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Solution of the traffic flow equation using the finite element method

Solución de la ecuación de flujo de tráfico usando el método de elementos finitos

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Abstract

In this document we will study and solve the nonlinear partial differential equation, with initial conditions for vehicle entry that serves to model the dynamics of traffic flow. To find a numerical solution to the dynamics that govern the behavior of traffic flow, the Finite Element Method in a spatial dimension was used. In accordance with the temporal dynamics, simulations were developed to know the flow in terms of time. The numerical solution is interesting for predicting the number of vehicles at the entrance to a high-flow road. Some theorems are enunciated that guarantee the existence of the solution and the uniqueness is given by the boundary conditions.

Keywords: combination linear; Dirichlet conditions; Neumann conditions; Robin conditions; contour; partial differential equation; traffic Flow; positive semidefinite matrix; finite element method; numerical solution; tridiagonal.

Resumen

En este documento estudiaremos y resolveremos la ecuación diferencial parcial no lineal, con condiciones iniciales de entrada de vehículos que sirve para modelar la dinámica del flujo de tráfico. Para encontrar una solución numérica de la dinámica que gobierna el comportamiento del flujo de tráfico, se usó el Método de Elementos Finitos en una dimensión espacial. De acuerdo con la dinámica temporal se desarrollaron simulaciones para conocer el flujo en términos del tiempo. La solución numérica resulta interesante para la predicción de la cantidad de vehículos a la entrada de una vía de alto flujo. Se enuncian algunos teoremas que garantizan la existencia de la solución y la unicidad viene dada por las condiciones de contorno.

Palabras clave: combinación lineal; condiciones de Dirichlet; condiciones de Neumann; condiciones de Robin; contorno; ecuación diferencial parcial; flujo de tráfico; matriz semidefinida positiva; método de elementos finitos; solución numérica; tridiagonal.

1. Introduction

The dynamics of traffic flow have been extensively studied in the literature. The conservation equations of the model (dynamics) come from a nonlinear partial differential equation. The solution develops under the initial conditions of the vehicle input density [1]. These initial conditions allow decoupling each of the equations

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for each input and solving independently. The function that determines the initial condition for the one-dimensional problem is known and defined on the solution interval, giving the solution to the Cauchy problem [2], [3]. Velocity is a function that depends on density. This scalar conservation law must be complemented with adequate initial conditions and with boundary conditions as developed in this document.

Traffic flow models and simulation tools are often used for traffic state estimation and prediction [4]. Traffic flow models such as the kinematic-wave model, "higher order" models, and car tracking models are more or less accurate representations of reality. In a simulation tool, the model equations are solved using numerical methods, again with near precision. On the other hand, the development of the finite element method in a spatial dimension allows calculating, from the point of numerical analysis, a system of linear equations that allows finding the value or prediction of flow in each of the points in space. The system of linear equations has the property of forming a tridiagonal positive semidefinite matrix whose inverse does not entail a higher computational cost. The formulation of the numerical solution to the traffic problem is obtained by comparing it to the fluid mechanics problem, reaching highly accurate results [5], [6]. By the similarity between fluid dynamics, traffic flow dynamics, and Newton's second law, the one-dimensional finite element method for velocity estimation is proposed as a numerical solution as an application of motion fluids mechanics [7]. Due to the nonlinearity in the temporal variable, the behavior of the solution from specific points of the spatial variable is evaluated in several temporal lines.

The theorems that allow us to guarantee the existence and uniqueness of the solution given by the Dirichlet [8], [9], [10], Robin [11] and Neumann boundary conditions will be presented in section 2 of this document.

2. Content

2.1. Problem model

In the domain $\Omega = (x_0, x_n)$, find u with $u(x_0) = u(x_n) = u_c$ for a given function f, such they satisfy the Dirichlet initial conditions.

$$u = \arg\min_{\phi \in X} J(\phi) \qquad X = H_0^1(\Omega) \tag{1}$$

Dirichlet minimization statement [5].

$$J(\phi) = \frac{1}{2}\int_{x_0}^{x_n}\left(\frac{d\phi}{dx}\right)^2 dx - \int_{x_0}^{x_n} f\phi dx \tag{2}$$

To find $\phi \in X$ such that:

$$\delta J_v(\phi) = 0, \qquad \forall v \in X \tag{3}$$

$$\int_{x_0}^{x_n} \frac{d\phi}{dx}\frac{dv}{dx} dx = \int_{x_0}^{x_n} f\phi dx \quad \forall v \in X \tag{4}$$

Noticing,

$$a(\phi, v) = \int_{x_0}^{x_n} \frac{d\phi}{dx}\frac{dv}{dx} dx \tag{5}$$

$$l(v) = \int_{x_0}^{x_n} f\phi dx \tag{6}$$

Should be minimized,

$$u = \arg\min_{\phi \in X} \ \frac{1}{2}a(\phi, v) - l(v) \qquad u \in X\colon\ a(\phi, v) = l(v)\,,\ \forall v \in X \tag{7}$$

Without losing generality, for any $l(v) \in H^{-1}(\Omega)$, then it is required to find $u \in H_0^1(\Omega)$ [6] such that:

$$u = \arg\min_{\phi \in H^{-1}(\Omega)} \ \frac{1}{2}a(\phi, \phi) - l(\phi) \tag{8}$$

For example, $l(v) = \langle \delta_{x0}, v\rangle = v(x_0)$ is admissible.

With the regularity theorems for boundary problems, the model takes the following form [7]:

Theorem 1. Let $\Omega \in \mathbb{R}^n$ be an open of class C^2 with bounded frontier, $u \in L^2(\Omega)$ and $f \in H_0^1(\Omega)$ be the solution of the Dirichlet problem. So, $f \in H_0^2(\Omega)$ given the fact that:

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(f(x,t)u(x)) = 0 & in\ \Omega \\ f = 0 & over\ the\ boundary\ \partial\Omega \end{cases} \tag{9}$$

$$\|f\|H^2(\Omega) \leq C\|u\|_{L^2(\Omega)} \tag{10}$$

With a constant $C > 0$ that only depends on Ω.

Furthermore, if Ω is of class C^{m+2} and $u \in H^m(\Omega)$, with $m \geq 1$ integer, then $f \in H^{m+2}(\Omega)$ and there is a constant $C_m > 0$ that only depends on m and Ω, in such a way that:

$$\|f\|H^{m+2}(\Omega) \leq C_m\|u\|_{H^m(\Omega)} \tag{11}$$

If we have that $m > n/2$ then $f \in C^2(\bar{\Omega})$, giving way to the following two theorems given the initial conditions of the problem [11].

Theorem 2. Let $\Omega \in \mathbb{R}^n$ be an open of class C^2 with bounded frontier, $u \in L^2(\Omega)$ $\sigma \in C^2(\bar{\Omega})$ and $f \in H_0^1(\Omega)$ be the solution to the Robin problem. So, $f \in H_0^2(\Omega)$ given the fact that:

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(f(x,t)u(x)) = 0 \quad in\ \Omega \\ \frac{\partial f}{\partial x} + \sigma(x) f = 0 \qquad over\ the\ boundary\ \partial\Omega \end{cases} \tag{12}$$

$$\|f\| H^2(\Omega) \leq C\|u\|_{L^2(\Omega)} \tag{13}$$

With a constant $C > 0$ that only depends on Ω.

On the other hand, if Ω is of class C^{m+2} and $u \in H^m(\Omega)$ and $\sigma(x) \in C^{m+1}(\bar{\Omega})$, with $m \geq 1$ integer, then $f \in H^{m+2}(\Omega)$ and there is a constant $C_m > 0$ that only depends on m y Ω, such that:

$$\|f\| H^{m+2}(\Omega) \leq C_m \|u\|_{H^m(\Omega)} \tag{14}$$

If $m > n/2$ then $f \in C^2(\bar{\Omega})$, giving way to the following two theorems given the initial conditions of the problem [12].

$$\|u\| H^{m+2}(\Omega) \leq C_m \|f\|_{H^m(\Omega)} \tag{15}$$

If we have $m > \frac{n}{2}$ then $u \in C^2(\bar{\Omega})$.

Theorem 3. Let $\Omega \in \mathbb{R}^n$ an open of class C^2 with bounded, $u \in L^2(\Omega)$ and $f \in H_0^1(\Omega)$ the solution of the Neumann problem. So, $f \in H_0^2(\Omega)$ given the fact that:

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(f(x,t)u(x)) = 0 \quad in\ \Omega \\ \frac{\partial f}{\partial x} = 0 \qquad over\ the\ boundary\ \partial\Omega \end{cases} \tag{16}$$

$$\|f\| H^2(\Omega) \leq C\|u\|_{L^2(\Omega)} \tag{17}$$

With a constant $C > 0$ that only depends on Ω.

Furthermore, if Ω is of class C^{m+2} and $u \in H^m(\Omega)$, with $m \geq 1$ integer, then $f \in H^{m+2}(\Omega)$ and there is a constant $C_m > 0$ that only depends on m and Ω, such that:

$$\|f\| H^{m+2}(\Omega) \leq C_m \|u\|_{H^m(\Omega)} \tag{18}$$

If we have that $m > n/2$ then $f \in C^2(\bar{\Omega})$, giving way to the following two theorems given the initial conditions of the problem [13].

2.2. Solution using finite elements

The domain is discretized through the triangulation T_h, in the space of X_h where it is generated by the basis $X_h = span\{\varphi_1, \dots, \varphi_n\}$ [14], [15], [16].

Be

$$u_h(\in X_h) = \sum_{j=1}^{n} u_{hj}\varphi_j(x) \tag{19}$$

Set u_{hj} such that:

$$\begin{gathered} \delta J_v(u_h) = 0, \ \forall v \in X_h \\ a(u_h, v) = l(v), \ \forall v \in X_h \end{gathered} \tag{20}$$

Since any $v \in X_h$ can be written as a linear combination of the form:

$$\begin{gathered} v = \sum_{i=1}^{n} v_i\varphi_i(x) \\ a(u_h, v) = l(v), \ \forall v \in X_h \end{gathered} \tag{21}$$

$$\begin{gathered} a(u_h, \textstyle\sum_{i=1}^{n} v_i\varphi_i(x)) = \\ l(\textstyle\sum_{i=1}^{n} v_i\varphi_i(x)), \forall v \in \mathbb{R}^n \end{gathered} \tag{22}$$

Knowing that: $u_h = \sum_{j=1}^{n} u_{hj}\varphi_j(x)$, then we have:

$$\begin{gathered} a\left(\textstyle\sum_{j=1}^{n} u_{hj}\varphi_j(x), \sum_{i=1}^{n} v_i\varphi_i(x)\right) = \\ l(\textstyle\sum_{i=1}^{n} v_i\varphi_i(x)), \ \forall v \in \mathbb{R}^n \end{gathered} \tag{23}$$

Written in matrix form, we get:

$$v^t A_h u_h = v^t F_h, \quad \forall v \in \mathbb{R}^n \tag{24}$$

Taking the following expansion, the above equations (24) can be developed as follows:

$$v = [1 \quad 0 \quad \dots \quad 0]^t \tag{25}$$

We have:

$$\sum_{j=1}^{n} A_{h1j} u_{hj} = F_{h1} \tag{26}$$

Then taking:

$$v = [0 \quad 1 \quad \dots \quad 0]^t \tag{27}$$

We have:

$$\sum_{j=1}^{n} A_{h2j} u_{hj} = F_{h2} \tag{28}$$

And so on $v^t A_h u_h = v^t F_h, \quad \forall v \in \mathbb{R}^n$, with $A_h u_h = F_h$

The elements of the Matrix A_h [17] are constructed with φ_i as linear functions (Figure 1) and the derivatives of $\varphi_i, \frac{d\varphi_i}{dx}$ for i=1 without losing generality (Figure 2).

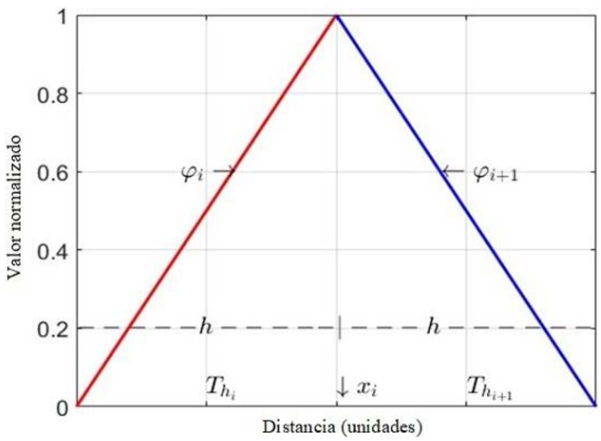

Figure 1. Functions in a linear way [17].

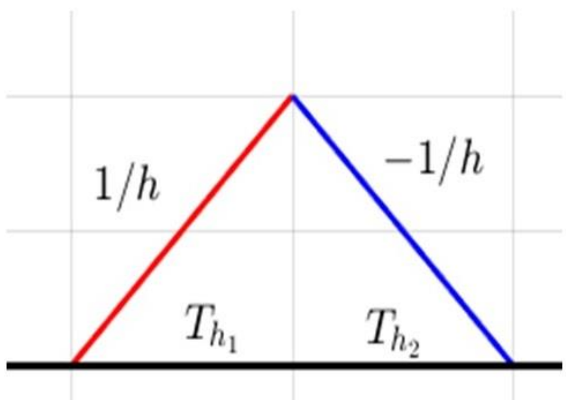

Figure 2. Derivative of the shape functions on the intervals $\frac{d\varphi_i}{dx} = \frac{1}{h}, \frac{d\varphi_{i+1}}{dx} = -\frac{1}{h}$ [17].

The A_{hij} elements are given by:

$$\begin{aligned} A_{hij} &= \int_{\Omega} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} dx \\ &= \iint_{T_{hi}} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} dx + \iint_{T_{hi+1}} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} dx \end{aligned} \tag{29}$$

The nonzero elements are $j = i-1, i, i+1$

$$A_{hii} = \frac{1}{h^2} h + \frac{1}{h^2} h = \frac{2}{h} \tag{30}$$

$$A_{hii-1} = \frac{1}{h}\left(\frac{-1}{h}\right) h = -\frac{1}{h} \tag{31}$$

$$A_{hii+1} = \left(\frac{-1}{h}\right)\frac{1}{h} h = -\frac{1}{h} \tag{32}$$

The border lines

$$A_{h11} = \frac{2}{h}, \quad A_{h12} = \frac{-1}{h}, \tag{33}$$

$$A_{h\,n\,n} = \frac{2}{h}, \quad A_{h\,n\,n-1} = \frac{-1}{h} \tag{34}$$

The formation of the elements of the A_h matrix characterizes this matrix as being positive definite [18] ($A_h \succ 0$), diagonally dominant, sparse, and tridiagonal as shown below:

$$A_h = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ 0 & 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{35}$$

The "charge" vectors are constructed in the general case as:

$$l(v): F_{hi} = l(\varphi_i) \tag{36}$$

$$l(v) = \int_{\Omega} f v dx \tag{37}$$

$$F_{hi} = \int_{T_{hi}} f\varphi_i dx + \int_{T_{hi+1}} f\varphi_i dx, \quad i = 1, \dots, n \tag{38}$$

Figure 3 describes the integration of f on $i \;\; e \;\; i+1$. On the other hand, we have that $u_h \in \mathbb{R}^n$ satisfies:

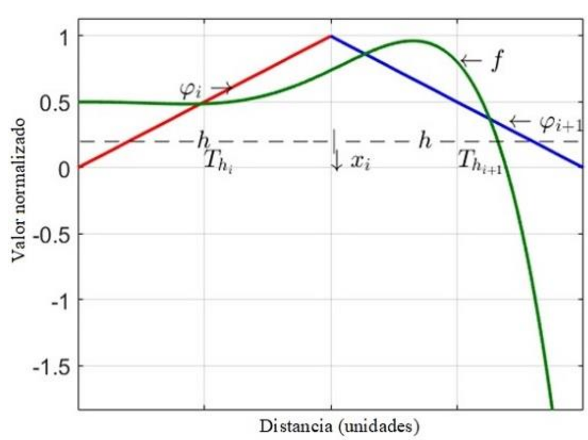

Figure 3. Integration of f in the elements $i, i+1$.

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ 0 & 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_{h1} \\ u_{h2} \\ \vdots \\ u_{hi} \\ \vdots \\ u_{hn} \end{bmatrix} = \begin{bmatrix} F_{h1} \\ F_{h2} \\ \vdots \\ F_{hi} \\ \vdots \\ F_{hn} \end{bmatrix} \quad (39)$$

To form the orthogonal basis of linear functions, we have the "mass" matrix, which has the property $M_h \in \mathbb{R}^{nxn} \succ 0$ to be positive semidefinite [19], the terms of this matrix are obtained by integrating the functions in a way, like this:

$$M_{hij} = \int_{\Omega} \varphi_i \varphi_j dx \quad (40)$$

$$v^t M_h v = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} v_j \int_{\Omega_i} \varphi_i \varphi_j dx \quad (41)$$

$$v^t M_h v = \int_{x_i}^{x_{i+1}} \sum_{i=1}^{n} v_i \, \varphi_i \sum_{j=1}^{n} v_j \varphi_j \, dx \quad (42)$$

$$v^t M_h v = \int_{x_i}^{x_{i+1}} (\sum_{i=1}^{n} v_i \, \varphi_i)^2 \, dx > 0 \quad (43)$$

$$\sum_{i=1}^{n} v_i \, \varphi_i \rightarrow v \in X_h \quad (44)$$

If $v \neq 0$, then φ_i are the bases in linear form. Therefore, for linear elements the nodal linear basis is given by the "mass" matrix [20]:

$$M_h = h\begin{bmatrix} 2/3 & 1/6 & 0 & 0 & \dots & 0 \\ 1/6 & 2/3 & 1/6 & 0 & \dots & 0 \\ 0 & 1/6 & 2/3 & 1/6 & \dots & 0 \\ 0 & 0 & 1/6 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2/3 & 1/6 \\ 0 & 0 & 0 & 0 & 1/6 & 2/3 \end{bmatrix} \quad (45)$$

In the temporary variable it is discretized with the recurrence equation:

$$\frac{df_i}{dt} = \frac{f_{i+1} - f_{i-1}}{2\Delta t} \quad (46)$$

In the case of linear basis elements with equal time division, the finite element approximation provides a coupled linear system of equations for each of the f_i with $i = 2, \dots, n$. In this way we arrive at the numerical solution of the finite element expansion, as shown below:

$$u(x,t) = \sum_{i=1}^{n} f_i(t) \varphi_i(x) \quad (47)$$

3. Analysis and results

Figure 4 shows the dynamics of the density. In the case study of this document, this function is taken as the initial condition for the Cauchy problem, posed through equations (9) and (24).

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(v(x,t)u(x)) = 0 & in \ \Omega \\ u = 0 & over \ the \ boundary \ \partial\Omega \end{cases} \quad (48)$$

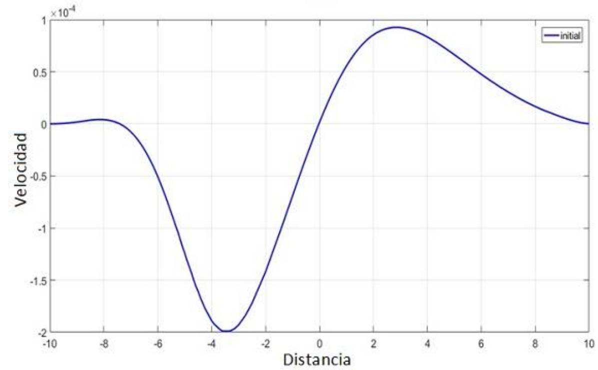

Figure 4. Initial density conditions.

v is the known velocity (initial conditions), and u is the density ($vehicles/m$).

The behavior of the velocity at different points in the spatial domain is shown in Figure 5, i.e., $(t,x), (t,0)$ blue curve, $(t,1)$ black curve, $(t,5)$ green curve, and $(t,10)$ red curve.

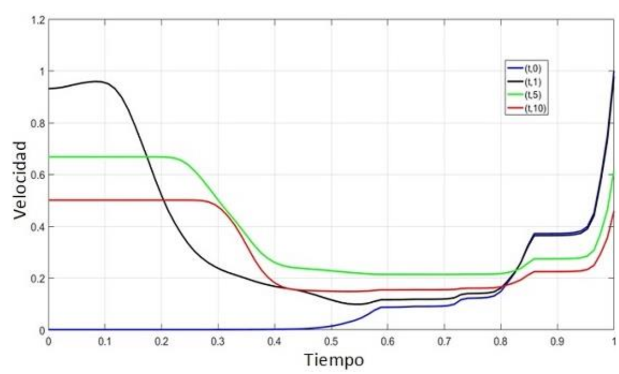

Figure 5. Velocity $v(t)$.

Using Figure 6, the numerical solution of the density of vehicles per meter can be verified. This solution is subject to the initial conditions established by the function proposed in Figure 4.

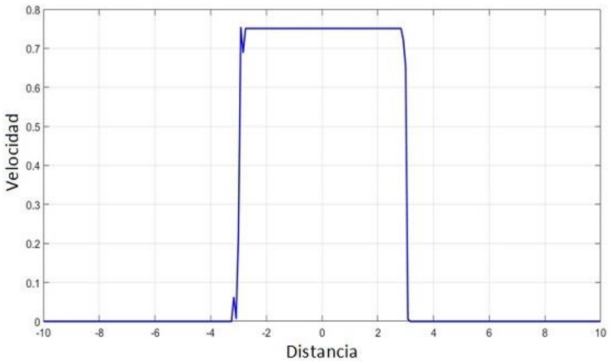

Figure 6. Input density $f(x)$.

4. Conclusions

Due to the velocity limit that must be met for the theorems outlined in this document, the solution is bounded between 0 and 1 vehicles/meter as shown in Figures 4 to 6. On the other hand, the non-linear solution given by the temporal variable is estimated by means of finite elements for 5 specific points in the space of the velocity dynamics.

The theorems described above are satisfied if $\Omega = \mathbb{R}^n$ and the solution obtained by any numerical method converges to the initial conditions and meets the initial conditions determined according to the problem posed (Dirichlet, Robin, or Neumann).

Due to the velocity of propagation, to solve the Cauchy problem for traffic flow in a whole network, it is proposed to build a local solution in a neighborhood. The solution found meets the conditions of the problem posed with equations (9) and (24) and the numerical solution given by equation (39).

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Autor Contributions

F. Mesa: Conceptualization, Data curation, Investigation, Writing – original draft, Writing – review & editing. D. Devia-Narváez: Investigation, Validation, Writing – review & editing. E. I. Gutierrez-Velasquez: Investigation, Validation, Writing – review & editing. R. Ospina-Ospina: Conceptualization, Methodology, Project administration, Supervision, Writing – original draft.

All authors have read an agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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