


Benford's Law with small sample sizes: A new exact test useful in health sciences during epidemics

Ley de Benford con muestras pequeñas: una prueba exacta nueva útil ciencias de la salud durante epidemias

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Abstract

Benford or "first digit" law has been used successfully to evaluate epidemiological surveillance systems, especially during epidemics. Conventional statistical methods for evaluation (x^2 and log-likelihood ratio) are controversial when the number of data is small ($n < 7$). In this methodological note a new test is proposed to evaluate compliance with Benford's law with small samples, which can be used with biomedical, medical and public health data.

Keywords: Analysis of data; Epidemics; COVID-19 infection; Public health emerging infections.

Resumen

La ley de Benford o de los "primeros dígitos" ha sido usada exitosamente para evaluar los sistemas de vigilancia epidemiológica, en especial durante epidemias. Los métodos estadísticos convencionales para la evaluación (x^2 y razón de log-verosimilitud) son controversiales cuando los datos son poco ($n < 7$). En esta nota metodológica se propone una nueva prueba para evaluar el cumplimiento de la ley de Benford con muestras pequeñas, que puede ser usada con datos de biomedicina, medicina y salud pública.

Palabras clave: Análisis de datos; Epidemia; Infección con COVID-19; Infecciones emergentes en salud pública.

In some articles published the Benford's or first-digit Law was proposed as a cost effective tool to evaluate data in biomedicine, medicine and public health¹⁻⁶ (see **Figure 1**). Its use could be very important in sanitary emergencies as COVID-19 epidemic, when rapid evaluation of epidemiological surveillance systems require to be evaluated⁷. Since the use can be

controversial when only few data are available ($n < 7$), we developed a new exact test to screen the fulfilment of Benford distribution. Under this law, the expected number digits for sample sizes varying from $n=1$ to 6 are in **Table 1**. In this case we assume that data come from an independent sequence of events (i.e., the occurrence of any particular digit doesn't depend on the occurrence

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of any previous one). For this particular case, it can be used according with the known probabilities given by Benford, and using it as the scenario corresponding to

the H_0 : “the data are Benford’s law distributed”. It can be expressed with the equation:

$$H_0: E[X_1, X_2, \dots, X_9 | n] = \left(n \log_{10} \left(1 + \frac{1}{1} \right), n \log_{10} \left(1 + \frac{1}{2} \right), \dots, n \log_{10} \left(1 + \frac{1}{9} \right) \right)$$

where

$$P[H_0] = \frac{n!}{X_1! X_2! \dots X_9!} \left(\log_{10} \left(1 + \frac{1}{1} \right) \right)^{X_1} + \left(\log_{10} \left(1 + \frac{1}{2} \right) \right)^{X_2}, \dots, \left(\log_{10} \left(1 + \frac{1}{9} \right) \right)^{X_9}$$

Naturally, data coming from no-Benford distribution are less probable to appear distributed like that, and consequently are more likely to reject the H_0 . Thus, the analysis does not depend on the sample size.

To observe the performance of this test we used data with small sample sizes ($n < 7$) from a previous publication⁶. This test was implemented in the R package, using the code: `dmultinom(x = c(#1,#2,#3,#4,#5,#6,#7,#8,#9), size = NULL, prob = c(0.301, 0.176, 0.125, 0.097, 0.079, 0.067, 0.058, 0.051, 0.046))`, where $\#i$ represents the absolute frequency of the correspondent digit $i=1, \dots, 9$. With this procedure we obtained exact probabilities. After, p values were calculated with the EMT package developed by Menzel⁸ using the code: `observed <- c(4,1,0,0,0,0,0,0,0) # observed data: underH0 <- c(0.301, 0.176, 0.125, 0.097, 0.079, 0.067, 0.058, 0.051, 0.046) # underH0 out <- multinomial.test(observed, underH0) # p.value`, where $\#i$ represents the absolute frequency of the correspondent digit $i=1, \dots, 9$. Comparisons with Kuiper’s tests were realized⁹.

With the new statistical test is possible to extend the use of Benford’s law to biomedical, medical and public health areas with small sample sizes.

References

1. Morag S, Salmon-Divon M. Characterizing human cell types and tissue origin using the Benford law. *Cells*. 2019;8(9): E1004. doi: <http://10.3390/cells8091004>
2. Pollach G, Brunkhorst F, Mipando M, Namboya F, Mndolo S, Luiz T. The “first digit law” - A hypothesis on its possible impact on medicine and development aid. *Med Hypotheses*. 2016; 97:102-

106. doi: <http://10.1016/j.mehy.2016.10.021>
3. Pinilla J, López-Valcárcel BG, González-Martel C, Peiro S. Pinocchio testing in the forensic analysis of waiting lists: using public waiting list data from Finland and Spain for testing Newcomb-Benford’s Law. *BMJ Open*. 2018;8(5):e022079. doi: <http://10.1136/bmjopen-2018-022079>
4. Manrique-Hernández EF, Fernández-Niño JA, Idrovo AJ. Global performance of epidemiologic surveillance of Zika virus: rapid assessment of an ongoing epidemic. *Public Health*. 2017;143:14-16. doi: <http://10.1016/j.puhe.2016.10.023>
5. Gómez-Camponovo M, Moreno J, Idrovo AJ, Páez M, Achkar M. Monitoring the Paraguayan epidemiological dengue surveillance system (2009-2011) using Benford’s law. *Biomedica*. 2016;36(4):583-592. doi: <http://10.7705/biomedica.v36i4.2731>
6. Idrovo AJ, Fernández-Niño JA, Bojórquez-Chapela I, Moreno-Montoya J. Performance of public health surveillance systems during the influenza A(H1N1) pandemic in the Americas: testing a new method based on Benford’s Law. *Epidemiol Infect*. 2011;139(12):1827-34. doi: <http://10.1017/S095026881100015X>
7. Zhu N, Zhang D, Wang W, et al. A novel coronavirus from patients with pneumonia in China, 2019. *N Engl J Med*. 2020;382:727-733.
8. Menzel U. EMT. Exact multinomial test: goodness-of-fit test for discrete multivariate data. R package version 1.0; 2012.
9. Kuiper NH. Tests concerning random points on a circle. *Proceedings Koninklijke Nederlandse Akademie van Wetenschappen, Series A* 1962;63:38–47.

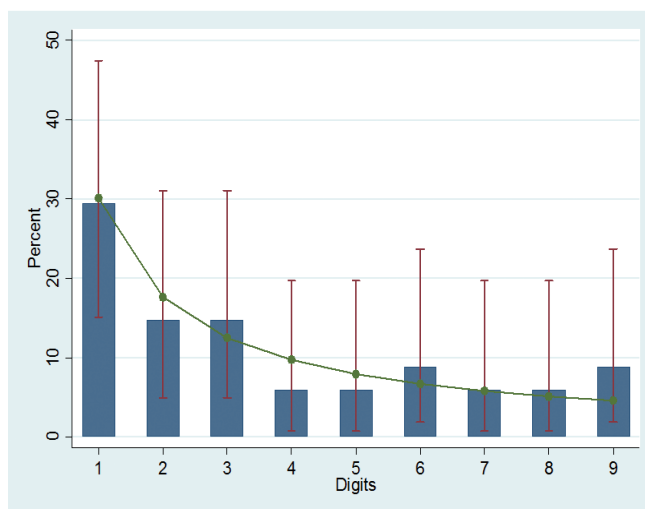


Figure 1. Fulfilment of Benford's law of data on COVID-19 outbreak from Chinese provinces, regions and cities – situation report 17 (n=34).

Table 1. Expected occurrence of first digits following Benford distribution with small sample sizes.

Sample size	Expected first digit								
	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
	(100%)								
3	1	1	0	0	0	0	0	0	0
	(50%)	(50%)							
4	1	1	1	0	0	0	0	0	0
	(33.33%)	(33.33%)	(33.33%)						
5	2	1	1	0	0	0	0	0	0
	(50%)	(25%)	(25%)						
6	2	1	1	1	0	0	0	0	0
	(40%)	(20%)	(20%)	(20%)					

Table 2. Fulfilment of Benford distribution by American countries reporting few data during the influenza A(H1N1) outbreak (epidemiological weeks 13–47, 2009).

Country	Weeks*	First digit (%)									Kuiper	Exact multinomial†
		1	2	3	4	5	6	7	8	9		
Antigua and Barbuda	3	2	1	0	0	0	0	0	0	0	<0.005	1
Saint Kitts and Nevis	3	1	2	0	0	0	0	0	0	0	<0.005	0.847
St. Vincent and Grenadines	3	3	0	0	0	0	0	0	0	0	<0.005	0.817
Surinam	4	2	1	0	0	0	0	1	0	0	<0.010	1
Belize	4	2	0	0	0	0	1	0	1	0	<0.010	0.824
Haiti	4	0	1	2	1	0	0	0	0	0	<0.010	0.344
Granada	4	4	0	0	0	0	0	0	0	0	<0.005	0.671
Santa Lucia	5	1	1	1	1	1	0	0	0	0	<0.010	0.847
Uruguay	5	3	1	1	0	0	0	0	0	0	<0.010	1
Dominica	5	2	1	0	0	0	1	0	1	0	<0.010	0.616
Guyana	5	0	2	1	0	2	0	0	0	0	<0.01	0.188

* Only weeks with report (one or more cases) to the Pan American Health Organization.

† estimated with EMT package.