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# Boundedness and Stability Properties of Some Integrodifferential Systems

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**Abstract**

In this paper we study the behavior of solutions of a Volterra integrodifferential system of the form (1).

## 1. Introduction

We consider a system of Volterra integrodifferential equations:

$$x'(t) = A(t)\,x(t) + \int_0^t B(t-s)\,x(s)\,ds + x(t)\,H(t, x(t), \sigma(t)), \qquad (1)$$

$$\begin{aligned} x(0) &= x_0, \\ \sigma(t) &= f(t) + \int_0^t k(t,s,x(s))\,ds, \end{aligned} \qquad (2)$$

where $0 \leq t < \infty$, $A(t)$ and $B(t)$ are $n \times n$ matices, x, $\sigma, f, k$, and $H$ are $n$–vector functions. The system will be studied as a perturbation of the linear system:

$$\begin{aligned} y'(t) &= A(t)\,y(t) + \int_0^t B(t-s)\,y(s)\,ds, \\ y(0) &= y_0. \end{aligned} \qquad (3)$$

Here $y$ is $n$–vector. The study of system (1)–(2) is motived by recent studies and applications (see [7] an their references).

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We know the solutions of (1) exist on $[0,\infty)$, and are continuous by converting (1)–(2) to an integral system of the form:

$$x(t) = F(t) + \int_0^t E(t-s)\, x(s)\, ds,$$

where

$$F(t) = x_0 + \int_0^t x(s)\, H(s, x(s), \sigma(s))\, ds,$$

and

$$E(t-s) = A(s) + \int_0^t B(u-s)\, du,$$

defining Picard's successive approximations and proving uniform convergence (see [1]).

Boundedness of solution of (1) is the key problem. A solution $x(t)$ of (1) satisfying the initial condition (2) may be expressed by the variation of parameters formula as:

$$x(t) = R(t)\, x_0 + \int_0^t R(t-s)\, x(s)\, H(s, x(s), \sigma(s))\, ds, \quad t \geq 0, \tag{4}$$

where $R(t-s)$ is an $n \times n$ matrix which is the unique solution of equation:

$$\begin{aligned} \frac{\partial R(t-s)}{\partial t} &= A(t)\, R(t-s) + \int_s^t B(u-s)\, R(t-u)\, du, \\ R(0) &= I, \end{aligned}$$

where $I$ is theidentity matrix (see Grossman and Miller [4] for details).

Let $\mathbb{R}^n$ denote real $n$–dimensional Euclidian space of column vectors with norm $|\cdot|$, $J$ denotes the set of all $t$ such that $0 \leq t < \infty$. For $p$ in the interval $1 \leq p < +\infty$, $L^p$ is the usual Lebesgue space of measurable functions $f$ with norm:

$$\|f\|_p = \left( \int_0^\infty |f(t)|\, dt \right)^{1/p} < +\infty.$$

$LL^p$ is the set of all functions wich are locally $L^p$ in $J$. Let $C[X,Y]$ denote the space of continuous functions from $X$ to $Y$, where $X$ and $Y$ are convenient spaces. We shall assume that $H \in C[J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $f \in C[J, \mathbb{R}^n]$ and $k \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n]$. Many stability results in integro–differential systems of the type:

$$x'(t) = A(t)\, x(t) + \int_0^t B(t,s)\, x(s)\, ds + f(t), \tag{5}$$

have been obtain by constructing Liapunov's functionals. Such functionals for (5) require that $A(t)$ be negative, however in this paper we present our results without this condition. It was investigated in [2] relations between stability properties of solutions of (5) under various assumptions on $A, B$ and the dimension $n$ ($f \equiv 0$), furthermore, asymptotic stability, uniform asymptotic stability and boundedness of all solutions of some perturbed stability and boundedness of all solutions fo some perturbed form of (5) are also presented.

In [1] Burton considered (5) with $A$ constant and $B = B(t-s)$ and showed that the theory of existence, uniqueness, dimensionality of solutions space, and the variation fo parameters formula are virtually indistinguishable from the corresponding elementary theory of ordinary differential equations.

In [5] Mauhfoud considered (5) with $f \equiv 0$ and studied the stability of solutions of (5) via the construction fo Liapunov's functionals for (5).

This approach is similar to the one used by Burton and Mahfoud in [3], but they wrote (5) in a general form, using a method of decomposition.

Mahfoud in [6] gave sufficient conditions to insure that (5) has bounded solutions, the method used in new and the main result unifies, improves, and extends earlier results.

Our results here are more general and apply to (1) whether $A$ is stable, identically zero or completely unstable.

The system (1)–(2) has been considered in [7] under the same assumption, but the techniques used here are quite different. In particular, our results contain those in [7].

## 2. Properties of solutions

The following lemma play a central role in this paper.

**Lemma 1.** *Let $x(t)$, $a(t)$ and $b(t)$ continuous and no–negative real functions on $J$. If $c(t)$ is a continuous and positive function, defined on $J$, for wich the inequality:*

$$(1-\gamma)\, c(t) \le \int_0^t b(s)\, x(s)\, ds, \quad t \in J, \ \gamma \in [0,1) \cap \mathbb{R}, \tag{6}$$

*holds. Then*

$$x(t) \le \frac{a(t)\, c(t)}{(1-\gamma)\left(1-\frac{P}{2}\right) + \left[(1-\gamma)^2 \left(\frac{P}{2}\right)^2 - \alpha \int_0^t a(s)\, b(s)\, ds\right]^{1/2}}, \tag{7}$$

where $0 \leq t < +\infty$,

$$t_1 = \max\left\{t \in J : \int_0^t a(s)\, b(s)\, ds \leq \frac{(1-\gamma)^2 P^2}{4\alpha}\right\}$$

with $0 < p \leq 2$, $\alpha > 0$.

**Proof.** Since $c(t)$ is positive and non decreasing then it follows from (6) that:

$$(1-\gamma)\frac{x(t)}{c(t)} \leq \frac{x(t)}{c(t)} \int_0^t \frac{b(s)\, x(s)}{c(t)} ds \leq \frac{x(t)}{c(t)} \int_0^t \frac{b(s)\, x(s)}{c(s)} ds. \tag{8}$$

Let

$$z(t) := \int_0^t \frac{b(s)\, x(s)}{c(s)} ds; \quad z(0) = 0, \tag{9}$$

then we have

$$z'(t) = \frac{b(t)\, x(t)}{c(t)}. \tag{10}$$

Multiplying (8) by $b(t)$ we obtain:

$$(1-\gamma)\frac{b(t)\, x(t)}{c(t)} \leq \frac{b(t)\, x(t)}{c(t)} \int_0^t \frac{b(s)\, x(s)}{c(s)} ds. \tag{11}$$

To prove (7), notice that from (9), (10) and (11) we deduce that $(1-\gamma)\, pz'(t) \leq 2z(t)\, z'(t)$; $0 < p \leq 2$, hence

$$(1-\gamma)\, pz'(t) \leq \alpha a(t)\, b(t) + \left[z^2(t)\right]'. \tag{12}$$

Integrating (12) from 0 to $t$ we have:

$$pz(t) \leq \alpha \int_0^t a(s)\, b(s)\, ds + z^2(t). \tag{13}$$

This last inequality can be written as $(z - z_1)(z - z_2) \geq 0$, where:

$$z_{1,2}(t) = \frac{(1-\gamma)\, P}{2} \pm \left[(1-\gamma)^2 \left(\frac{P}{2}\right)^2 - \alpha \int_0^t a(s)\, b(s)\, ds\right]^{1/2}.$$

Thus,

$$z(t) \leq \frac{(1-\gamma)\, P}{2} - \left[(1-\gamma)^2 \left(\frac{P}{2}\right)^2 - \alpha \int_0^t a(s)\, b(s)\, ds\right]^{1/2}.$$

Using (8) we obtain:

$$(1-\gamma)\frac{x(t)}{c(t)} \leq a(t) + \frac{x(t)}{c(t)} z(t), \tag{14}$$

and thus:

$$(1-\gamma)\frac{x(t)}{c(t)} \leq$$

$$\leq \; a(t) + \frac{x(t)}{c(t)}\left[\frac{(1-\gamma)P}{2} - \left[(1-\gamma)^2\left(\frac{P}{2}\right)^2 - \alpha\int_0^t a(s)\,b(s)\,ds\right]^{1/2}\right],$$

yielding the desired inequality. ▮

**Remark 1.** *when* $\gamma \equiv 0$, *from condition* (6) *we obtain the inequality* (33), *Theorem* 4 *of* [8], *on the other, our result is obtained under milder conditions.*

**Remark 2.** *If* $\gamma \equiv 0$, $P \equiv 2$ *and* $\alpha \equiv 2$ *we obtain the inequality* (33), *Theorem* 4 *of* [8].

**Remark 3.** *The value*

$$z(t) = 1 + \left[1 - 2\int_0^t a(s)\,b(s)\,ds\right]^{1/2}$$

*given in* (36) *of* [8] *was not admissible, since this value of* $z(t)$ *will give us a lower bound for* $x(t)$; *however, we can consider*

$$z(t) = \frac{(1-\gamma)P}{2} + \left[(1-\gamma)^2\left(\frac{P}{2}\right)^2 - \alpha\int_0^t a(s)\,b(s)\,ds\right]^{1/2}$$

*when* $0 < p < 1$, *because* $(1-\gamma) - (1-\gamma)\frac{P}{2} > (1-\gamma)\frac{P}{2}$, *and then:*

$$(1-\gamma)\left(1 - \frac{P}{2}\right) > \left[(1-\gamma)^2\left(\frac{P}{2}\right)^2 - \alpha\int_0^t a(s)\,b(s)\,ds\right]^{1/2}.$$

*In this case we obtain:*

$$x(t) \leq \frac{a(t)\,c(t)}{(1-\gamma)\left(1-\frac{P}{2}\right) + \left[(1-\gamma)^2\left(\frac{P}{2}\right)^2 - \alpha\int_0^t a(s)\,b(s)\,ds\right]^{1/2}}.$$

Now, we give some simple criteria on the boundedness of solutions of (1)–2.

**Theorem 1.** *Let $y(t)$ be a solution of (3) such that the corresponding solution $x(t)$ of (1)–(2) is a continuous function. In addition we assume that the conditions:*

$$|R(t-s)H(s,x(s),\sigma(s))| \le b(s), \tag{15}$$

$$|x(t)| > (1-\gamma)c(t), \tag{16}$$

$$|y(t)| \le \frac{\alpha a(t)c(t)}{2(1-\gamma)} \le K, \quad \alpha > 0, \ K \in \mathbb{R}_+^*, \tag{17}$$

$$\int_0^\infty a(s)b(s)\,ds = M \le \frac{(1-\gamma)^2 P^2}{4\alpha}, \tag{18}$$

*are fulfilled, and $a(t), b(t), c(t)$ are the same of Lemma 1, then $x(t)$ is bounded on $0 \le t \le t_1 < +\infty$, where:*

$$t_1 = \max\left\{t \in J : \int_0^t a(s)b(s)\,ds \le \frac{(1-\gamma)^2 P^2}{4\alpha}\right\}, \quad 0 < p < 2; \ \alpha > 0.$$

**Proof.** Since

$$x(t) = y(t) + \int_0^t R(t-s)x(s)H(s,x(s),\sigma(s))\,ds,$$

then by (15), (16) and (17) we obtain:

$$|x(t)| \le \frac{\alpha a(t)c(t)}{2(1-\gamma)} + \frac{|x(t)|}{(1-\gamma)c(t)}\int_0^t b(s)|x(s)|\,ds \tag{19}$$

From Lemma 1:

$$|x(t)| \le \frac{a(t)c(t)}{(1-\gamma)\left(1-\frac{P}{2}\right) + \left[(1-\gamma)^2\left(\frac{P}{2}\right)^2 - \alpha\int_0^t a(s)b(s)\,ds\right]^{1/2}},$$

now, by (17) and (18), we conclude that $x(t)$ is bounded. ∎

**Corollary 1.** *If in the Theorem 1, we put:*

$$|R(t-s)H(s,x(s),\sigma(s))| \le b(s)\exp(\beta s), \quad \beta > 0, \tag{20}$$

$$|y(t)| < \frac{a(t)c(t)}{2(1-\gamma)}\exp(-\beta t), \tag{21}$$

*and in addition, we suppose the condition:*

$$a(t)\,c(t) \le k_1, \quad k_1 \in \mathbb{R}_+^* \tag{22}$$

*then*

$$x(t) \to \infty \quad \text{as } t \to \infty. \tag{23}$$

**Proof.** As

$$x(t) = y(t) + \int_0^t R(t-s)\,x(s)\,H(s, x(s), \sigma(s))\,ds,$$

using (20) and (22) we obtain:

$$\begin{aligned} |x(t)|\exp(\beta t) &\le \\ &\le \frac{a(t)\,c(t)}{2(1-\gamma)} + \\ &+ \frac{|x(t)|}{(1-\gamma)\,c(t)}\exp(\beta t)\int_0^t b(s)\,|x(s)|\exp(\beta s)\,ds. \end{aligned}$$

Let $x_1(t) = x(t)\exp(\beta t)$. From Theorem 1 we see that $|x_1(t)| \le K$, from which (23) follows. The Theorem is proved. ∎

**Remark 4.** *Notice the advantage of condition* (20) *over* (15).

**Theorem 2.** *Let* $y(t)$ *be a solution of* (3) *and* $x(t)$ *as in the above theorem. If the following conditions:*

$$\left|\frac{\partial R}{\partial t}\right| \le b(t), \tag{24}$$

$$|x(t)\,H(s, x(s), \sigma(s))| \le c(t)\left[|x(t)| + |x'(t)|\right], \tag{25}$$

$$b(t)\,|x_0| + |x(t)\,H(s, x(s), \sigma(s))| \le a(t), \tag{26}$$

$$\int_0^\infty [a(s) + b(s)\,E(s, r, \lambda)]\,d\lambda ds < \infty, \tag{27}$$

*where*

$$E(s, r, \lambda) = \int_0^s c(\lambda)\left[D(\lambda) + B(\lambda)\right]\exp\left(\int_\lambda^s c(r)\left[b(r) - 1\right]dr\right)d\lambda,$$

$$B(t) = \int_0^t c(s)\,D(s)\exp\left(\int_s^t \left[b(\lambda) + c(\lambda) + b(\lambda)\,c(\lambda)\right]d\lambda\right)ds$$

*and*

$$D(t) = x_0 + a(t) + \int_0^t a(s)\,ds,$$

*hold.*

*Then from boundedness of* $y(t)$ *we obtain the boundedness of* $x(t)$.

**Proof.** Since

$$x(t) = R(t)\,x_0 + \int_0^t R(t-s)\,x(s)\,H(s, x(s), \sigma(s))\,ds,$$

then it follows from here:

$$x'(t) = x_0 \frac{\partial R}{\partial t} + \int_0^t \frac{\partial R}{\partial t} x(s)\,H(s, x(s), \sigma(s))\,ds + x(t)\,H(t, x(t), \sigma(t)).$$

Thus, in view of (24)–(26) we obtain:

$$|x'(t)| \le a(t) + b(t) \int_0^t c(s)\left[|x(s)| + |x'(s)|\right] ds.$$

Now, from Theorem 1 of [8] we have

$$|x'(t)| \le a(t) + b(t)\,E(s, r, \lambda).$$

Integration of both sides from 0 to $t$ yields:

$$|x(t)| \le \int_0^\infty \left[a(s) + b(s)\,E(s, r, \lambda)\right] ds + |x_0|,$$

then, it follows that (27) implies that $|x(t)| \le k$, $k \in \mathbb{R}_+^*$, i.e., $x(t)$ is bounded. The proof is now complete. ∎

**Corollary 2.** *If the condition:*

$$\exists M > 1 : \beta\,|x(t)| \le (M-1)\,|x'(t)|, \tag{28}$$

$$\left|\frac{\partial R}{\partial t}\right| \le \frac{b(t)}{M} \exp(-\beta t), \quad \beta > 0, \tag{29}$$

$$|x(t)\,H(t, x(t), \sigma(t))| \le c(t)\left[x(t) + |x'(t) + \beta x(t)|\right], \tag{30}$$

$$b(t)\,|x_0| + |x(t)\,H(t, x(t), \sigma(t))| \exp(\beta t) \le \frac{a(t)}{M}, \tag{31}$$

*and* (27) *hold. Then* $x(t) \to 0$ *as* $t \to \infty$.

**Proof.** From (29), (30) and (31) we obtain:

$$\begin{aligned} |x'(t)| &\le \\ &\le \frac{a(t)}{M} \exp(-\beta t) \\ &\quad + \frac{b(t)}{M} \exp(-\beta t) \int_0^t c(s)\left[|x(t)| \exp(\beta s) + |x'(s) + \beta x(s)| \exp(\beta s)\right] ds, \end{aligned} \tag{32}$$

but this is equivalent to:

$$|x'(t)|\exp(\beta t) \le$$
$$\le \frac{a(t)}{M} + \frac{b(t)}{M}\int_0^t \left[|x(t)|\exp(\beta s) + \left|[x(s)\exp(\beta s)]'\right|\right] ds.$$

From (28) we have:

$$\begin{aligned} M|x'(t)|\exp(\beta t) &\ge [|x'(t)| + \beta|x(t)|]\exp(\beta t) \\ &\ge |x'(t)\exp(\beta t) + \beta x(t)\exp(\beta t)| \\ &= \left|[x(t)\exp(\beta t)]'\right|. \end{aligned} \quad (33)$$

Therefore we transforms (33) in

$$\left|[x(t)\exp(\beta t)]'\right| \le$$
$$\le a(t) + b(t)\int_0^t \left\{|x(s)|\exp(\beta s) + \left|[x(s)\exp(\beta s)]'\right|\right\} ds.$$

From this and the notation $x_1(t) = x(t)\exp(\beta t)$ it follows that

$$|x_1'(t)| \le a(t) + b(t)\int_0^t c(s)\left[|x_1(s)| + |x_1'(s)|\right] ds. \quad (34)$$

Thus

$$|x_1'(t)| \le a(t) + b(t)E(s, r, \lambda). \quad (35)$$

After integrating both sides from 0 to $t$ by (33) we deduce:

$$\begin{aligned} \int_0^t M|x'(s)|\exp(\beta s)\,ds &\ge \int_0^t \left|[x(s)\exp(\beta s)]'\right| ds \\ &\ge \left|\int_0^t [x(s)\exp(\beta s)]\,ds\right| \end{aligned}$$

and hence

$$M\int_0^t |x'(s)|\exp(\beta s)\,ds \ge |x(t)\exp(\beta t) - x_0|. \quad (36)$$

The integration of (36) between 0 and $t$, (33) and (36) yields:

$$|x(t)\exp(\beta t) - x_0| \le M\int_0^t [a(s) + b(s)E(r, s, \lambda)]\,ds.$$

Therefore:

$$|x(t)|\exp(\beta t) \le M\int_0^\infty [a(s) + b(s)E(r, s, \lambda)]\,ds + |x_0|.$$

Hence $|x(t)|\exp(\beta t) \le Mk + |x_0|$, for some $k$, and we conclude that $|x(t)| \le [Mk + |x_0|]\exp(-\beta t) \to 0$ as $t \to \infty$. ∎

**Remark 5.** *Notice that in Theorem 2 and your corollary, it is not necessary the monotony of $c(t)$.*

Now, may be easily obtained the following result:

**Corollary 3.** *Under conditions of Corollary 1 or 2, the zero solution of (1)–(2) is asymptotically stable.*

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