \\ z' &= -qy. \end{aligned} \tag{6}$$

2. Results

The problem of boundedness of solutions is of particular importance in the qualitative theory, and if has received a considerable amount of attention in the last three decades. So, we begin with some of the simpler possibilities.

Theorem 1. *Under assumptions* i) *and* ii) *every solution of* (2) *satisfies the following conditions:*

1. *y is bounded for all $x \geq a$;*
2. *py' is bounded for all $x \geq a$.*

Proof. Let us consider the following Liapunov's function:

$$E(x) = (z+y)^2 + y^2. \tag{7}$$

The derivate E' of the function E defined by (7) along any solution of system (6) satisfies:

$$\frac{dE}{dx} = -2qyz - 2qy^2 + \frac{2z^2}{p} + \frac{4yz}{p}. \tag{8}$$

Employing ii) we get:

$$\begin{aligned}\frac{dE}{dx} &\leq -2qyz + 2\frac{z^2}{p} + \frac{2}{p}2yz \\ &\leq -2qyz + \frac{2}{p}\left(z^2 + 2yz\right) \\ &\leq q\left(\frac{E}{2^{1/2}-1}\right) + \frac{2}{p}E \\ &\leq \left(\frac{q}{2^{1/2}-1} + \frac{2}{p}\right)E.\end{aligned}$$

Now note that $\left(\frac{q}{2^{1/2}-1} + \frac{2}{p}\right) \in L^1[a,\infty)$ and $w(E) = E \in G$.
Therefore, by Lemma 1, E es bounded. Using Lemma 2, we can conclude that y and z are bounded. ∎
In [13] Wintner proved that if there exist constants a and $w \neq 0$ such that either:

$$\int_a^\infty \left|q - w^2\right| < \infty \tag{9}$$

or

$$\int_a^\infty |q'| < \infty \quad \text{and } q(w) \to w^2 \text{ as } x \to \infty, \tag{10}$$

then both y and y' are bounded on $[a, \infty)$ for any solution y of (2) with $p \equiv 1$.

Remark 1. *It is seen easy that Theorem* 1 *extend* (9) *to* $w \equiv 0$ *and the condition of regularity* (10) *are not used.*

The above remark remain true if are consider the results [1] and [2] taking q bounded.

Theorem 2. *Under conditions* i) *and* ii) *we have* $q^{1/2} \in L^2[a, \infty)$.

Proof. From (8) we obtain:

$$\begin{aligned} 2qy^2 &= -\frac{dE}{dx} - 2qyz + \frac{2}{p}(y+z)^2 \\ &\leq -\frac{dE}{dx} + q\left(\frac{E}{2^{1/2}-1}\right) + \frac{2}{p}E \\ &\leq -\frac{dE}{dx} + \left(\frac{q}{2^{1/2}-1} + \frac{2}{p}\right)E. \end{aligned}$$

Integrating this inequality between 0 and x we have:

$$\int_0^x qy^2 dt \leq E(0) - E(x) + \int_0^x \left(\frac{q}{2^{1/2}-1} + \frac{2}{p}\right) E\, dt.$$

Since E is bounded, there is a constant k such that $\int_0^x qy^2 dt \leq k$. The result now follows easily. ∎

Theorem 3. *If in addition to conditions* i)–ii), *we assume that l.u.b.* $p(t) = M < \infty$, *then* $py' \in L^2[a, \infty)$.

Proof. From (8) we obtain:

$$\begin{aligned} \frac{2z^2}{p} &= \frac{dE}{dx} + 2qyz + 2qy^2 - \frac{4yz}{p} \\ &\leq \frac{dE}{dx} + 3qE + \frac{2}{(2^{1/2}-1)\,p}E. \end{aligned}$$

Integrating this inequality between 0 and x we have:

$$2\int_0^x \frac{2z^2}{p}dt \le E(x) - E(0) + \int_0^x \left(3q + \frac{2}{(2^{1/2}-1)\,p}\right) E\,dt.$$

From this, since E is bounded we have $\int_0^x \frac{z^2}{p}dt \le K$.

If condition l.u.b. $p(t) = M < \infty$ then $\int_0^x \frac{z^2}{p}dt \le \infty$. This completes proof. ∎

Theorem 4. *If in addition to conditions* i)–ii), *we assume that* $|q| \le M$ $(M > 0)$ *then* $\lim_{x\to\infty} z(x) = 0$.

Proof. First note the following:

$$\left|\frac{d(z^2)}{dx}\right| = |2zz'| = |2qyz| \le |q|\left(y^2 + z^2\right).$$

The expression on the right–hand side is bounded by a positive constant for y and z bounded. The use of an elementary Lemma [5, p.261] yield the started result. ∎

Theorem 5. *If under the conditions of the above theorem, we assume that* $0 < m \le |q| \le M$ *then* $\lim_{x\to\infty} y(x) = 0$.

Proof. Since y is bounded, we can choose a sequence x_n such that:

$$\lim_{n\to\infty} |y(x_n)| = \overline{\lim_{x\to\infty}} |y(x)| = \bar{y}.$$

We shall show that $\bar{y} = 0$.

If follows from the last equation of (6) that:

$$\begin{aligned} z(x_n+1) - z(x_n) &= \\ &= -\int_{x_n}^{x_n+1} q(\tau)\,y(\tau)\,d\tau \\ &= -\int_{x_n}^{x_n+1} [q(\tau)\,y(\tau) - q(\tau)\,y(x_n)]\,d\tau - \int_{x_n}^{x_n+1} q(\tau)\,y(x_n)\,d\tau. \end{aligned}$$

Therefore

$$m\,|y(x_n)| \le \int_{x_n}^{x_n+1} q(\tau)\,d\tau\,|y(x_n)|$$

$$\leq |z(x_n+1) - z(x_n)| + \int_{x_n}^{x_n+1} |q(\tau)| |y(\tau) - y(x_n)| \, d\tau$$
$$\leq M \max_{\xi \in [x_n, x_n+1]} |y'(\xi)|$$
$$\leq \frac{M}{\delta} \max_{\xi \in [x_n, x_n+1]} |z(\xi)|.$$

Since $z(x)$ tends to zero as x tends to infinity we have that $|y(x_n)| \to 0$ as $n \to \infty$, that is $\lim_{n\to\infty} |y(x_n)| = \overline{\lim_{x\to\infty}} |y(x)| = 0$. ∎

Remark 2. *Further results on the boundednees and asymptotic properties of z can be obtained by imposing a very restrictive set of conditions on the differential equations. For example, if $q \in L^1[a,\infty)$ then py' is bounded on $[a,\infty)$ whenever y is bounded real–valued solution of* (2).

3. On the limit point–limit circle classification

From equation (1), we define the second order, ordinary differential expressions τ defined by

$$\tau y = -(py')' + qy, \tag{11}$$

with Δ as your maximal domain

$$\Delta := \left\{ y \in [a,\infty) \to \mathbb{R} : y,\ py \in AC_{Loc}[a,\infty);\ y, \tau y \in L^2[a,\infty) \right\}. \tag{12}$$

Following previous autors, we define the following terms, the expressions (11) is:

a) *limit–circle* (LC) if all solutions of it are $L^2[a,\infty)$,

b) *limit–point* (LP) if there is a solution wich is not $L^2[a,\infty)$,

c) *Dirichlet* (D) at ∞ if $|p|^{1/2} f$, and $|q|^{1/2} f \in L^2[a,\infty)$; $f \in \Delta$.

The cases a) and b) is due to Weyl and it has important consequences in the spectral theory associated with (1) in Hilbert space $L^2[a,\infty)$. No necessary and sufficient condition on p and q is known distinguishes between the LP and LC types, but a number of sufficient conditions for the two types are known. From previous results (see [5, 9] and [15]) and Theorem 2, we can easily derive that:

Corollary 1. *Under condition* i) *and* $|q| \geq m$ *the equation* (2) *is* LC *on* $[a, \infty)$.

So, our results is a LC criterion under milder condition, because decomposition methods or regularity conditions are not used. In this sense, this criterion is the best possible.

From [12, Lemma 3.1] and the same Theorem 2 we obtain the following result:

Corollary 2. *Suppose* $\int_a^x |p|^{-1} \notin L[a, \infty)$ *then expressions* τ *is* D *at* ∞.

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